# Nonlinear Effective Masses Schrödinger Equation with Harmonic Oscillator

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With a nonlinear effective mass, we study the quantum particles described by the generalized Schrödinger equation in a harmonic oscillator. Using the Gauss-Hermite functions, an infinite dimensional discrete dynamical system and its stationary solutions are obtained. Furthermore, we also study the perturbation on any eigenmode of the corresponding potential under the influence of nonlinear effective masses.

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## I. INTRODUCTION

For particles (electrons or holes) moving inside a periodic potential, or interacting with other identical particles, their motions differ from those in a vacuum, resulting in *effective masses* denoted as $m^*$ [1]. In particular, with a nonuniform composition in potential or particle distributions, a position-dependent effective mass (PDEM) Schrödinger equation has gained much interest for its applications from semiconductors to quantum fluids [2–6]. Recently, a PDEM Schrödinger equation exhibiting a similar position dependence for both the potential and mass was exactly solved [7].

By extending this concept, in this article, we consider the nonlinear effective mass Schrödinger equation with a harmonic oscillator ($\omega > 0$)

$$i\Psi_t + \Psi_{xx} - b|\Psi|^2\Psi_{xx} - \omega x^2\Psi = 0, \tag{1}$$

and it also has the conserved density of Eq. (1)

$$\frac{1}{b}\int_{-\infty}^{\infty} \ln|-1 + b|\Psi|^2|dx. \tag{2}$$

As one can see, when the nonlinear effective mass term is zero $b = 0$, Eq. (1) is reduced to the well known scenario for a quantum particle in a parabolic potential. On the contrary, in electromagnetically induced transparency (EIT) configuration, considering atomic media trapped near a nano-waveguide, one can also arrive at a similar equation [8]:

$$i\partial_t\Psi = \gamma\Delta(\Psi)\Psi + [1 - b\Delta(\Psi)]\Psi_{xx}, \tag{3}$$

where $\Psi$ is the EIT polariton field and $\Delta(\Psi)$ accounts an effective nonlinearity induced due to the interaction. Compared to the nonlinear Schrödinger (NLS) or 1D Gross-Pitaevskii (GP) equation, the existence and linear stability of dark solitons, as well as the proof of periodic solutions, are examined [9, 10]. After introducing a potential, Eq. (1) has a similar structure as the GP equation (see below). Then, we can study the perturbation of any eigenmode of the corresponding potential under the influence of the nonlinear effective mass.

The article is organized as follows: in Section II, we introduce the quantum oscillator into this nonlinear effective mass Schrödinger equation and reduces Eq. (2) to an infinite dynamical system. Then, we investigate its stationary solutions. Section III is given for concluding remarks.

## II. NONLINEAR EFFECTIVE MASS SCHRÖDINGER EQUATION WITH HARMONIC OSCILLATOR

Suppose the stationary solution of Eq. (1) is $\Psi = P(x; c)e^{-ict}$. Then, one has

$$cP(x; c) + P''(x; c) - bP(x)^2P''(x; c) - \omega x^2P(x; c) = 0. \tag{4}$$

From Eq. (2), the conserved density becomes

$$Q(c) = \frac{1}{b}\int_{-\infty}^{\infty} \ln|-1 + b|P(x; c)^2|dx. \tag{5}$$

If $b = 0$ and $\omega = 1$, then Eq. (1) becomes the well-known equation for the quantum harmonic oscillator. And its stationary localized solutions of Eq. (4) are the Gauss-Hermite functions [11]

$$\phi_n(x) = \mu_n e^{-x^2/2}H_n(x), \tag{6}$$

where $\mu_n = (2^n n!\sqrt{\pi})^{-1/2}$ and $H_n(x)$ is the Hermite polynomials defined by the Rodrigues' formula

$$H_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}e^{-x^2/2}.$$

For example, $H_0(x) = 1, H_1(x) = 2x, H_2(x) = 4x^2 - 2, H_3(x) = 8x^3 - 12$. The corresponding eigenvalue of $\phi_n(x)$ is $c_n = 2n + 1, n = 0, 1, 2, 3\cdots$, that is,

$$-\frac{d^2\phi_n(x)}{dx^2} + x^2\phi_n(x) = (2n+1)\phi_n(x). \tag{7}$$

To investigate Eq. (1), we use the perturbation theory based on the expansion of the solution on the eigenfunction $\phi_n(x)$, that is,

$$\Psi = \sum_{n=0}^{\infty} B_n(t)\phi_n(x) \tag{8}$$

Plugging this expansion into Eq. (1), one has

$$\begin{aligned}
& i\sum_{n=0}^{\infty}\frac{dB_n(t)}{dt}\phi_n(x) + \sum_{n=0}^{\infty}B_n(t)\frac{d^2\phi_n(x)}{dx^2} \\
- & \sum_{k=0,l=0}^{\infty}B_k(t)\bar{B}_l(t)\phi_n(x)\phi_l(x)(b\sum_{n=0}^{\infty}B_n(t)\frac{d^2\phi_n(x)}{dx^2}) - \omega x^2\sum_{n=0}^{\infty}B_n(t)\phi_n(x) \\
= & i\sum_{n=0}^{\infty}\frac{dB_n(t)}{dt}\phi_n(x) + \sum_{n=0}^{\infty}B_n(t)[x^2\phi_n(x) - c_n\phi_n(x)] \\
- & \sum_{k=0,l=0}^{\infty}B_k(t)\bar{B}_l(t)\phi_n(x)\phi_l(x)[b\sum_{n=0}^{\infty}B_n(t)(x^2\phi_n(x) - c_n\phi_n(x))] - \omega x^2\sum_{n=0}^{\infty}B_n(t)\phi_n(x) \\
= & i\sum_{n=0}^{\infty}\frac{dB_n(t)}{dt}\phi_n(x) + (1-\omega)\sum_{n=0}^{\infty}x^2B_n(t)\phi_n(x) - \sum_{n=0}^{\infty}c_nB_n(t)\phi_n(x) \\
+ & b\sum_{n=0,k=0,l=0}^{\infty}c_nB_n(t)B_k(t)\bar{B}_l(t)\phi_n(x)\phi_k(x)\phi_l(x) \\
- & bx^2\sum_{n=0,k=0,l=0}^{\infty}B_n(t)B_k(t)\bar{B}_l(t)\phi_n(x)\phi_k(x)\phi_l(x) = 0.
\end{aligned} \tag{9}$$

Multiplying Eq. (9) by $\phi_m(x)$, averaging and using the orthonormal property of $\phi_m(x)$, one obtains

$$\begin{aligned}
& i\frac{dB_m(t)}{dt} - c_mB_m(t) + (1-\omega)\sum_{n=0}^{\infty}\Gamma_{m,n}B_n(t) \\
+ & b\sum_{n=0,k=0,l=0}^{\infty}(c_nV_{m,n,k,l} - W_{n,m,k,l})B_n(t)B_k(t)\bar{B}_l(t) = 0,
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
\Gamma_{m,n} &= \int_{-\infty}^{\infty}\phi_m(x)\phi_n(x)dx, \\
V_{m,n,k,l} &= \int_{-\infty}^{\infty}\phi_m(x)\phi_n(x)\phi_k(x)\phi_l(x)dx, \\
W_{m,n,k,l} &= \int_{-\infty}^{\infty}x^2\phi_m(x)\phi_n(x)\phi_k(x)\phi_l(x)dx.
\end{aligned}$$

From Eq. (8) and plugging it into Eq. (1), one reduces the PDE to the infinite discrete dynamical system given in Eq. (10). Letting $\omega = 1$ and when comparing Eq. (10) with the GP equation in [10] by the expansion given in Eq.

(8), we see that there is no term $W_{n,m,k,l}$ which comes from the second order derivative in Eq. (1) via the relation shown in Eq. (7).

One notices that from the recursive relation of Hermite polynomial $H_m(x)$ (Chapter 13 in [11])

$$x^2 H_m(x) = m(m-1)H_{m-2}(x) + (m+1/2)H_m(x) + 1/4H_{m+2}(x), \tag{11}$$

it's not difficult to obtain

$$W_{m,n,k,l} = \frac{\sqrt{m(m-1)}}{2} V_{m-2,n,k,l} + (m+1/2)V_{m,n,k,l} + \frac{\sqrt{(m+1)(m+2)}}{2} V_{m+2,n,k,l}. \tag{12}$$

Also, it is known that

$$\Gamma_{n-2,n} = \frac{\sqrt{n(n-1)}}{2}, \Gamma_{n,n} = \frac{2n+1}{2}, \Gamma_{n+2,n} = \frac{\sqrt{(n+1)(n+2)}}{2}, \quad otherwise = 0. \tag{13}$$

To obtain the stationary solution of Eq. (4), we assume

$$P(x;c) = e^{-ict} \sum_{n=0}^{\infty} B_n \phi_n(x), \quad B_n \in R, \tag{14}$$

Then from Eq. (10), given a value $c$, one yields

$$(c - c_m)B_m + (1-\omega) \sum_{n=0}^{\infty} \Gamma_{m,n} B_n + b \sum_{n=0,k=0,l=0}^{\infty} (c_n V_{m,n,k,l} - W_{n,m,k,l}) B_n B_k B_l = 0. \tag{15}$$

For simplicity, we assume $P(x)$ is even function, i.e., $P(x) = P(-x)$, and then $B_{2n+1} = 0, n = 0, 1, 2, 3, \cdots$.

Next, we consider perturbation for any Gauss-Hermite eigenmode $\phi_{2n}(x)$. As in the case of GP equation [12, 13], by substituting $P(x;c) = \sqrt{Q(c)}\phi(x)$ with $\|\phi(x)\| = 1$ into Eq. (4), one obtains

$$c\phi(x) + (1 - bQ(c)\phi^2(x))\phi^{''}(x) - wx^2\phi = 0. \tag{16}$$

From Eq. (16), we see that if $Q(c) \to 0$ as $c \to c_{2n} = 4n+1$, then the non-linear term in Eq. (16) can be neglected and $\phi(x)$ approximated by the Gauss-Hermite function $\phi_{2n}(x)$. By substituting $\phi(x)$ with $\phi_{2n}(x)$ defined in Eq. (6) into Eq. (16), we obtain the following relation between $c$ and $Q(c)$ near $c = c_{2n} = 4n+1$

$$\begin{aligned} c &\approx c_{2n} + bQ(c)[wW_{2n,2n,2n,2n} - (4n+1)V_{2n,2n,2n,2n}] \\ &= c_{2n} + bQ(c)\mu_{2n}^4[w\int_{-\infty}^{\infty} x^2 e^{-x^2} H_{2n}(x)^4 dx - (4n+1)\int_{-\infty}^{\infty} e^{-x^2} H_{2n}(x)^4 dx]. \end{aligned} \tag{17}$$

To compute the integrals in Eq. (17), noticing that Eq. (11), one can utilize the Feldheim Identity ( Chapter 13 in [11] ) for the Hermite polynomials

$$H_m(x)H_n(x) = \sum_{\nu=0}^{min(m,n)} H_{m+n-2\nu}(x) 2^\nu \nu! \begin{pmatrix} m \\ \nu \end{pmatrix} \begin{pmatrix} n \\ \nu \end{pmatrix}, \tag{18}$$

and the Titchmarsh's integral formula (p. 804 in [14] )

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{-2x^2} H_{2m}(x)H_{2n}(x)H_{2p}(x)dx \\ &= \pi^{-1} 2^{m+n+p-1/2} \Gamma(n+p+\frac{1}{2}-m)\Gamma(m+p+\frac{1}{2}-n)\Gamma(m+n+\frac{1}{2}-p), \end{aligned} \tag{19}$$

where $n+p \geq m, m+p \geq n, m+n \geq p$; otherwise, the integral is zero. From Eqs. (18) and (19), a direct calculation can yield

$$\int_{-\infty}^{\infty} e^{-2x^2} H_{2m}(x) H_{2n}(x)^3 dx = \frac{1}{\pi} 2^{m+3n-1/2} \sum_{\nu=0}^{min(2m,2n)} \nu! \begin{pmatrix} 2m \\ \nu \end{pmatrix} \begin{pmatrix} 2n \\ \nu \end{pmatrix} \Gamma(n+\nu+1/2-m)\Gamma(m+n+1/2-\nu)^2. \tag{20}$$

Let $w=1$ in Eq. (17). We can verify by the Maple software that

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} H_{2n}(x)^4 dx - (4n+1) \int_{-\infty}^{\infty} e^{-x^2} H_{2n}(x)^4 dx < 0 \tag{21}$$

or

$$W_{2n,2n,2n,2n} - (4n+1) V_{2n,2n,2n,2n} < 0 \tag{22}$$

for several values of $n \geq 0$. The prove using Eqs. (11) and (20) for any positive integer $n$ is given in the appendix.

Now, we can formulate the

**Conjecture**: Let $P(x;c) = P(-x;c), w=1$, and $b>0$ in Eq. (4). Then under the condition given in Eq. (21) solutions with linear limit $P(x;c) \approx \sqrt{Q(c)}\phi_{2n}(x)$ exist only if $c < 4n+1$ and $Q(c) \to 0$ as $c \to 4n+1$.
We give some examples for this conjecture.

- **n=0: (one-soliton case)** We start with the ground state $\phi_0(x)$. Assuming $B_0 >> B_{2n}, n = 1,2,3,\cdots$, from Eq. (15), one has

$$B_0^2 \approx \frac{c - c_0 + \Gamma_{0,0}(1-w)}{b(W_{0,0,0,0} - V_{0,0,0,0})} \tag{23}$$

and

$$\begin{aligned} B_{2n} &\approx \frac{bB_0^3 (W_{2n,0,0,0} - c_{2n} V_{2n,0,0,0})}{[c - c_{2n} + \Gamma_{2n,2n}(1-w)]}. \\ &= B_0 \frac{[c - c_0 + \Gamma_{0,0}(1-w)](W_{2n,0,0,0} - c_{2n} V_{2n,0,0,0})}{[c - c_{2n} + \Gamma_{2n,2n}(1-w)](W_{0,0,0,0} - V_{0,0,0,0})}. \end{aligned} \tag{24}$$

By Eqs. (12) and (13), a simple calculation yields

$$\begin{aligned} &\Gamma_{00} = 1/2, W_{0,0,0,0} = \frac{1}{4\sqrt{2\pi}}, V_{0,0,0,0} = \frac{1}{\sqrt{2\pi}} \\ &\Gamma_{2n,2n} = 2n + \frac{1}{2}, V_{2n,0,0,0} = \frac{(-1)^n}{\sqrt{\pi 2^{2n+1}(2n)!}}(2n-1)!! \\ &W_{2n,0,0,0} = \frac{(-1)^{n+1}}{\sqrt{\pi 2^{2n+5}(2n)!}}(2n-1)!!(2n-1), n \geq 1 \end{aligned}$$

Therefore, from Eqs. (23) and (24), we obtain, noting that $c_0 = 1, c_{2n} = 4n+1$,

$$\begin{aligned} B_0^2 &\approx \frac{-4\sqrt{2\pi}}{3b}[c - \frac{1}{2}(1+\omega)], \\ B_{2n} &\approx bB_0^3 \frac{(-1)^{n+1}(\frac{9}{2}n + \frac{3}{4})(2n-1)!!}{\sqrt{\pi 2^{2n+1}(2n)!}[c - 4n - 1 + (2n+\frac{1}{2})(1-\omega)]}, n \geq 1 \\ &\approx bB_0^3 \frac{(-1)^{n+1}(\frac{9}{2}n + \frac{3}{4})(2n-1)!!}{\sqrt{\pi 2^{2n+1}(2n)!}[(2n+\frac{1}{2})(1-\omega) - 4n - 1]}, \quad as \quad n \to \infty. \end{aligned} \tag{25}$$

Using Eq. (25), we could obtain the perturbation of $\phi_0(x)$ as

$$\psi_0(x) \approx B_0\phi_0(x) + B_2\phi_2(x) + B_4\phi_4(x) + B_6\phi_6(x) + \cdots \approx B_0\phi_0(x). \tag{26}$$

By the Ratio Test for $B_{2n}$ and boundedness of $\phi_{2n}(x)$, it is not difficult to see that Eq. (26) is convergent. Let $w=1$ and we can obtain the convergence of stationary solution given in Eq. (26) near $c=1$ for the conserved density given in Eq. (2) assuming $b>0, c<0$. It can be seen as follows. We notice that $\psi_0(x)\to 0$ as $c\to 1$ from Eqs. (25) and (26). Using the orthonormality of $\phi_{2n}(x)$ and a simple calculation, one has, noticing that $(2n-1)!!=\frac{(2n)!}{2^n n!}$,

$$\begin{aligned} Q(c) &= \frac{1}{b}\int_{-\infty}^{\infty} \ln|-1+b\psi_0^2(x)|dx \approx \int_{-\infty}^{\infty}\psi_0^2(x)dx = B_0^2+B_2^2+B_4^2+B_6^2+\cdots. \\ &= \frac{-4\sqrt{2\pi}}{3b}(c-1)[1+\frac{9}{16}(c-1)^2\sum_{n=1}^{\infty}\frac{(2n)!}{(n!2^{2n})^2}]. \end{aligned} \tag{27}$$

Therefore, we see that $Q(c)\to 0$ as $c\to c_0=1$

- **n=2:** Assuming $B_2 >> B_{2n}, n=0,2,3,\cdots$, also from Eq. (15), one has after a direct calculation

$$B_2^2 \approx \frac{c-c_2+\Gamma_{2,2}(1-w)}{b[\frac{\sqrt{2}}{2}V_{0,2,2,2}+(\frac{5}{2}-c_2)V_{2,2,2,2}+\sqrt{3}V_{4,2,2,2}]} \tag{28}$$

and

$$B_{2n}\approx \frac{bB_2^3(W_{2n,2,2,2}-c_{2n}V_{2n,2,2,2})}{[c-c_{2n}+\Gamma_{2n,2n}(1-w)]}. \tag{29}$$

By the product formula given in Eq. (20), a direct calculation yields

$$V_{2n,2,2,2}=\frac{(-1)^{n-3}(2n-1)!!(8n^3-60n^2+94n-1)}{\sqrt{\pi 2^{2n+10}(2n)!}}. \tag{30}$$

Consequently, we have

$$B_2^2\approx \frac{-256\sqrt{2\pi}}{435b}[c-c_2+(1-w)\Gamma_{2,2}], \tag{31}$$

and, using Eqs. (29) and (12), one also has

$$\begin{aligned} B_{2n}^2 &\approx \frac{b^2B_2^6(2n)!(4n^4-28n^3+8n^2+119n+\frac{23}{4})^2}{2^{4n+10}\pi(n!)^2[c-c_{2n}+\Gamma_{2n,2n}(1-w)]^2} \\ &\approx b^2B_2^6\frac{(2n)!(4n^4-28n^3+8n^2+119n+\frac{23}{4})^2}{2^{4n+10}\pi(n!)^2[(2n+\frac{1}{2})(1-\omega)-4n-1]^2}. \end{aligned} \tag{32}$$

Then the perturbation of $\phi_2(x)$ is

$$\psi_2(x)\approx B_0\phi_0(x)+B_2\phi_2(x)+B_4\phi_4(x)+B_6\phi_6(x)+\cdots\approx B_2\phi_2(x). \tag{33}$$

Also, the Ratio Test for $B_{2n}$ and boundedness of $\phi_{2n}(x)$ show that Eq. (33) is convergent. Letting $w=1$, from Eqs. (31) and (31), we also obtain $\psi_2(x)\to 0$ as $c\to c_2=5$. As in the case $n=0$ for $b>0, c<0$, we see that

$$\begin{aligned} Q(c) &= \frac{1}{b}\int_{-\infty}^{\infty} \ln|-1+b\psi_2^2(x)|dx \approx \int_{-\infty}^{\infty}\psi_2^2(x)dx = B_0^2+B_2^2+B_4^2+B_6^2+\cdots. \\ &\approx \frac{-256\sqrt{2\pi}}{435b}(c-5)[1+\frac{2\times 256^2}{435^2}(c-5)^2\sum_{n=1}^{\infty}\frac{(2n)!n^6}{(n!2^{2n+5})^2}]. \end{aligned} \tag{34}$$

Therefore, we see that $Q(c)\to 0$ as $c\to c_2=5$.

For general case $n$, the formula shown in Eq. (34) is similar but it is more involved.

## III. CONCLUDING REMARKS

By using the orthonormal property of Gauss-Hermite functions, we reduce the PDE given in Eq. (1) to the infinite discrete dynamical system given in Eq. (10). Then, the corresponding stationary solution are studied by using the expansion (14), resulting in obtaining the infinite dimensional equation given in Eq. (15) for $(B_0, B_1, B_2, \cdots, B_n, \cdots)$. Inspired by the results from nonlinear Schrödinger (NLS) or 1D Gross-Pitaevskii (GP) equation [9, 12], we also investigate the perturbation of any eigenmode $\phi_{2n}(x)$ using Eq. (15) and propose the conjecture of critical value for the perturbation. On the other hand, compared to NLS equation [10], one sees that the dynamical system revealed in Eqs. (10) and (15) needs further investigation. Also, the equation (1) with $\omega = 0$ is interesting [15].

## IV. APPENDIX:

We prove the inequality given in Eq. (22).
From Eq. (12), we see that

$$\begin{aligned} & W_{2n,2n,2n,2n} - (4n+1)V_{2n,2n,2n,2n} \qquad (35)\\ = & \frac{\sqrt{2n(2n-1)}}{2}V_{2n-2,2n,2n,2n} - (2n+1/2)V_{2n,2n,2n,2n} \\ + & \frac{\sqrt{(2n+1)(2n+2)}}{2}V_{2n+2,2n,2n,2n} \\ < & nV_{2n-2,2n,2n,2n} - (2n+1/2)V_{2n,2n,2n,2n} + (n+1)V_{2n+2,2n,2n,2n} \\ = & n(V_{2n-2,2n,2n,2n} - V_{2n,2n,2n,2n}) + n(V_{2n+2,2n,2n,2n} - V_{2n,2n,2n,2n}) \\ + & (V_{2n+2,2n,2n,2n} - \frac{1}{2}V_{2n,2n,2n,2n}). \end{aligned} \qquad (36)$$

Using Eq., (20) and the formula

$$\Gamma(h+\frac{1}{2}) = \begin{pmatrix} h+\frac{1}{2} \\ h \end{pmatrix} h!\sqrt{\pi} = \frac{(2h-1)!!}{2^h h!}h!\sqrt{\pi}, \qquad (37)$$

one has

$$V_{2n,2n,2n,2n} = \frac{1}{\sqrt{2\pi}}\sum_{\nu=0}^{2n}\begin{pmatrix} \nu-\frac{1}{2} \\ \nu \end{pmatrix}\begin{pmatrix} 2n-\nu-\frac{1}{2} \\ 2n-\nu \end{pmatrix}^2. \qquad (38)$$

Similarly, by Eqs. (20) and (37), a simple calculation yields

$$V_{2(n-1),2n,2n,2n} = \frac{1}{\sqrt{2\pi}}\sum_{\nu=0}^{2(n-1)}\begin{pmatrix} \nu-\frac{1}{2} \\ \nu \end{pmatrix}\begin{pmatrix} 2n-\nu-\frac{1}{2} \\ 2n-\nu \end{pmatrix}^2 \frac{(2n-\nu)(2n-\nu-1)}{(2n-\nu-\frac{1}{2})^2}.$$

Since $\frac{(2n-\nu)(2n-\nu-1)}{(2n-\nu-\frac{1}{2})^2} < 1$, we know that

$$V_{2(n-1),2n,2n,2n} < V_{2n,2n,2n,2n} - \frac{1}{\sqrt{2\pi}}\sum_{\nu=2n-1}^{2n}\begin{pmatrix} \nu-\frac{1}{2} \\ \nu \end{pmatrix}\begin{pmatrix} 2n-\nu-\frac{1}{2} \\ 2n-\nu \end{pmatrix}^2. \qquad (39)$$

Also,

$$\begin{aligned} V_{2(n+1),2n,2n,2n} &= \frac{1}{\sqrt{2\pi}}\sum_{\nu=1}^{2n}\begin{pmatrix} \nu-\frac{1}{2} \\ \nu \end{pmatrix}\begin{pmatrix} 2n-\nu-\frac{1}{2} \\ 2n-\nu \end{pmatrix}^2 \frac{2}{2\nu-1}\frac{(2n-\nu+\frac{1}{2})^2}{(2n-\nu+1)(2n-\nu+2)} \\ &< \frac{1}{\sqrt{2\pi}}[\sum_{\nu=1}^{2}\begin{pmatrix} \nu-\frac{1}{2} \\ \nu \end{pmatrix}\begin{pmatrix} 2n-\nu-\frac{1}{2} \\ 2n-\nu \end{pmatrix}^2 \frac{2}{2\nu-1}\frac{(2n-\nu+\frac{1}{2})^2}{(2n-\nu+1)(2n-\nu+2)} \\ &+\frac{1}{2}\sum_{\nu=3}^{2n}\begin{pmatrix} \nu-\frac{1}{2} \\ \nu \end{pmatrix}\begin{pmatrix} 2n-\nu-\frac{1}{2} \\ 2n-\nu \end{pmatrix}^2] \end{aligned} \qquad (40)$$

since $\frac{(2n-\nu+\frac{1}{2})^2}{(2n-\nu+1)(2n-\nu+2)} < 1$. Now, a simple calculation yields

$$\sum_{\nu=1}^{2} \begin{pmatrix} \nu-\frac{1}{2} \\ \nu \end{pmatrix} \begin{pmatrix} 2n-\nu-\frac{1}{2} \\ 2n-\nu \end{pmatrix}^2 \frac{2}{2\nu-1} \frac{(2n-\nu+\frac{1}{2})^2}{(2n-\nu+1)(2n-\nu+2)}$$
$$= \pi^{3/2}[\frac{(4n-5)!!}{2^{2n}(2n)!}]^2[\frac{2n(4n-1)^2(4n-3)^2}{2n+1} + \frac{4n(2n-1)^2(4n-3)^2}{2n-1}]$$
$$< \pi^{3/2}[\frac{(4n-5)!!}{2^{2n}(2n)!}]^2[(4n-1)^2(4n-3)^2 + 3(2n-1)^2(4n-3)^2].$$
$$= \pi^{3/2}[\frac{(4n-5)!!}{2^{2n}(2n)!}]^2(448n^4 - 992n^3 + 796n^2 - 276n + 36) \quad (41)$$

On the other hand,

$$\sum_{\nu=0}^{2} \begin{pmatrix} \nu-\frac{1}{2} \\ \nu \end{pmatrix} \begin{pmatrix} 2n-\nu-\frac{1}{2} \\ 2n-\nu \end{pmatrix}^2$$
$$= \pi^{3/2}[\frac{(4n-5)!!}{2^{2n}(2n)!}]^2[576n^4 - 896n^3 + 472n^2 - 96n + 9]. \quad (42)$$

From Eqs. (41) and (42), one obtains

$$\sum_{\nu=1}^{2} \begin{pmatrix} \nu-\frac{1}{2} \\ \nu \end{pmatrix} \begin{pmatrix} 2n-\nu-\frac{1}{2} \\ 2n-\nu \end{pmatrix}^2 \frac{2}{2\nu-1} \frac{(2n-\nu+\frac{1}{2})^2}{(2n-\nu+1)(2n-\nu+2)}$$
$$< \sum_{\nu=0}^{2} \begin{pmatrix} \nu-\frac{1}{2} \\ \nu \end{pmatrix} \begin{pmatrix} 2n-\nu-\frac{1}{2} \\ 2n-\nu \end{pmatrix}^2 \quad (43)$$

when $n \geq 1$. Consequently, using Eqs. (40) and (43), we have

$$V_{2(n+1),2n,2n,2n} < \frac{1}{2} V_{2n,2n,2n,2n} + \frac{1}{2} \sum_{\nu=0}^{2} \begin{pmatrix} \nu-\frac{1}{2} \\ \nu \end{pmatrix} \begin{pmatrix} 2n-\nu-\frac{1}{2} \\ 2n-\nu \end{pmatrix}^2. \quad (44)$$

From Eqs. (36), (39) and (44), we obtain

$$W_{2n,2n,2n,2n} - (4n+1)V_{2n,2n,2n,2n} < -n \sum_{\nu=2n-1}^{2n} \begin{pmatrix} \nu-\frac{1}{2} \\ \nu \end{pmatrix} \begin{pmatrix} 2n-\nu-\frac{1}{2} \\ 2n-\nu \end{pmatrix}^2$$
$$-\frac{n}{2} \sum_{\nu=3}^{2n} \begin{pmatrix} \nu-\frac{1}{2} \\ \nu \end{pmatrix} \begin{pmatrix} 2n-\nu-\frac{1}{2} \\ 2n-\nu \end{pmatrix}^2 + \frac{1}{2} \sum_{\nu=0}^{2} \begin{pmatrix} \nu-\frac{1}{2} \\ \nu \end{pmatrix} \begin{pmatrix} 2n-\nu-\frac{1}{2} \\ 2n-\nu \end{pmatrix}^2. \quad (45)$$

One considers the last two terms in Eq. (45). Let's take $\nu = 3, 4$. Then

$$-\frac{n}{2} \sum_{\nu=3}^{4} \begin{pmatrix} \nu-\frac{1}{2} \\ \nu \end{pmatrix} \begin{pmatrix} 2n-\nu-\frac{1}{2} \\ 2n-\nu \end{pmatrix}^2$$
$$= -\pi^{3/2}[\frac{(4n-5)!!}{2^{2n}(2n)!}]^2[10\frac{n(2n-2)^2(2n-1)^2(2n)^2}{(4n-5)^2}$$
$$+35\frac{n(2n-3)^2(2n-2)^2(2n-1)^2(2n)^2}{(4n-5)^2(4n-7)^2}]. \quad (46)$$

Comparing with Eq. (42) and using the Maple software, we obtain that the last two terms in Eq. (45) is negative when $n \geq 2$. For $n = 0, 1$ in Eq. (22), it can be verified directly.
This completes the proof.

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