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Adding or subtracting a single photon is the same for pure squeezed vacuum states

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ABSTRACT

The addition of a single photon to a light field can lead to exactly the same *outcome* as the subtraction of a single photon: starting from the same initial state, both procedures can generate the same final quantum state. We prove that this identity-of-outcome is true for pure squeezed vacuum states of light, and in some sense only for those. We show that mixed states can show this identity-of-outcome for the addition or subtraction of a photon if they are generated from the incoherent sums of pure squeezed vacuum states with the same squeezing. We point out that our results give a reinterpretation to the fact that pure squeezed vacuum states, with squeezing e^{-z}, are formally annihilated by Bogoliubov-transformed annihilation operators: $\hat{a}_z = \hat{a}\cosh(z) - \hat{a}^\dagger \sinh(z)$.

I. INTRODUCTION

Adding a single photon to a light field and subtracting a single photon are different processes—so different that it seems the question as to whether a quantum state can exist that gives the same *outcome* after a photon is subtracted from it, or a photon is added to it, has not been investigated, or at least not been answered in the affirmative.[1–3]

It might appear surprising that adding a photon or subtracting one can give the same outcome, for correctly chosen input states, seemingly violating energy conservation. Yet, here, in Sec. II, using Schrödinger wave functions, we prove that this surprising fact holds true for all *pure squeezed vacuum* states. Then, in Sec. III, we introduce Wigner's phase space formulation and apply it in Sec. IV to consider the case of mixed Gaussian states as input states and show that this identity-of-outcome is only true for pure squeezed vacuum input states. In Sec. V, we explicitly show that this identity-of-outcome also holds when using the Fock representation. We then show in Sec. VI that only incoherent sums of pure squeezed vacuum states with the same amount of squeezing, as input states, can give the identity-of-outcome and prove this result using the Bargmann representation in Sec. VII. We explain how this result connects with the behavior of Bogoliubov-transformed annihilation operators in Sec. VIII and conclude with some observations about the low efficiency for standard approaches to investigate the identity-of-outcome in Sec. IX and consider the technological relevance and meaning of our results in Sec. X.

The Conclusions specifically discuss that Sec. IX displays low probabilities in a possible experimental setup to be successful in adding or subtracting a photon. Hence, energy is always conserved[4] and for the subensemble of successful outcomes, the identity-of-outcome is guaranteed; see Appendix A.

II. USING WAVE FUNCTIONS

We consider a bosonic single mode system, specifically a harmonic oscillator whose creation operator is $\hat{a}^\dagger(\hat{x},\hat{p})$

$= \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - \frac{i\hat{p}}{m\omega})$. Setting $m = 1$ and $\omega = 1$, $\hat{a}^\dagger$ can describe the excitation of an optical mode[5] by the addition of a single photon; for brevity, we often also set $\hbar = 1$.

Since the momentum operator has the form $\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$, the corresponding annihilation operator, describing the removal of a single photon, is $\hat{a} = (\hat{x} + \frac{\partial}{\partial x})/\sqrt{2}$.

It is well known that the removal of a single photon has the optical vacuum state as its trivial eigensolution; in Fock language, $\hat{a}|0\rangle = 0$. The addition of a photon has no eigenstate (formally, it leads to a non-normalizable Gaussian state[6]).

Here, instead of finding eigenstates, we identify input states for which the addition or subtraction of a photon leads to an identity-of-outcome, we ask:

Do quantum states, ψ, exist, which, after addition, ψ_+, or subtraction, ψ_-, of a photon, give the same outcome state: $\psi_+ = \psi_-$?

Formally, for such pure input states $\psi(x)$, this means we have to equate the operations,

$$\hat{a}^\dagger \cdot \psi(x) = r\, \hat{a} \cdot \psi(x), \tag{1}$$

for the resulting *renormalized* outcome states. Namely, we have to confirm that the identity-of-outcome condition (1) is fulfilled, while

$$r = \sqrt{\|\hat{a}^\dagger \psi\|_2 / \|\hat{a}\psi\|_2}, \tag{2}$$

of Eq. (1), forms the ratio of norms (2), where $\|\psi\|_2 = \int_{-\infty}^{\infty} dx |\psi(x)|^2$ is the regular L_2-norm.

Multiplying both sides with $\sqrt{2}$ and separating terms, Eq. (1) amounts to

$$(1 - r)\; x\; \psi(x) = (1 + r)\frac{\partial}{\partial x}\psi(x). \tag{3}$$

This is obviously solved by the Gaussian wave functions of the form

$$\psi(x, \sigma_x) = \exp\left[-x^2/(2\sigma_x^2)\right] \Big/ \sqrt{\sigma_x \sqrt{\pi}}. \tag{4}$$

In phase space, state (4) is aligned with the x-axis.

We emphasize that, for $\sigma_x = 1$, $\psi(x, 1) = \psi_{|0\rangle}(x)$ is the vacuum state, since $|\psi_{|0\rangle}(x)|^2 = \exp\left[-x^2/(2\Sigma_0^2)\right]/\sqrt{2\pi\Sigma_0^2}$ correctly implies that $\Sigma_0^2 = \sigma_x^2/2 = \frac{1}{2}\frac{\hbar}{m\omega}$.

Checking whether r represents the ratio of norms (2), we find that, when using pure normalized squeezed vacuum states (4) as input states, Eq. (3) yields the conditions that $\sigma_x > 1$ and $r(\sigma_x) = \sqrt{\frac{(\sigma_x{}^2+1)^2}{(\sigma_x{}^2-1)^2}} = \frac{(\sigma_x{}^2+1)}{|\sigma_x{}^2-1|}$.

This expression for $r(\sigma_x)$ is what Eq. (2) demands, and it establishes that all pure squeezed states, anti-squeezed in position (for them $\sigma_x > 1$), are states for which the addition of a photon or the subtraction of a photon gives the identity-of-outcome.

A. Resolving the sign problem

This result for $r(\sigma_x)$, just given, appears peculiarly limited since in quantum phase space, position and momentum essentially play equivalent roles,[5] and it is therefore unclear why only states anti-squeezed in position $(\sigma_x > 1)$ and thus squeezed in momentum should show this identity-of-outcome. What about states squeezed in position $(0 < \sigma_x < 1)$ and anti-squeezed in momentum?

Some consideration reveals that, in Eq. (2), we dropped the negative branch of the square root. The substitution of r with $-r$ in Eq. (1) addresses this oversight: with this sign convention, all position-squeezed pure states $(0 < \sigma_x < 1)$ fulfill the desired identity-of-outcome as well.

We conclude that all pure squeezed vacuum input states, as long as $\sigma_x \neq 1$, yield an identity-of-outcome, regardless of whether a photon is added or subtracted.

There are two good ways to circumvent this sign problem we have just identified:

For position-squeezed input states $(0 < \sigma_x < 1)$, the outcome of the operation $\hat{a}\psi(x)$ switches the sign such that the resulting wave functions have negative values for $x < 0$. This is in contravention of the sign convention of Hermite's polynomials of order n for the harmonic oscillator, which are adorned by the coefficient $(-1)^n$, and this switching-problem can therefore be "repaired" by multiplying with the sign-function $\operatorname{sgn}(\sigma_x^2 - 1)\; \hat{a} \cdot \psi(x)$.

Instead, here we take the concise approach of taking the branches of the square root (2) such that the sign automatically switches by itself; we use

$$r(\sigma_x) = \frac{(\sigma_x{}^2 + 1)}{(\sigma_x{}^2 - 1)}. \tag{5}$$

This expression for the ratio-of-norms r (2) leads to a formal fulfillment of the identity-of-outcome condition (1) for all pure squeezed vacuum states, at the price of the outcome state $\hat{a} \cdot \psi(x)$ having the "wrong" orientation for position-squeezed input states.

B. No vacuum state, no displaced states

The case $\sigma_x = 1$ is obviously excluded since that constitutes the vacuum state in Eq. (4), and while in this case, the application of $\hat{a}^\dagger$ generates the first excited single-photon Fock state $|1\rangle$, the application of $\hat{a}$ yields zero: formally, expression (5) diverges.

We notice that the displaced states (for which $\langle \hat{x} \rangle \neq 0$), such as squeezed coherent states, cannot be the solutions of the identity-of-outcome condition (3), because its left-hand-side term, purely linear in x, is incompatible with such displacements of the input states.

III. STUDYING MIXED GAUSSIAN STATES IN THE WIGNER FORMULATION

Mixed squeezed vacuum states are conveniently studied using Wigner's phase space formulation, since in phase space, such states have a real Gaussian form.

We will now briefly introduce this formulation and its star-product.

Let us emphasize that Wigner's phase space formulation is entirely equivalent to the Schrödinger-von Neumann approach since they are connected by the unitary and invertible Fourier-transform (6), as follows:

Consider a single-mode operator, $\hat{O}$, given in the coordinate representation $\langle x - y|\hat{O}|x + y\rangle = O(x - y, x + y)$.

To map it to phase space, we employ the Wigner-transform, $\mathcal{W}[\hat{O}]$,[7–9]

$$\mathcal{W}[\hat{O}](x,p) = \int_{-\infty}^{\infty} dy\, O\left(x - \frac{y}{2}, x + \frac{y}{2}\right) e^{\frac{i}{\hbar}py}. \tag{6}$$

If $\hat{O}$ is a normalized single-mode density matrix $\hat{\rho}$, then the associated *normalized* distribution in the Wigner representation is $W(x,p) \equiv \mathcal{W}[\hat{\rho}]/(2\pi\hbar)$, which can represent a mixed state and fulfills $\int\int dx\, dp\, W = 1$.[7,9]

We also need to know how Hilbert space operator products translate into Wigner's phase space. Wigner transforming such products gives us $\mathcal{W}[\hat{A}\cdot\hat{B}](x,p) = \mathcal{W}[\hat{A}](x,p) \star \mathcal{W}[\hat{B}](x,p)$, where $\star$ stands for Groenewold's star product.[7,9,10] Its explicit form is

$$\star \equiv \exp\left[\frac{i\hbar}{2}\overleftrightarrow{\partial}\right] = \sum_{n=0}^{\infty} \frac{(i\hbar\overleftrightarrow{\partial})^n}{2^n n!}, \tag{7}$$

where the differential operator $\overleftrightarrow{\partial} = \left(\overleftarrow{\partial}_x\overrightarrow{\partial}_p - \overleftarrow{\partial}_p\overrightarrow{\partial}_x\right)$ is the Poisson bracket of classical mechanics, namely, the overhead arrows indicate the "direction" of differentiation:
$f\overrightarrow{\partial}_x g = g\overleftarrow{\partial}_x f = f(x,p)\left(\frac{\partial}{\partial x}g(x,p)\right)$.

A good mnemonic for Eq. (7) is to think of star-products as the Wigner transforms of operator compositions, specifically, of the operator multiplication signs:[7] $\mathcal{W}[\cdot] = \star$.

Groenewold's star product captures how the non-commutativity of Hilbert space operators gets mapped into the quantum phase space of Wigner's formulation; it implements the operator non-commutativity in phase space.[7,9,10]

IV. PHOTON ADDITION AND SUBTRACTION IN PHASE SPACE

According to Sec. III, the photon addition operator applied to the Wigner distribution is[11]

$$\frac{\mathcal{W}[\hat{a}^\dagger\cdot\hat{\rho}\cdot\hat{a}]}{2\pi\hbar} = a^* \star W \star a = \frac{1}{2}\left(p^2 + x^2 + 1\right)W \tag{8a}$$

$$+\left[-\frac{1}{2}\nabla\cdot\begin{pmatrix}xW\\ pW\end{pmatrix} + \frac{\Delta W}{8}\right]. \tag{8b}$$

Here, the phase space gradient $\nabla = (\partial_x, \partial_p)$ forms a scalar product with the column vector $(x,p)^\top W$ and Δ is the Laplacian $(\partial_x^2 + \partial_p^2)$ in phase space.

Similarly to Eq. (8), the phase space form for photon removal from the light field W is given by the expression[11]

$$\frac{\mathcal{W}[\hat{a}\cdot\hat{\rho}\cdot\hat{a}^\dagger]}{2\pi\hbar} = a \star W \star a^* = \frac{1}{2}\left(p^2 + x^2 - 1\right)W \tag{9a}$$

$$+\left[+\frac{1}{2}\nabla\cdot\begin{pmatrix}xW\\ pW\end{pmatrix} + \frac{\Delta W}{8}\right]; \tag{9b}$$

both Eqs. (8) and (9) are second-order partial differential expressions.

A. Impure squeezed vacuum states with photon added or subtracted

In phase space, the identity-of-outcome condition equivalent to Eq. (1), but now with the ratio of the norms of the form $R = \|\hat{a}^\dagger\psi\|_2/\|\hat{a}\psi\|_2$, reads

$$\begin{aligned} &a^* \star W(x,\ p) \star a - R\, a \star W(x,p) \star a^* \\ &= (1-R)\left(\frac{p^2+x^2+1}{2} + \frac{\Delta}{8} + \frac{p}{2}\frac{\partial}{\partial p} + \frac{x}{2}\frac{\partial}{\partial x}\right)W(x,p) \\ &\quad - \left(1 + p\frac{\partial}{\partial p} + x\frac{\partial}{\partial x}\right)W(x,p) \ = 0. \end{aligned} \tag{10}$$

Using *impure* squeezed vacuum states $(\sigma_x\sigma_p > 1)$,

$$W(x,p,\sigma_x,\sigma_p) = \exp\left[-\frac{x^2}{\sigma_x{}^2} - \frac{p^2}{\sigma_p^2}\right]/(\pi\sigma_p\sigma_x), \tag{11}$$

as inputs for Eq. (8), yields, for *normalized* photon-added outcome states, the expression

$$W_+(x,p,\sigma_x,\sigma_p) = f_+(x,p,\sigma_x,\sigma_p)W(x,p,\sigma_x,\sigma_p), \tag{12a}$$

where

$$f_+(x,p,\sigma_x,\sigma_p) = \frac{2p^2\left(\sigma_p{}^2+1\right)^2}{\sigma_p{}^4\left(\sigma_p{}^2+\sigma_x{}^2+2\right)} \tag{12b}$$

$$-\frac{\sigma_p{}^2\left(2\sigma_x{}^2+1\right)+\sigma_x{}^2}{\sigma_p{}^2\sigma_x{}^2\left(\sigma_p{}^2+\sigma_x{}^2+2\right)} \tag{12c}$$

$$+\frac{2\left(\sigma_x{}^2+1\right)^2x^2}{\sigma_x{}^4\left(\sigma_p{}^2+\sigma_x{}^2+2\right)}, \tag{12d}$$

and Eq. (9), for *normalized* photon-subtracted outcome states, yields the expression

$$W_-(x,p,\sigma_x,\sigma_p) = f_-(x,p,\sigma_x,\sigma_p)W(x,p,\sigma_x,\sigma_p), \tag{13a}$$

where

$$f_-(x,p,\sigma_x,\sigma_p) = \frac{2p^2\left(\sigma_p{}^2-1\right)^2}{\sigma_p{}^4\left(\sigma_p{}^2+\sigma_x{}^2-2\right)} \tag{13b}$$

$$+\frac{\sigma_p{}^2\left(2\sigma_x{}^2-1\right)-\sigma_x{}^2}{\sigma_p{}^2\sigma_x{}^2\left(\sigma_p{}^2+\sigma_x{}^2-2\right)} \tag{13c}$$

$$+\frac{2\left(\sigma_x{}^2-1\right)^2x^2}{\sigma_x{}^4\left(\sigma_p{}^2+\sigma_x{}^2-2\right)}. \tag{13d}$$

Generally, expressions (12) and (13) are different from each other.

Using the input state (11) in the identity-of-outcome condition (10) returns two constraints as follows:

$$R(\sigma_x) = (\sigma_x^2+1)^2/(\sigma_x^2-1)^2 \tag{14}$$

and $\sigma_p = 1/\sigma_x$.

We notice that the second returned constraint implies that the state is pure: $\sigma_p\sigma_x = 1$, fulfilling Heisenberg's uncertainty principle[12] $\langle x^2\rangle\langle p^2\rangle = 1/4$; see Fig. 1.

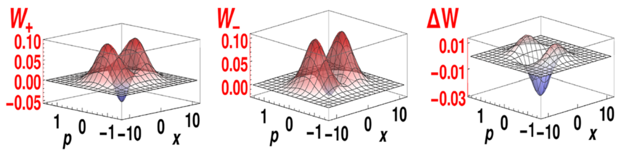

FIG. 1. Using a squeezed vacuum $W(x, p, \sigma_x = 4, \sigma_p = 1/2)$ input state (11) that is *impure*: Plots of the *renormalized* outcome states after a photon is added [W_+ of Eq. (12), left panel] or subtracted [W_- of Eq. (13), middle panel] show that their difference is non-zero [$\Delta W = W_+ - W_- \neq 0$, right panel]. The impurity violates the identity-of-outcome condition (10).

The normalization constraint, R of (14), contains both branches $R = (\pm r)^2$, thus "automatically" giving us the full set of solutions: $0 < \sigma_x < 1$ and $1 < \sigma_x$, without forcing us to have to resolve the sign-problem of Sec. II A.

As an aside, we mention that the normalization constraint $R(\sigma_x)$ agrees with the general commutator expression $\langle \hat{a}\hat{a}^\dagger \rangle = \langle \hat{a}^\dagger \hat{a} \rangle + 1$. In phase space language, this can also be shown by subtracting Eq. (9) from Eq. (8) [i.e., setting $R = 1$ in Eq. (10)] and taking into account that the divergence terms vanish at infinity.

Applying the purity constraint to the squeezed vacuum input states [setting $\sigma_p = 1/\sigma_x$ in (11)], we get the identity-of-outcome, since $f_+ = f_- = 2p^2{\sigma_x}^2 + \frac{2x^2}{{\sigma_x}^2} - 1$ are identical, and so $W_+ = W_-$, with values $W_+ = W_- = -\frac{1}{\pi}$ at the origin of phase space.

Adding up the Wigner distributions of input states (i.e., adding states incoherently) with varying degrees of squeezing, σ_x, violates the identity-of-outcome condition (10), since f_+ and f_- depend on σ_x in different ways.

B. Impure Gaussian states

In order to treat a slightly more general mixed-state case, we consider the general expression $W(x, p) = \exp[g(x, p)]$. Since we deal with the second order linear partial differential equation (10), we expect three integration constants plus a fourth constant term in the exponent, which is fixed by the normalization condition.

We expand g to the fourth order in x and p and find that fulfilling Eq. (10) implies that only the quadratic terms can be non-zero: $g = -\frac{1}{2}(x^2/\sigma_x^2 + c\, x\, p + p^2/\sigma_p^2)$.

Checking on the conditions for the three integration constants, $\{\sigma_x^2, c, \sigma_p^2\}$, we find, again, that the state has to be pure $\sigma_p = 1/\sigma_x$ and that the diagonal term, c, in g has to be a "trigonometric mean." Therefore, the general family of pure states that obey the identity-of-outcome condition (10) are randomly oriented pure squeezed vacuum state of the form

$$W\left(x, p, \theta, \sigma_x, \sigma_p = \frac{1}{\sigma_x}\right) = \frac{1}{\pi} \exp\left[-\frac{x'^{\,2}}{\sigma_x^2} - \sigma_x^2\, p'^{\,2}\right], \qquad (15)$$

where $x' = x\cos\theta + p\sin\theta$ and $p' = p\cos\theta - x\sin\theta$ are rotated coordinates.[13]

V. PHOTON ADDITION AND SUBTRACTION IN THE FOCK REPRESENTATION

In the Fock representation, pure squeezed vacuum states, $\psi(x, \sigma_x)$ of Eq. (4), with the positive squeezing parameter $1/\sigma_x = e^z$, or $\frac{1-\sigma_x^2}{1+\sigma_x^2} = \tanh z = \tanh(\ln(1/\sigma_x))$, have the form[12]

$$|\psi(\sigma_x)\rangle = \frac{1}{\sqrt{\cosh z}} \sum_{m=0}^{\infty} (-\tanh z)^m \frac{\sqrt{(2m)!}}{2^m m!} |2m\rangle. \qquad (16)$$

Forming the ratio of photon-added and photon-subtracted states yields, after a few steps of calculation (see Appendix A), that

$$\frac{\hat{a}^\dagger\, |\psi(\sigma_x)\rangle}{\hat{a}\, |\psi(\sigma_x)\rangle} = r(\sigma_x) = \frac{-1}{\tanh z}. \qquad (17)$$

This shows that despite the fact that photon numbers are not conserved, the identity-of-outcome is guaranteed. We note that a similarly surprising effect is predicted to exist for two-mode vacuum squeezed states, where the removal of a photon in one mode adds a photon in the other;[2] similarly, photon numbers can change without adding or subtracting a photon.[14]

In the language of Sec. IV B, this ratio (17) is, as expected, since, according to Eq. (5), $r = -1/\tanh(z)$.

The ratio conditions (5) and (17) also make plausible that mixed states, even impure squeezed vacuum states (11) or sums of pure squeezed vacuum states with different squeezing ratios, cannot fulfill the identity-of-outcome condition (10), since the simultaneous presence of different squeezing levels, σ_x, cannot be accommodated by the single ratio (17): to graphically illustrate this finding, Fig. 1 displays the case of an impure squeezed input state (11) with $\sigma_x \sigma_p = 2$.

VI. IMPURE SQUEEZED VACUA

The ratio conditions (5), (14), and (17) make plausible that only pure squeezed vacuum states are likely to fulfill the identity-of-outcome condition (10).

In addition, in Sec. IV B, we showed that the expansion to fourth order in the exponent shows that pure rotated squeezed vacuum states (15) fulfill the identity-of-outcome condition (10), but, for example, impure squeezed vacuum states of the Gaussian form (11) do not.

Furthermore, at the end of Sec. II, we pointed out that pure displaced states, for which $\langle \hat{x} \rangle \neq 0$ [or, in light of the rotation invariance of state (15), states for which $\langle \hat{p} \rangle \neq 0$], can also not fulfill the identity-of-outcome condition (10).

All this leads us to focus on pure squeezed vacuum states, with the same squeezing, σ_x, as candidates for the fulfillment of the identity-of-outcome condition (10).

We notice that, in Wigner phase space, the identity-of-outcome condition (10) is a *linear* map of the input state W. It can therefore accommodate any two (or more) states simultaneously, when they are added up *incoherently*, assuming that those form the same ratios of norms, R; see Fig. 2.

Then, the following statement is the central conclusion of this work:

Any *incoherent* sum of pure squeezed vacuum states (15) with the same squeezing, σ_x, but any phase space rotation angle, θ, can

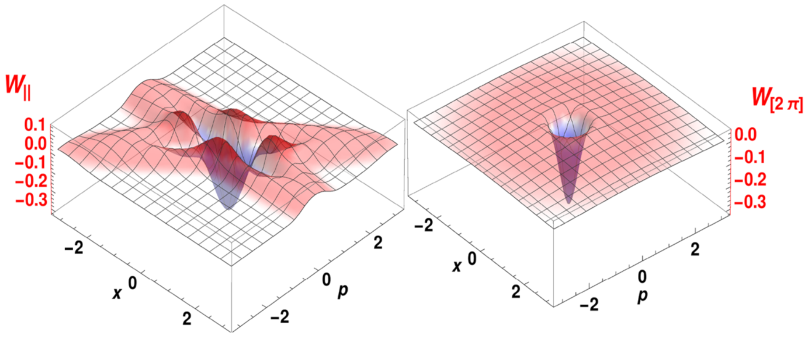

FIG. 2. Plot of photon-added or photon-subtracted cases for input states $W_{\text{II}}(\sigma_x = 2.2)$ (19), with parameters $P = 0.5$ and orientation angles $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{4}$ [left panel] and maximally mixed state $W_{[2\pi]}(\sigma_x = 2.2)$ (20) [right panel]. In both cases, $W(x = 0, p = 0) = -\frac{1}{\pi}$ and both fulfill the identity-of-outcome condition (10).

be added up to give impure states obeying the identity-of-outcome condition (10); they generally have the form

$$W_P(\sigma_x) = \int_0^{2\pi} d\theta\, P(\theta)\, W\left(x, p, \theta, \sigma_x, \frac{1}{\sigma_x}\right), \tag{18}$$

where $\int_0^{2\pi} d\theta\, P(\theta) = 1$ is a (positive) probability distribution in the distribution-sense: With "δ" for Dirac's delta-distribution, the specific case $P(\theta) = \delta(\theta - \theta_0)$ describes the pure squeezed vacuum states such as (15).

A. Mixtures of rotated pure vacuum states with equal squeezing in phase space

As a special case of state (18), we consider the two-component states,

$$W_{\text{II}}(\sigma_x) = P\, W(\theta_1, \sigma_x) + (1 - P)\, W(\theta_2, \sigma_x), \tag{19}$$

with probabilities $0 \le P \le 1$ parameterizing the incoherent addition, for any orientation θ at equal squeezing σ_x.

A specific case of the input state W_{II} is displayed in Fig. 2. Expanding it to fifth order or higher order shows that its exponent function, g, see Sec. IV B, contains the terms of order 5 and above, for both x and p. Therefore, a proof of our main conjecture just given above, based on the approach of Sec. IV B, fails and we need to use a different approach; see Sec. VII.

In this context, it is interesting to observe that a simple expression for the angular average of pure states (15) with fixed squeezing, σ_x, exists. $W_{[2\pi]}(\sigma_x) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\, W(x, p, \theta, \sigma_x, \frac{1}{\sigma_x})$ yields the maximally mixed input state,

$$W_{[2\pi]}(\sigma_x) = \frac{1}{\pi} \exp\left[-\frac{(x^2+p^2)(\sigma_x^4+1)}{2\sigma_x^2}\right] I_0\left(\frac{(x^2+p^2)(\sigma_x^4-1)}{2\sigma_x^2}\right), \tag{20}$$

with I_0 being the zeroth-order modified Bessel function of the first kind.

Its purity, $\mathcal{P}[W_{[2\pi]}(\sigma_x)] = \int dx \int dp\, 2\pi W_{[2\pi]}^2(\sigma_x) = (4\sigma_x^2 K[(1-\sigma_x^4)^2/(1+\sigma_x^4)^2])/(\pi(1+\sigma_x^4))$, where K is the complete elliptic integral of the first kind, drops with increasing squeezing; see Fig. 3.

This purity expression (should and) does obey the symmetry relation $\mathcal{P}[W_{[2\pi]}(\sigma_x)] = \mathcal{P}[W_{[2\pi]}(1/\sigma_x)]$; see Fig. 3.

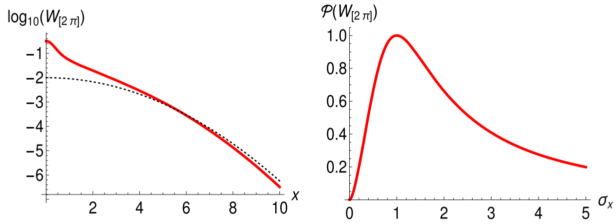

FIG. 3. Radial profile, $\log_{10}[W_{[2\pi]}(x, 0, \sigma_x)]$, of maximally mixed *input* state (20) with fixed $\sigma_x = 2.2$ [left panel, red curve; for contrast, the black dotted curve shows a Gaussian] demonstrating clear deviation from a Gaussian profile. Purity, $\mathcal{P}(W_{[2\pi]}(\sigma_x))$, of the maximally mixed *input* state as a function of squeezing σ_x (right panel).

Similarly to the case of W_{II}, also $W_{[2\pi]}(\sigma_x)$, expanded to high order, contains the terms of sixth and higher orders in x and p in its exponent function, g; see Fig. 3.

VII. IDENTITY-OF-OUTCOME IN THE BARGMANN REPRESENTATION

In order to prove that the identity-of-outcome condition is only met by the states of the form (18), we now switch to the Bargmann representation.[15] This allows us to write out the identity-of-outcome condition as a partial differential equation, which factorizes into two first-order ordinary differential equations.

Forming overlaps with Glauber coherent states $|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle$ maps density matrices $\hat{\rho} \mapsto F$, where F is its Bargmann representation [see Eq. (90) of Ref. 15], the so-called stellar function.[16,17] Specifically, we form

$$F(\alpha, \beta) = e^{\frac{|\alpha|^2+|\beta|^2}{2}} \langle \alpha^* | \hat{\rho} | \beta \rangle = \sum_{n,m=0}^{\infty} \rho_{nm} \frac{\alpha^n}{\sqrt{n!}} \frac{\beta^m}{\sqrt{m!}}, \tag{21}$$

where "$*$" stands for complex conjugation and F is a two-variable analytic function obeying $\rho_{nm} = \rho_{mn}^*$. To preserve the commutation relations, the map from $\hat{\rho}$ to F simultaneously maps the operators $\hat{a} \mapsto \partial_\alpha$ and $\hat{a}^\dagger \mapsto \alpha\times$ (multiplication with α), and the same for β.

Correspondingly, using stellar functions to represent states, the identity-of-outcome condition (10) becomes

$$\alpha\beta F(\alpha,\beta) = R\ \partial_\alpha\partial_\beta F(\alpha,\beta). \tag{22}$$

This form factorizes; the two equations in α and β only couple to each other through a positive R [compare (14)].

Now, it is straightforward to solve the identity-of-outcome condition generally; we find

$$F(\alpha,\beta) = \sqrt{1-\frac{1}{R}}\exp\left[\frac{1}{2\sqrt{R}}\left(e^{i\theta}\alpha^2 + e^{-i\theta}\beta^2\right)\right]. \tag{23}$$

Note that this is a pure state, confirming that Eq. (14) holds and Eq. (15) together with Eq. (18) describes general solutions; for more details, see Appendix B.

VIII. BOGOLIUBOV-TRANSFORMED ANNIHILATION OPERATORS

Formally, a squeezed vacuum state (4) can be generated by the action of the squeezing operator on the true vacuum $\hat{S}(z)|0\rangle = |\psi(\sigma_x)\rangle$, with $z = \ln(1/\sigma_x)$ real, and

$$\hat{S}(\zeta) = \exp\left[(\zeta\hat{a}^2 - \zeta^*\hat{a}^{\dagger 2})/2\right]. \tag{24}$$

Here, $\zeta = ze^{i\phi}$ is the *squeezing parameter*, with phase ϕ.[12] This phase ϕ determines the angle of the quadrature that is being squeezed, but for simplicity, we can set it to zero for what follows.

It is known that, in the Heisenberg picture, squeezing maps position and momentum according to

$$\hat{x}_z = \hat{S}^\dagger(z)\ \hat{x}\ \hat{S}(z) = \hat{x}\ e^{-z}, \tag{25a}$$

$$\hat{p}_z = \hat{S}^\dagger(z)\ \hat{p}\ \hat{S}(z) = \hat{p}\ e^{+z}, \tag{25b}$$

which describes a passive transformation for the position squeezing by the factor $1/\sigma_x = e^z$ and corresponding momentum antisqueezing.

The passive squeezing transformations of the creation and annihilation operators are given by

$$\hat{a}_z = \hat{a}\cosh z - \hat{a}^\dagger \sinh z, \tag{26a}$$

$$\hat{a}_z^\dagger = \hat{a}^\dagger \cosh z - \hat{a}\sinh z, \tag{26b}$$

the well-known *Bogoliubov transformations*.[12]

Our Eq. (1) can also be interpreted as the quest for finding the states that are annihilated by the σ-parameterized annihilation operator,

$$\hat{a}_\sigma = \hat{a}^\dagger - r(\sigma)\ \hat{a}. \tag{27}$$

This operator annihilates the pure squeezed vacuum states of the form (4): $\hat{a}_\sigma\psi(\sigma) = 0$ (their phase is $\theta = 0$).

Because of Eq. (5), with $r = -1/\tanh(z)$, this is of Bogoliubov-form [where the phase $\phi \neq \theta$ since Eq. (26) describes passive transformations].

A related result was given by Miller and Mishkin in 1966.[6,18]

What we try to emphasize is this: for the generalization to the mixed-state case (here in the language of phase space), neither the combination $a_z \star W \star a_z^*$ nor $a_\sigma \star W \star a_\sigma^*$ can be used because they contain second-order terms, proportional to $a^* \star W \star a^*$ and $a \star W \star a$.

In phase space, the correct form for the identity-of-outcome condition is given by Eq. (10), which annihilates all θ-rotated pure squeezed vacuum states (15) and mixed states such as (19) and (20).

Therefore, generally speaking, the annihilation operators of Bogoliubov form only annihilate properly aligned pure states, unlike the more general operator $a^* \star W \star a - R(\sigma_x) a \star W \star a^*$ for the identity-of-outcome condition (10).

IX. EFFICIENCIES OF STATE GENERATIONS

Photon added or subtracted squeezed states can be generated experimentally by mixing squeezed vacuum states $W = W(x,p,\theta,\sigma_x,\sigma_p)$, see Eq. (15), at a mode-matched[19] two-mode beam splitter with reflection probability ϵ (and transmission probability $1-\epsilon$); see Ref. 20.

If the squeezed state enters through mode $\hat{a}_{in}$ and the other port $\hat{b}_{in}$ is either empty or a single photon enters it, then upon detecting, respectively, a single photon in the outgoing port $\hat{b}$ or no photon, we generate, respectively, a photon-subtracted (9) or a photon-added (8) squeezed state in the outgoing port $\hat{a}$.

The respective state generation success probabilities are

$$P_+ = \frac{\sigma_p\sigma_x(1-\epsilon)\left(\sigma_p^2 + \sigma_x^2\left(\sigma_p^2(\epsilon+1)+1\right) - \epsilon + 1\right)}{\pi\left(\left(\sigma_p^2 + \left(\sigma_p^2-1\right)\epsilon+1\right)\left(\sigma_x^2 + \left(\sigma_x^2-1\right)\epsilon+1\right)\right)^{3/2}} \tag{28}$$

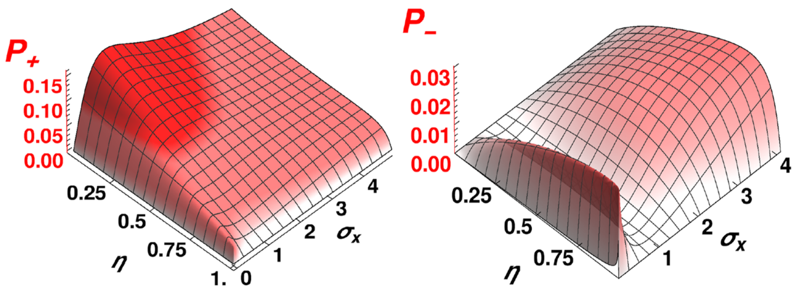

FIG. 4. Plots of state generation success probabilities P_+ (28) and P_- (29) for *pure* squeezed input states $(\sigma_p = 1/\sigma_x)$. Consistent with the energy-budget depending on operational details, the probabilities are much less than unity.

and

$$P_- = \frac{\sigma_p\sigma_x(\epsilon-1)\left(\sigma_p^2\sigma_x^2+\left(\sigma_p^2-1\right)\left(\sigma_x^2-1\right)\epsilon-1\right)}{\pi\left(\left(\sigma_p^2+\left(\sigma_p^2-1\right)\epsilon+1\right)\left(\sigma_x^2+\left(\sigma_x^2-1\right)\epsilon+1\right)\right)^{3/2}}. \tag{29}$$

For *pure* squeezed input states, they are depicted in Fig. 4.

The processes associated with the state generation success probabilities P_+ (28) and P_- (29) are quite different from each other, and each has fairly low success probabilities. Consequently, Fig. 4 does not only illustrate the behavior of P_+ and P_- but also serves to remind us that while overall all processes are strictly energy-conserving,[4] the subset of those processes of the experimental ensemble that give a "successful" outcome are a small subensemble. This explains why arguments based on naïve energy conservation assumptions, as alluded to in the Introduction, cannot invalidate our result that the identity-of-outcome is possible.

X. CONCLUSIONS AND OUTLOOK

Single photon states promise to play a crucial role as photonic qubits,[21] as well as provide nonclassical resources for quantum computing.[22] Quantum state engineering at the level of a few photons also gives rise to the generation of optical cat states.[23–25]

Photon-subtraction is typically realized through a conditional measurement at a beam splitter,[26–28] but photon-addition[26] can also be realized through the injection of a state into parametric down-converters.[14,29–31]

It might appear surprising that adding a photon or subtracting it can give the same outcome, seemingly violating energy conservation. However, our calculations using the Fock representation confirm our results. To consider energy conservation, the energy-budget for an entire ensemble subjected to the subtraction or addition of photons, and its environment, would have to be considered. This is illustrated in Sec. IX.

We emphasize that this work does, of course, not show any violation of energy conservation nor is it our intention to suggest that this might be an interesting question to study—far from it, within standard quantum theory energy is always strictly conserved.[4] What this work establishes is that the identity-of-outcome results when a single photon is added to or subtracted from the same pure squeezed state (or suitably chosen mixtures thereof). By this, we mean the *successful* addition or subtraction of a photon. In addition, this can (and is typically being) operationally checked when either the subtracted photon is detected or the added photon is detected by a photon missing.

Our results do not describe the large number of attempts, contributing to the energy-budget, where photon addition or subtraction is unsuccessful.

Similarly, our results do not carry over to further repeated addition or subtraction of photons, since, after the first round, the resulting state is no longer a squeezed vacuum state.

Probably, there exist other interesting "generalized" symmetries, annihilation operators, or eigenrelations like the identity-of-outcome for single-photon addition-versus-subtraction studied here.

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AUTHOR DECLARATIONS

The authors have no conflicts to disclose.

Conflict of Interest Author Contributions

Ole Steuernagel: Conceptualization (equal); Formal analysis (equal); Funding acquisition (supporting); Writing – original draft (lead); Writing – review & editing (equal). **Shao-Hua Hu**: Formal analysis (supporting); Validation (equal); Writing – review & editing (supporting). **Ray-Kuang Lee**: Conceptualization (equal); Funding acquisition (lead); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

APPENDIX A: DERIVATION OF Eq. (17) USING THE FOCK-STATE REPRESENTATION

Rewrite Eq. (16) as $|\psi(\sigma_x)\rangle = \frac{1}{s}\sum_{m=0}^{\infty} c_m|2m\rangle$, where $s = \sqrt{\cosh z}$ and $c_m = (-\tanh z)^m \frac{\sqrt{(2m)!}}{2^m m!}$. With $\hat{a}^\dagger|2m\rangle = \sqrt{2m+1}\,|2m+1\rangle$ and $\hat{a}|2m\rangle = \sqrt{2m}\,|2m-1\rangle$, we have for the left hand side of Eq. (17)

$$\frac{\hat{a}^\dagger\,|\psi(\sigma_x)\rangle}{\hat{a}\,|\psi(\sigma_x)\rangle} = \frac{\sum_{m=0}^{\infty} c_m\sqrt{2m+1}|2m+1\rangle}{\sum_{n=1}^{\infty} c_n\sqrt{2n}|2n-1\rangle} \tag{A1}$$

$$= \frac{\sum_{m=0}^{\infty} c_m\sqrt{2m+1}|2m+1\rangle}{\sum_{n=0}^{\infty} \frac{c_{n+1}}{c_n} c_n\sqrt{2n+2}|2n+1\rangle}. \tag{A2}$$

Since $\sqrt{2n+2}\frac{c_{n+1}}{c_n} = (-\tanh z)\sqrt{2n+2}\,\frac{\sqrt{(2n+2)(2n+1)}}{2n+2}$, expression (A2) equals the right hand side of Eq. (17). ■

We would like to emphasize that this result in no way should be seen to invalidate energy conservation. Energy is strictly conserved;[4] for more details, see Sec. IX and Sec. X.

APPENDIX B: SOLVING Eq. (22)

Equation (22) has the product form, allowing us to use the ansatz $F(\alpha,\beta) = A(\alpha)B(\beta)$ such that (22) becomes

$$(\alpha A)(\beta B) = R(\partial_\alpha A)(\partial_\beta B). \tag{B1}$$

The solutions have two integration constants, the real and imaginary parts of $C \in \mathbb{C}$ in

$$A(\alpha) = \exp\left[\frac{C}{2R}\alpha^2\right], \quad B(\beta) = \exp\left[\frac{1}{2C}\beta^2\right]. \tag{B2}$$

These are further constrained by the normalization and hermiticity of ρ_{mn} (21). With $\hat{\rho} = \hat{\rho}^\dagger$, we have $(F(\alpha,\beta))^* = F(\beta^*,\alpha^*)$, or, in terms of (B2)

$$\exp\left[\frac{C^*}{2R}\alpha^{*2} + \frac{1}{2C^*}\beta^{*2}\right] = \exp\left[\frac{1}{2C}\alpha^{*2} + \frac{C}{2R}\beta^{*2}\right], \tag{B3}$$

so $|C|^2 = R$. This implies that F, structurally, must have the form (23). For the reader's benefit, we will now show that the normalization constant, in (23), is $\sqrt{1-\frac{1}{R}}$, where, without loss of generality, we assume $R > 1$; see below.

Starting out from Eq. (21), the normalization condition is $\int_{\mathbb{C}} \frac{d^2\alpha}{\pi} e^{-|\alpha|^2} F(\alpha,\alpha^*) = 1$ [note that $\frac{1}{\pi}e^{-|\alpha|^2}F(\alpha,\alpha^*)$ is Husimi's Q-function]. We have

$$\begin{aligned}\int_{\mathbb{C}} \frac{d^2\alpha}{\pi} e^{-|\alpha|^2} e^{\frac{\alpha^2 e^{i\vartheta} + \alpha^{*2} e^{-i\vartheta}}{2\sqrt{R}}} &= \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdy \\ &\quad \times e^{\frac{(x+iy)^2+(x-iy)^2}{2\sqrt{R}}} e^{-x^2-y^2} \\ &= \frac{1}{\pi}\left(\int_{-\infty}^{\infty} dx\, e^{-\left(1-\frac{1}{\sqrt{R}}\right)x^2}\right) \\ &\quad \times \left(\int_{-\infty}^{\infty} dy\, e^{-\left(1+\frac{1}{\sqrt{R}}\right)y^2}\right) \\ &= \left(1-\frac{1}{R}\right)^{-\frac{1}{2}} = \cosh(z). \end{aligned} \tag{B4}$$

This links the solutions (23), with (15) and (16). The proportionality constant $R = \frac{1}{\tanh^2(z)} > 1$ agrees with expression (17) and is via expression (5) linked to expression (14), with the squeezing level z given in Eq. (24).

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