Quantum Channels as Temporal Correlations in Quantum Mechanics

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Since the eminent work of J. S. Bell, spatial correlations in quantum mechanics have gained much progress. Nevertheless, up to now, there is no agreement on the nature of temporal correlations. In this paper, based on the entangled-history theory, we prove that temporal correlations are just quantum channels. Moreover, by using the state tomography technology, the quantum channel can be uniquely determined. What is more, through the entanglement of formation, temporal correlations can be quantified. Cases when the temporal correlation is strongest and weakest are investigated, too.

Introduction: Even though the terminology, "entanglement", was introduced by E. Schrödinger in the early days of quantum mechanics [1], only after the famous work by J. S. Bell [2], people started to appreciate extraordinary features of quantum spatial correlations. On the other hand, it has already been observed that observables of a single system at different instants have correlations. However, compared with spatial correlations, temporal correlations did not draw attention until the Leggett-Garg inequality [3], almost 20 years after J. S. Bell's work [2]. In this inequality, four different instants, not two, have to be considered. Furthermore, due to the strong action of measurements, the Leggett-Garg inequality is unable to reflect well the temporal correlation of the original states. To tackle this problem, approaches based on the two-vector formalism [4–7] and the entangled-history theory [8–11] have been introduced. In the framework of entangled-history theory, a kind of temporal CHSH inequality [9] has been obtained to avoid the strong action of measurements, as well as the monogamy relation discovered in [7]. Besides, works like [12] also discussed the temporal correlations. In [12], although relationship between spatial correlations and temporal correlations is studied through transformations between them, however, whatever temporal correlations are is still not clear. In this letter, we prove that temporal correlations are just quantum channels. Thus, the physical entity corresponding to temporal correlations is found. We completely characterize temporal correlations.

Quantum channels as temporal correlations: In quantum teleportation [13–15], spatial correlations can be used to send states in one place to another place. And what is more, in teleportation, similarity between initial states and outcome states is directly connected with the strength of spatial correlations of bipartite states [14]. By comparison, the temporal correlations between two instants t_0 and t_1 can be regarded as a transformation Φ. This transformation maps states at the instant t_0 to states at the instant t_1. And quantum mechanics principle tell us that the transformation should be linear. Now, by the entangled-history theory [11], firstly, we show that the transformation Φ is just a quantum channel.

Now, following the entangled-history theory [11], we show how to use a bipartite state to correlate two instants and related protocols to achieve this bipartite state.

Suppose a system is initialized in a state ρ at t_0. Using the Hilbert space H_0 to represent the system at t_0, $\{|\alpha_i\rangle\}_i$ is chosen as the orthonormal basis for H_0. And let $E_{ij} = |\alpha_i\rangle\langle\alpha_j|, \forall i, j$, then the initial state ρ can be expressed as $\rho = \sum_{i,j}\rho_{ij}E_{ij}$. Similarly, use the Hilbert space H_1 to represent the system at t_1 and $\{|\beta_j\rangle\}_j$ are chosen as the orthonormal basis for H_1. And let $F_{kl} = |\beta_k\rangle\langle\beta_l|, \forall k, l$. Furthermore, by the stinespring representation theorem [16], $dim(H_0) = dim(H_1) = d$ can be assumed.

Firstly, add two auxiliary systems A, B to the original system H_0, each system is initialized in a pure state, denoted as $|0_p\rangle$, $p = A, B$. For them, $dim(H_A) = dim(H_0) = d = dim(H_1) = dim(H_B)$. Before the experiment begins, the whole system is initialized in the state

$$\sum_{i,j}\rho_{ij}E_{ij}\otimes|0_B\rangle\langle 0_B|\otimes|0_A\rangle\langle 0_A|.$$

At t_0, firstly a unitary gate U_0 is practiced between the system H_0 and the auxiliary system A. The unitary gate acts in the following way $U_0|\alpha_i\rangle|0_A\rangle = |\alpha_i\rangle|\alpha_i\rangle, \forall i$. After this, the whole system is transformed into

$$\sum_{i,j}\rho_{ij}E_{ij}\otimes|0_B\rangle\langle 0_B|\otimes E_{ij}.$$

Keeping the auxiliary systems unchanged between t_0 and t_1. Then at t_1, because of the temporal correlation Φ, the whole system is transformed into

$$\sum_{i,j} \rho_{ij} \Phi(E_{ij}) \otimes |0_B\rangle \langle 0_B| \otimes E_{ij}.$$

Now use another unitary gate U_1 between the system H_1 and the auxiliary system B. The unitary gate U_1 acts as $U_1 |\beta_k\rangle |0_B\rangle = |\beta_k\rangle |\beta_k\rangle, \forall k$. Then after this, the state of the whole system becomes

$$\sum_{i,j,k,l} \rho_{ij} \Phi_{kl,ij} F_{kl} \otimes F_{kl} \otimes E_{ij},$$

where $\Phi_{kl,ij} = tr(F_{kl}^\dagger \Phi(E_{ij}))$. Now project the system H_1 onto the state $\frac{1}{\sqrt{d}} \sum_k |\beta_k\rangle$, and then remove the first system H_1, the state on auxiliary systems B and A will be

$$\widetilde{\rho} = \sum_{i,j,k,l} \rho_{ij} \Phi_{kl,ij} F_{kl} \odot E_{ij}. \quad (1)$$

Here, following conventions in the entangled-history theory [11], to differentiate the temporal structure of the system to the spatial structure of the system, the signature $\odot$ is used to represent the tensor product structure, instead of $\otimes$. By $\widetilde{\rho}$ in Eq. (1), the system H_0 at t_0 and the system H_1 at t_1 are correlated.

Note that the transformation Φ is linear and maps states to states, and $\Phi(E_{ij}) = \sum_{k,l} \Phi_{kl,ij} F_{kl}, \forall i,j$. Therefore, $\Phi(\rho) = \sum_{i,j} \rho_{ij} \Phi(E_{ij}) = \sum_{i,j,k,l} \rho_{ij} \Phi_{kl,ij} F_{kl}$ is a quantum state for arbitrary states $\rho = \sum_{i,j} \rho_{ij} E_{ij}$ on H_0. Moreover, if ρ is $\frac{1}{\sqrt{d}} \sum_i |\alpha_i\rangle$, then its corresponding $\widetilde{\rho}$ in Eq. (1) is just the Choi matrix [17] of the transformation Φ. And because operators in Eq. (1) are just states of auxiliary systems, this verifies the complete positivity of Φ. Thus, Φ is just a quantum channel.

Moveover, the above also shows how to prepare the state $\widetilde{\rho}$ experimentally for each input state ρ at instance t_0.

And we have seen that if the initial state ρ is the state $|\mu\rangle = \frac{1}{\sqrt{d}} \sum_i |\alpha_i\rangle$, then the final state on auxiliary systems will be

$$\rho_{\mu,\Phi} = \frac{1}{d} \sum_{i,j,k,l} \Phi_{kl,ij} F_{kl} \odot E_{ij}. \quad (2)$$

Note that $\{F_{kl} \odot E_{ij}\}_{i,j,k,l}$ is a basis for operators on auxiliary systems B and A. Therefore, once the state $\rho_{\mu,\Phi}$ is decided by the state tomography technical [18], then all values of $\Phi_{kl,ij}$ will be known. Thus, the quantum channel Φ can be uniquely determined by tomography state $\rho_{\mu,\Phi}$.

Thus, for an initial state ρ on H_0, a bipartite state $\widetilde{\rho}$ on the big system $H_1 \odot H_0$ can be made to correlate two instants. In particular, for the special initial state $|\mu\rangle = \frac{1}{\sqrt{d}} \sum_i |\alpha_i\rangle$, their $\rho_{\mu,\Phi}$ can uniquely determine the temporal correlation between t_0 and t_1.

Quantify temporal correlations: Now, we use the entanglement ability of $\rho_{\mu,\Phi}$ to reflect the strength of the temporal correlation, that is, we quantify the temporal correlation.

Let $E(\cdot)$ be the entanglement of formation [19–23]. Note that $|\mu\rangle = \frac{1}{\sqrt{d}} \sum_i |\alpha_i\rangle$ is a maximally coherent state in the l_1 norm measure [24] of coherence of states for the basis $\{|\alpha_i\rangle\}_i$. In fact, for all maximally coherent states in the l_1 norm measure of coherent of state on the basis $\{|\alpha_i\rangle\}_i$, the entanglement of formation of their $\widetilde{\rho}$ is the same. Suppose we have two initial states $|\nu_k\rangle = \frac{1}{\sqrt{d}} \sum_j e^{i\theta_j^k} |\alpha_j\rangle, k \in \{0,1\}$, then their $\widetilde{\rho}$ can be expressed as $\rho_{\nu_k,\Phi} = \frac{1}{d} \sum_{m,n,k,l} e^{i(\theta_m^k - \theta_n^k)} \Phi_{kl,mn} F_{kl} \odot E_{mn}$, then

$$\rho_{\nu_1,\Phi} = I \odot U(\rho_{\nu_0,\Phi}) I \odot U^\dagger, \quad (3)$$

where U is a unitary operator with the following action $\langle \alpha_n | U | \alpha_m \rangle = e^{i(\theta_m^1 - \theta_m^0)} \delta_{mn}$. So from the property of $E(\cdot)$, we get the conclusion.

Thus, once the bases $\{|\alpha_i\rangle\}_i$ for H_0 and $\{|\beta_i\rangle\}_i$ for H_1 are chosen, for Φ, we define

$$Q_{\{|\alpha_i\rangle\}_i, \{|\beta_j\rangle\}_j}(\Phi) = E(\rho_{\mu,\Phi}).$$

$Q_{\{|\alpha_i\rangle\}_i, \{|\beta_j\rangle\}_j}(\Phi)$ describes the strength of the temporal correlation Φ between instants t_0 and t_1 with respect to the bases $\{|\alpha_i\rangle\}_i$ and $\{|\beta_j\rangle\}_j$.

To get rid of the dependence on choices of orthonormal bases, the quantity $Q(\Phi)$ is defined as

$$Q(\Phi) = \inf_{\{|\alpha_i\rangle\}_i, \{|\beta_j\rangle\}_j} Q_{\{|\alpha_i\rangle\}_i, \{|\beta_j\rangle\}_j}(\Phi)$$

, where the inf runs over all possible orthonormal bases $\{|\alpha_i\rangle\}_i$ and $\{|\beta_j\rangle\}_j$ for Hilbert spaces H_0 and H_1, respectively.

It follows from the definition of $Q(\cdot)$ that its value range is $[0, logd]$. Moreover, from the invariance of the entanglement of formation under the local unitary transformation, we have

$$Q(\Phi) = Q(\mathcal{V} \circ \Phi), \quad (4)$$

where $\mathcal{V}$ is an arbitrary unitary channel on H_1.

Now, we study when the temporal correlation is the strongest and the weakest, respectively.

Theorem 1 *If ρ is a state on H_0, then $\widetilde{\rho}$ is a maximally entangled state if and only if Φ is a unitary channel and ρ can be expressed as the state $|\mu\rangle = \frac{1}{\sqrt{d}} \sum_i |\alpha_i\rangle$. In particular, $Q(\Phi) = logd$ if and only if Φ is a unitary channel.*

Proof. For fixed orthonormal bases $\{|\alpha_i\rangle\}_i$ on H_0 and $\{|\beta_j\rangle\}_j$ on H_1, if ρ is a state on H_0, then the fact that $\widetilde{\rho}$ is a maximally entangled state means that $\widetilde{\rho}$ is pure and $\mathrm{tr}_{H_0}(\widetilde{\rho}) = \mathrm{tr}_{H_1}(\widetilde{\rho}) = \frac{1}{d} I$. To be pure, it is equivalent to $tr(\widetilde{\rho}^2) = \sum_{i,j} |\rho_{ij}|^2 (\sum_{k,l} |\Phi_{kl,ij}|^2) = 1$, where $\Phi_{lk,ji} = tr(F_{lk}^\dagger \Phi(E_{ji})) = \overline{tr(F_{kl}^\dagger \Phi(E_{ij}))}$. On the other hand, because Φ is a channel, it can have the Kraus operator representation:

$$\Phi(M) = \sum_p A_p M A_p^\dagger, \forall M \in L(H_0), \quad (5)$$

and $\Phi(E_{ii})$ is a quantum state for every i. Thus, we have $tr(\Phi(E_{ii})^\dagger\Phi(E_{ii})) \leq 1, \forall i$. Now, let us define $|a_{i,p}\rangle = A_p|\alpha_i\rangle$ and the vector $|\overline{a_{i,p}}\rangle$ whose coordinates are conjugate of coordinates of the vector $|a_{i,p}\rangle$.

Then for every i, j, we have

$$\sum_{k,l}|\Phi_{kl,ij}|^2 = \sum_{p,q}\langle a_{i,p}|a_{i,q}\rangle\overline{\langle a_{j,p}|a_{j,q}\rangle}. \quad (6)$$

On the other hand, for every i, we have

$$\sum_{p,q}|\langle a_{i,p}|a_{i,q}\rangle|^2 = \mathrm{tr}(\Phi(E_{ii})^\dagger\Phi(E_{ii})) \leq 1. \quad (7)$$

So, combined this fact with the Cauchy-Schwarz inequality, we have $\sum_{k,l}|\Phi_{kl,ij}|^2 \leq 1, \forall i, j$. Therefore, $\mathrm{tr}(\widetilde{\rho}^2) = \sum_{i,j}|\rho_{ij}|^2(\sum_{k,l}|\Phi_{kl,ij}|^2) \leq \sum_{i,j}|\rho_{ij}|^2$. Since $\widetilde{\rho}$ is a pure state, so ρ has to be pure state and Φ need to satisfy $\sum_{k,l}|\Phi_{kl,ij}|^2 = 1, \forall i, j$. In particular, $\Phi(E_{ii})$ is a pure state, $\forall i$.

Next, let us consider the second requirement for $\widetilde{\rho}$ being a maximally entangled state,

$$\mathrm{tr}_{H_1}(\widetilde{\rho}) = \mathrm{tr}_{H_0}(\widetilde{\rho}) = \frac{1}{d}I. \quad (8)$$

Note that for every i, $\sum_k \Phi_{kk,ii} = \mathrm{tr}(\Phi(E_{ii})) = 1$. Combined this with Eq. (8), by comparing coefficients, we have $\rho_{ii} = \frac{1}{d}, \forall i$. Nevertheless, as we have verified that ρ has to be pure, so $\rho = |\alpha\rangle\langle\alpha|$. Thus, we have $|\alpha\rangle = \frac{1}{\sqrt{d}}\sum_j e^{i\theta_j}|\alpha_j\rangle$.

Now, we have

$$\widetilde{\rho} = \Phi \odot I(\frac{1}{d}\sum_{m,m} e^{i(\theta_m-\theta_n)}|\alpha_m\alpha_m\rangle\langle\alpha_n\alpha_n|). \quad (9)$$

As $\frac{1}{\sqrt{d}}\sum_m e^{i\theta_m}|\alpha_m\alpha_m\rangle$ is a maximally entangled state on $H_1 \odot H_0$, so in this case, $\widetilde{\rho}$ is proportional to the Choi matrix of Φ. From the isomorphism between channels and their Choi matrices, we know that for channel Φ, $\Phi \odot I$ maps some maximally entangled state into another maximally entangled state if and only if Φ is a unitary channel.

Thus, we prove the following conclusion: for fixed orthonormal bases $\{|\alpha_i\rangle\}_i$ and $\{|\beta_j\rangle\}_j$, $\widetilde{\rho}$ is a maximally entangled state implies that Φ is a unitary channel and the initial state is maximally coherent in the l_1 norm measure of coherence.

Conversely, for every unitary channel, it can be verified easily that for arbitrary orthonormal bases $\{|\alpha_i\rangle\}_i$ on H_0 and $\{|\beta_j\rangle\}_j$ on H_1, $\rho_{\mu,\Phi}$ is always a maximally entangled state.

In particular, note that $Q(\Phi) = logd$ means that for arbitrary orthonormal bases $\{|\alpha_i\rangle\}_i$ on H_0 and $\{|\beta_j\rangle\}_j$ on H_1, the quantity $Q_{\{|\alpha_i\rangle\}_i,\{|\beta_j\rangle\}_j}(\Phi)$ is always $logd$. By [22], this implies that for arbitrary orthonormal bases $\{|\alpha_i\rangle\}_i$ on H_0 and $\{|\beta_j\rangle\}_j$ on H_1, the corresponding state $\rho_{\mu,\Phi}$ is always a maximally entangled state. The above shows that the theorem is proved. ■

Theorem 2 *If $\rho_{\mu,\Phi}$ is a classical-classical state, then Φ is the coherence destroying channel, and its temporal correlation is the weakest, that is $Q(\Phi) = 0$.*

In fact, since every classical-classical state [25, 26] is a separable state, so if $\rho_{\mu,\Phi}$ is a classical-classical state, then $Q(\Phi) = 0$. Moreover, since $\rho_{\mu,\Phi}$ is a classical-classical state, so $\rho_{\mu,\Phi} = \frac{1}{d}\sum_{i,j,k,l}\Phi_{kl,ij}F_{kl} \odot E_{ij} = \sum_{i,k}\lambda_{k,i}F_{kk} \odot E_{ii}$. Thus, we have $\Phi(E_{ij}) = 0$ for every $i \neq j$ and $\Phi(E_{ii})$ is a diagonal matrix for every i. This shows that the quantum channel Φ is a coherence destroying channel [27].

Remarks: Although the equivalence of temporal correlations and quantum channels is based on states on the big Hilbert space $H_1 \odot H_0$, which is similar to spatial correlations. However, there is a notable difference between temporal correlations and spatial correlations. Temporal correlations is a relative concept, it is dependent on choices of bases.

Firstly, seeing orthonormal bases $\{|\alpha_i\rangle\}_i$ and $\{|\beta_j\rangle\}_j$ as eigenvectors of nondegenerate observables M at t_0 and N at t_1, by using a bipartite state to correlate two different instants, the quantity $Q_{\{|\alpha_i\rangle\}_i,\{|\beta_j\rangle\}_j}(\Phi)$ measures the temporal correlation in terms of observables M at t_0 and N at t_1. In this sense, for a quantum channel Φ, $Q(\Phi)$ measures at least how strong the temporal correlation is. This is one aspect that temporal correlations are dependent on choices of bases.

Secondly, for an initial ρ and one choice of $\{|\alpha_i\rangle\}_i$ and $\{|\beta_j\rangle\}_j$, its $\widetilde{\rho}$ in Eq. (2) may be separable, and for another choice of $\{|\alpha_i\rangle\}_i$ and $\{|\beta_j\rangle\}_j$, its $\widetilde{\rho}$ may be entangled. In spatial cases, the fact that a bipartite state is entangled or not is not influenced by local unitary transformations. This is another aspect that temporal correlations are dependent on choices of bases.

What is more, there is another big difference between temporal correlations and spatial correlations. This difference is rooted in the nature of time. Time has its order, while space does not have such order. In spatial correlations, for a bipartite system, the role of two subsystems are symmetric. However, in temporal correlations, things are different. For Alice at t_0, once she knows what the channel between t_0 and t_1 is, she can always definitely know the state at t_1 from her state. However, suppose that Bob at t_1 knows the channel between t_0 and t_1 is Φ, and he measures the observable N and obtains a outcome j corresponding to the eigenvector $|\beta_j\rangle$. In this case, generally, he can not say definitely what the state at t_0 is from knowing $|\beta_j\rangle$ and Φ. In particular, if Φ is unitary, Bob can certainly know what the state at t_0 should be from his $|\beta_j\rangle$. This shows that the strongest temporal correlation has to be unitary channels in intuition. Our Thm. 1 asserted the fact strictly.

Conclusion: In this letter, by the entangled-history theory, we verify that temporal correlations are quantum channels. Thus, if we consider spatial correlations are about quantum states, then temporal correlations are just about quantum channels. Our results fit physical intuitions and pave the way to further study many instants temporal correlation problems.

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[1] E. Schrodinger, Math. Proceed. Cambridge Phil. Soc. **31** 555-563 (1935); E. Schrodinger, “Probability relations between separated systems,” Math. Proceed. Cambridge Phil. Soc. **32** 446-452 (1935); R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, “Quantum entanglement,” Rev. Mod. Phys. **81**, 865 (2009).

[2] J. S. Bell,
Phys. Physique Fizika, **1**, 195 (1964).

[3] A. J. Leggett, A. Garg, Phys. Rev. Lett. **54**, 857 (1985).

[4] Y. Aharonov, S. Popescu, and J. Tollaken, in *Quantum Theory: A Two-Time Success Story,* Ed. by D. C. Struppa and J. M. Tollaksen, pp. 21-36 (Springer, New York, 2013).

[5] Y. Aharonov, S. Popescu, J. Tollaksen, L. Vaidman, Phys. Rev. A **79**, 052110 (2009).

[6] Y. Aharonov and L. Vaidman, in *Time in Quantum Mechanics,* Ed. by J. G. MugaR. Sala MayatoI. L. Egusquiza, (Springer 2002), pp. 369-412.

[7] M. Nowakowski, E. Cohen, and P. Horodecki, Phys. Rev. A **98**, 032312 (2018).

[8] J. Cotler, F. Wilczek, Phys. Scripta, **T168**, 014004 (2016).

[9] J. Cotler, F. Wilczek, arXiv: quant-ph/1503.06458 (2015).

[10] J. Cotler, L.-M. Duan, P.-Y. Hou, F. Wilczek, D. Xu, Z.-Q. Yin, and C. Zu, Annal. Phys. **387**, 334-347 (2017).

[11] J. Cotler and F. Wilczek, arXiv: quant-ph/1702.05838 (2017).

[12] F. Costa, M. Ringbauer, M. E. Goggin, A. G. White, and A. Fedrizzi, Phys. Rev. A. **98**, 012328 (2018).

[13] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).

[14] R. Horodecki, M. Horodecki, P. Horodecki, Phys. Lett. A. **222**, 21 (1996).

[15] M. Horodecki, P. Horodecki, R. Horodecki, Phys. Rev. A. **60**, 1888 (1999).

[16] W. F. Stinespring, Proc. Amer. Math. Soc. **6**, 211 (1955).

[17] M. D. Choi, Linear Algebras and its Applications, 10, 285-290 (1975).

[18] U. Fano, Rev. Mod. Phys. **29**, 74 (1957), Sec. 6

[19] C. H. Bennett, D. P. DiVincenzo, J. Smolin and W. K. Wootters, Phys. Rev. A. **54**, 3824 (1996).

[20] S. Hill, W. K. Wootters, Phys. Rev. Lett. **78**, 5022 (1997).

[21] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).

[22] D. Spehner, J. Math. Phys. **55**, 075211 (2014).

[23] A. Winter, Commun. Math. Phys. **347**, 291 (2016).

[24] T. Baumgratz, M. Cramer, and M. B. Plenio, Phys. Rev. Lett. **113**, 140401 (2014).
J. Phys. B 43, 215508 (2010).

[25] K. Modi, A. Brodutch, H. Cable, T. Paterek, and V. Vedral, Rev. Mod. Phys **84**(4), 1655 (2012).

[26] A. Ferraro, L. Aolita, D. Cavalcanti, F. M. Cucchietti, and A. Acin, Phys. Rev. A. **81**, 052318 (2010).

[27] Z.-W. Liu, X. Hu, and S. Lloyd, Phys. Rev. Lett. **118**, 060502 (2017).