EE 2015
(Partial) Differential Equations and Complex Variables

Ray-Kuang Lee*

Institute of Photonics Technologies,
Department of Electrical Engineering, and Department of Physics,
National Tsing-Hua University

*rklee@ee.nthu.edu.tw

Course Description:

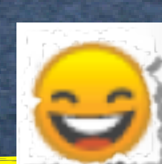
★ **Time:** T5T6R5R6 (1:10PM-3:00PM, Tuesday and Thursday)

★ This course is one of "Engineering (Applied) Mathematics":

- Vector Calculus (required), [Textbook] PART B
- Linear Algebra (required), [Textbook] PART B

- Ordinary Differential Equations, ODEs, [Textbook] PART A
- Partial Differential Equations, PDEs, [Textbook] PART C
- Fourier Analysis (moved to "Singnals and Systems," required)
- Complex Analysis, [Textbook] PART D

- Numeric Analysis, [Textbook] PART E
- Optimization and Graphs, [Textbook] PART F
- Probability and Statistics (required), [Textbook] PART G

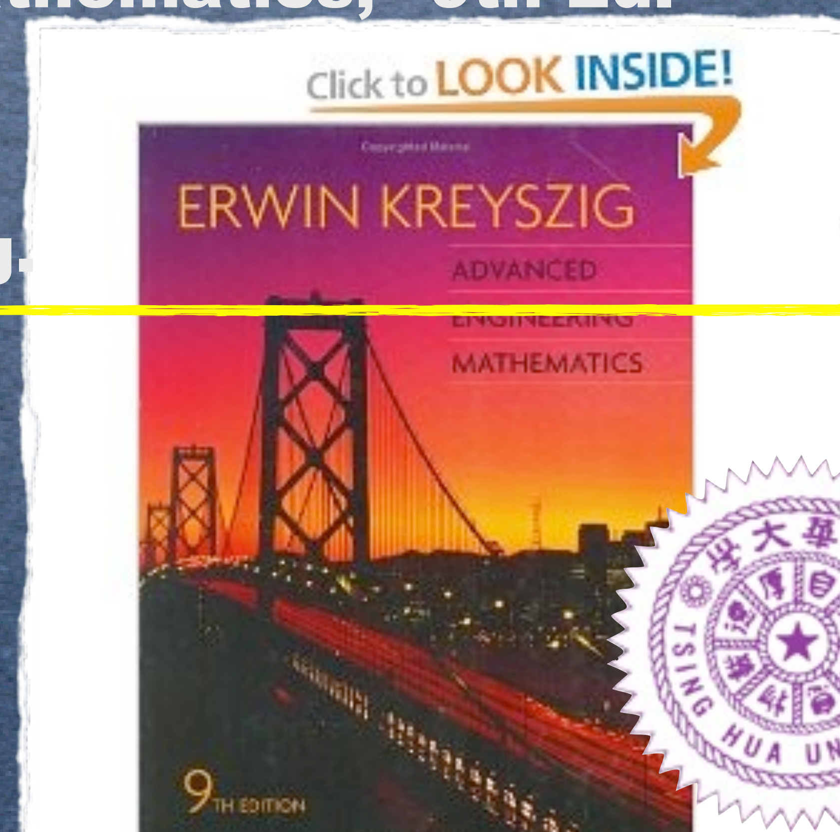
★ 3 Credits, but 4 Hours 

Textbook:

- ★ **No background, but "Calculus" is required.**
- ★ **Teaching Method: in-class lectures with examples.**
- ★ **Textbook:**

Erwin Kreyszig, "Advanced Engineering Mathematics," 9th Ed. John Wiley & Sons Inc., (2006).

- ★ **Office hours: T78R78 at R523, EECS bldg.**



Syllabus:

★ Course description and Introduction, 9/14

1. **Ordinary Differential Equations: 4 weeks**
   - First-order ODEs, Ch. 1: 9/16, 9/21
   - Second-order ODEs, Ch. 2: 9/23, ~~9/28~~, 9/30, 10/5, 10/7
   - Higher-order ODEs, Ch. 3: 10/12
   - Systems of ODES, Ch. 4: 10/14
   - 1st EXAM, 10/15 (Friday night)
2. **Transform Methods: 4 weeks**
   - Laplace Transforms, Ch. 6: 10/19 - 11/11
   - 2nd EXAM, 11/12 (Friday night)
3. **Series and Complex Variables: 9 weeks**
   - Power Series, Ch. 5: 11/16, 11/18
   - Fourier Series, Ch. 11: 11/23, 11/25, 11/30, 12/2
   - 3rd EXAM, 12/3 (Friday night)
   - PDE by Fourier Series, Ch. 12, 12/7 - 12/22
   - 4th EXAM, 12/23 (in Class)
   - Taylor and Laurent Series, Ch. 13-16: 12/28, 12/30
   - Complex and Residue Integrations, Ch. 16: 1/4 - 1/13
   - 5th EXAM, 1/14 (Friday night)

Evaluation:

1. **Homework:** 30%
2. **EXAMS:** 70%
   - 1st EXAM: 20%, 10/15 (Friday night)
     Ordinary Differential Equations, [Textbook] Ch.1 - Ch. 4
   - 2nd EXAM: 15%, 11/12 (Friday night)
     Laplace Transforms, [Textbook] Ch. 6
   - 3rd EXAM: 10%, 12/3 (Friday night)
     Power and Fourier Series, [Textbook] Ch. 5, Ch. 11
   - 4th EXAM: 10%, 12/23 (in class)
     Partial Differential Equations, [Textbook] Ch.12
   - 5th EXAM: 15%, 1/14 (Friday night)
     Complex Variables, [Textbook] Ch. 13 - Ch. 16
3. **Bonus:** 10%
   - Quiz and Questions in the classroom

3 Credits = 17 Weeks*4 Hours + Homework (>12) + 5 EXAMS + ...

TSING HUA UNIVERSITY

Engineering (Applied) Mathematics

Modeling → Solving → Interpretation

- **Analytical approach**
- **Numerical approach**

Maxwell's equations :

(1831-1879)

- Gauss's law for the electric field:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \Longleftrightarrow \oint_S \mathbf{E} \cdot \mathrm{d}A = \frac{q}{\epsilon_0},$$

- Gauss's law for magnetism:

$$\nabla \cdot \mathbf{B} = 0 \Longleftrightarrow \oint_S \mathbf{B} \cdot \mathrm{d}A = 0,$$

- Faraday's law of induction:

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B} \Longleftrightarrow \oint_C \mathbf{E} \cdot \mathrm{d}l = -\frac{\partial}{\partial t}\Phi_B,$$

- Ampére's circuital law:

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J} + \frac{\partial}{\partial t}\mathbf{D}) \Longleftrightarrow \oint_C \mathbf{B} \cdot \mathrm{d}l = -\mu_0(\mathbf{I} + \frac{\partial}{\partial t}\Phi_D)$$

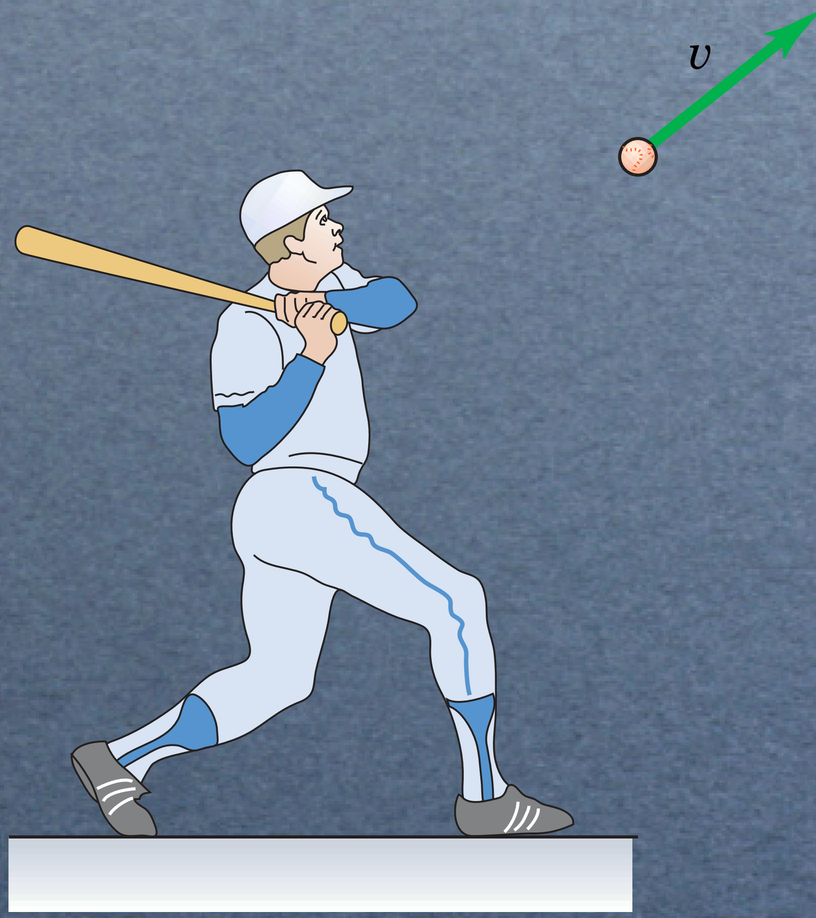
QUIZ: Differential or Integral Equations?

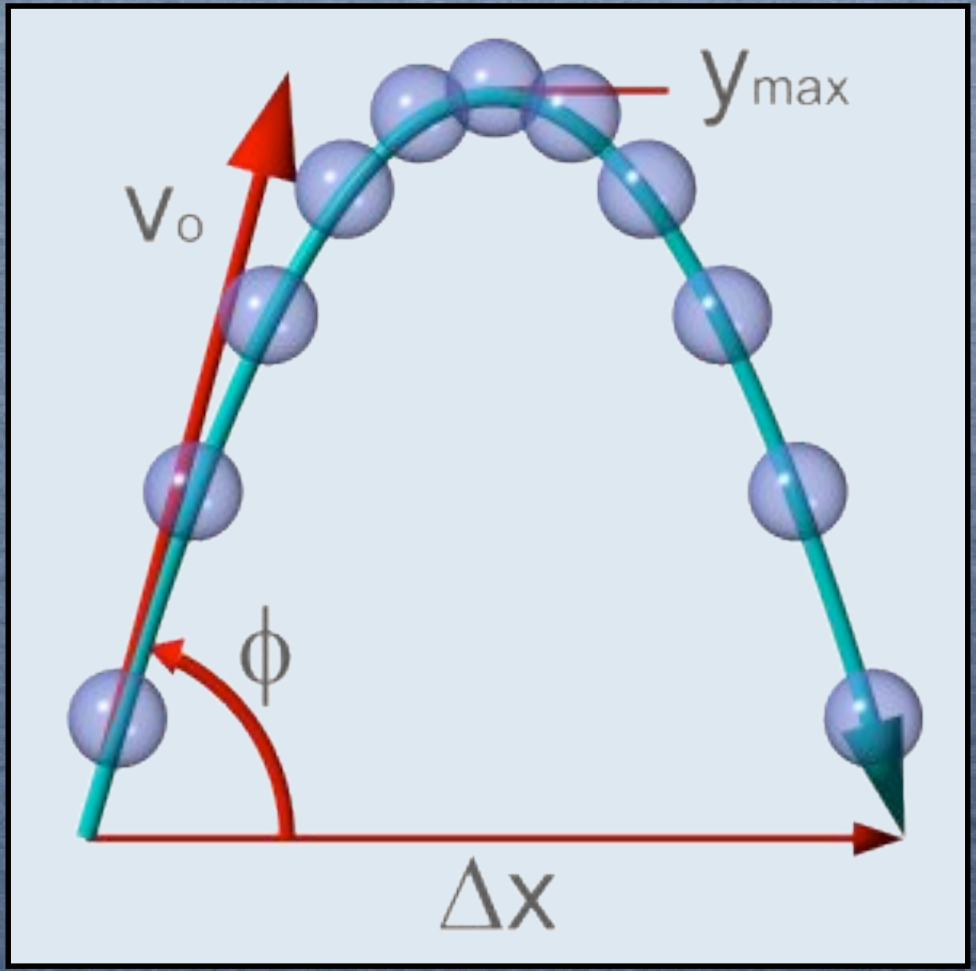
Differential v.s. **Integral**

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \qquad \int f(x), \mathrm{d}x$$

Differential	Integral
Change Rate	Total Sum
Local Information	Global Information
Initial Value Problem	Boundary Condition

Modeling: Projectile motion





Newton's mechanics: $m\vec{a} = \vec{F}$, i.e.,

$$\frac{dv_y}{dt} = -g, \qquad \Rightarrow \qquad v_y(t) = v_0 - gt.$$

Solving: Projectile motion without Air Resistance

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = -g \quad \Rightarrow \quad v_y(t) = v_0 \sin\theta - g\,t,$$
$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = 0 \quad \Rightarrow \quad v_x(t) = v_0 \cos\theta,$$

- Analytically approach:

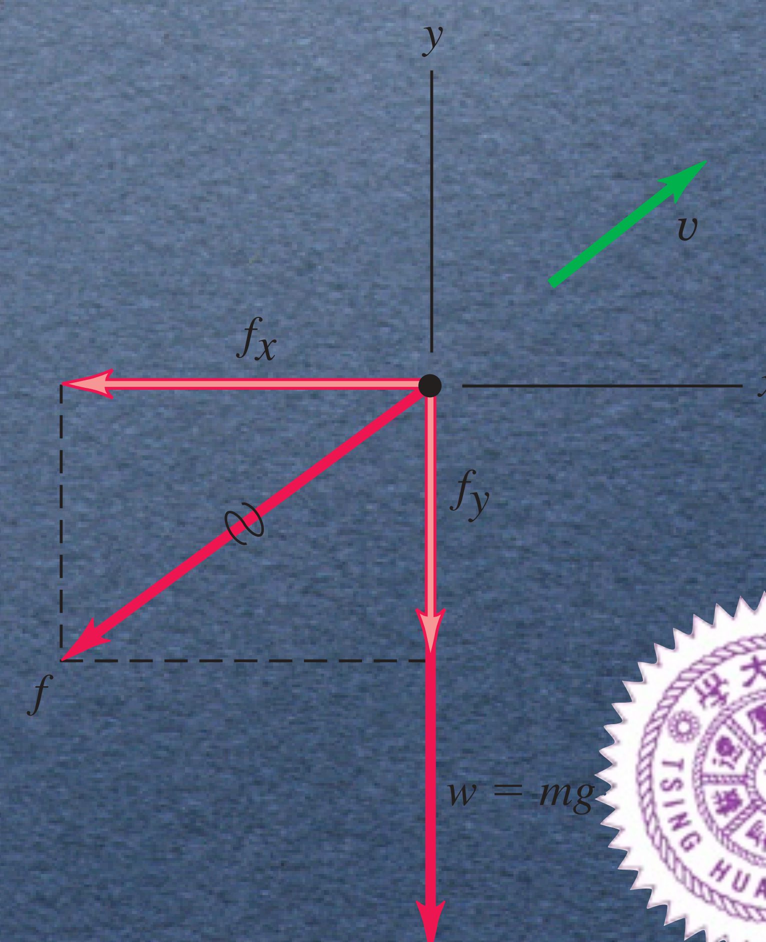
$$x(t) = \int_0^t v_x(t)\,\mathrm{d}t = x(0) + v_0 \cos\theta\, t$$
$$y(t) = \int_0^t v_y(t)\,\mathrm{d}t = y(0) + v_0 \sin\theta\, t - \frac{1}{2} g\, t^2,$$

- Numerical approach:
  - Solve Differential Eq.: Finite-Difference, Finite-Element, ...
  - Solve Integral Eq.: Finite-Volume, Moment methods, ...

Modeling: Projectile motion with Air Resistance

Assumption 1:
that there exists an air drag force,

$$m\frac{\mathrm{d}v_x}{\mathrm{d}t} = -|\vec{F}^D_x(v_x, v_y, t)|,$$
$$m\frac{\mathrm{d}v_y}{\mathrm{d}t} = -mg - |\vec{F}^D_y(v_x, v_y, t)|.$$



Modeling: Projectile motion with Air Resistance, cont.

Assumption 2:

- Assume that the magnitude of the air drag force $\vec{F}^D$ is approximately proportional to the square of the projectile's speed relative to the air, i.e., $|\vec{F}^D| \approx v^2$, or

$$\vec{F}^D \equiv C\,|v|\vec{v} = C\,\sqrt{v_x^2 + v_y^2}\vec{v},$$

where the constant C depends on the density ρ of air, the silhouette area A of the body (its area as seen from the front), and a dimensionless constant C_d called the *drag coefficient* that depends on the shape of the body, i.e., $C = C_d\,\rho\,A$.

$$\frac{dv_x}{dt} = -\frac{C}{m}|v|\,v_x = -\frac{C}{m}\sqrt{v_x^2 + v_y^2}v_x,$$
$$\frac{dv_y}{dt} = -g - \frac{C}{m}|v|\,v_y = -\frac{C}{m}\sqrt{v_x^2 + v_y^2}v_y,$$

Modeling: Projectile motion with Air Resistance, cont.

Assumption 2:

Model 1: $$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{C}{m}v^2,$$

Model 2: $$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{C}{m}v,$$

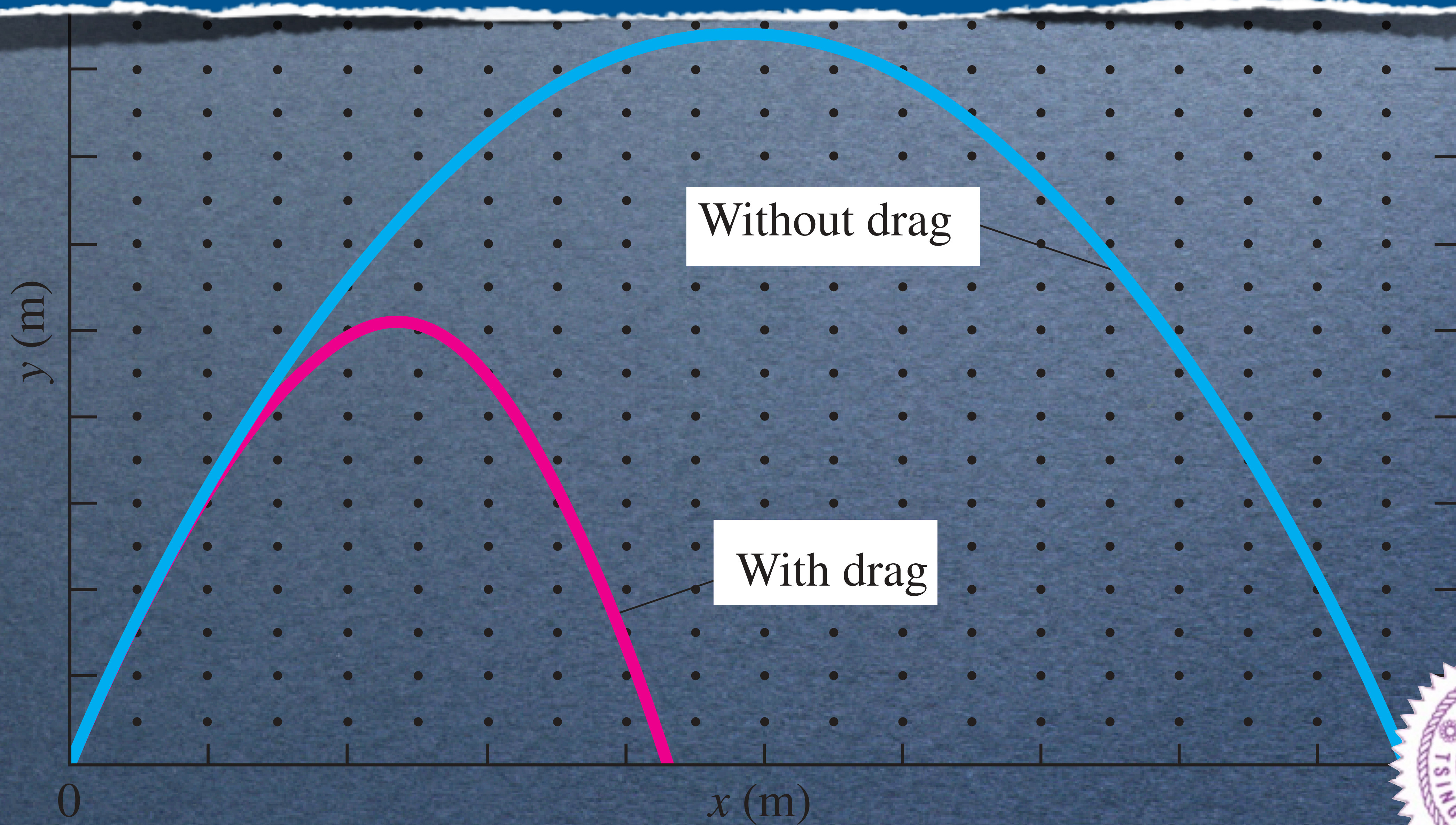
Model 3: $$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{C}{m}\sqrt{v},$$

Model 4: $$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{C(t)}{m}f(v),$$

QUIZ: Which model supports the longest projectile motion distance?

Interpretation: Projectile motion with drag

QUIZ: The projectile angle to support a farest projectile motion is the same as the case without a drag resistance?



Terminologies:

ONE independent variable → Ordinary Differential Equation, **ODE**

$$y'(x) \equiv \frac{\mathrm{d}}{\mathrm{d}x} y(x)$$

Ch. 1 - 5

More than ONE independent variable → Partial Differential Equation, **PDE**

$$f_x \equiv \frac{\partial}{\partial x} f(x, y, \dots)$$

Ch. 12

First-order ODEs: Order

- If the nth derivative $y^{(n)} = \mathrm{d}^n y/\mathrm{d}\,x^n$ of the unknown function $y(x)$ is the highest occurring derivative, it is called an ODE of nth-order:

$$F(x, y, y', \cdots, y^{(n)}) = 0, \qquad \text{where} \quad y^{(n)} = \frac{\mathrm{d}^n y}{\mathrm{d}\,x^n},$$

- Linear nth-order ODE:

$$y^{(n)} + p_{n-1}(x) y^{(n-1)} + \cdots + p_1(x) y' + p_0(x) y = r(x),$$

- Explicit form:

$$y' = f(x, y)$$

- Implicit form:

$$F(x, y, y') = 0$$

Family of solutions:

$$y' = \frac{\mathrm{d}y}{\mathrm{d}t} = \pm\gamma y,$$

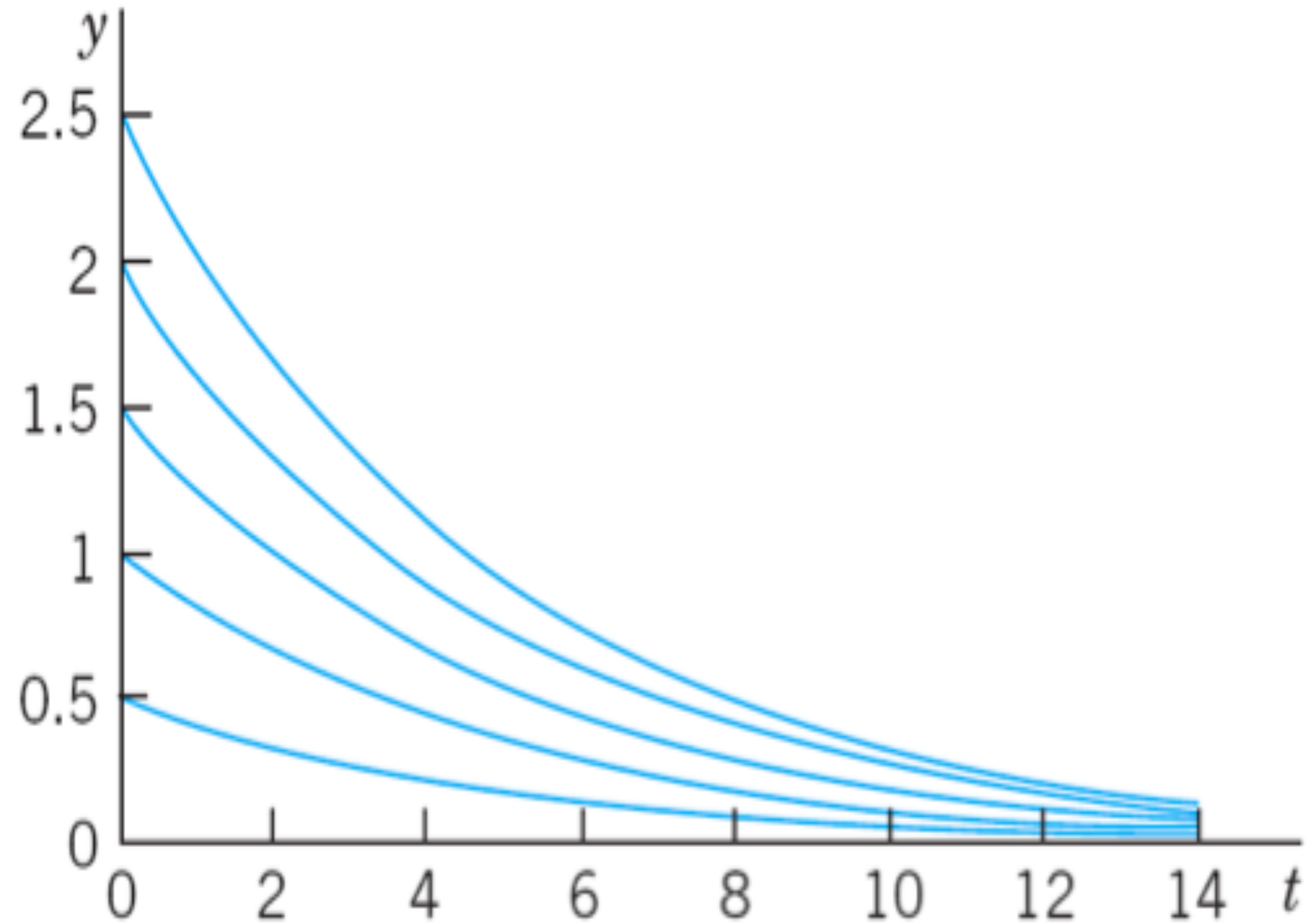
- General solutions:

$$y(t) = c\, e^{\pm\gamma t},$$

- $\pm\gamma$ denotes the growth/decay rate.
- c is an arbitrary constant.
- $y(t) = c\, e^{\pm\gamma t}$ is a family of solutions.
- Initial value problem, $y(t = 0)$ is given.
- Boundary value problem, $y(t_1)$ is given.

Family of solutions: Exponential Decay

$$y' = -0.2\,y$$



First-order ODEs: Separable equations

$$y' = f(x, y) = f_1(x)\, f_2(y), \quad \text{or equivalently}$$

$$g(y)\,\mathrm{d}\,y = f(x)\,\mathrm{d}\,x, \Rightarrow \int_{y_0}^{y} g(y_1)\,\mathrm{d}y_1 = \int_{x_0}^{x} f(x_1)\,\mathrm{d}x_1.$$

Example: $$y' = 1 + y^2$$

Hint: Integrals:

$$\int \frac{1}{a^2 + x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a},$$

$$\int \frac{1}{a^2 - x^2}\,\mathrm{d}x = \frac{1}{a}\tanh^{-1}\frac{x}{a},$$

Solution:

$$y = \tan(x + c) \quad \text{or} \quad y = \tan x + c \quad ?$$

First-order ODEs: Reducible to Separable Form

Example:

$$2xyy' = y^2 - x^2$$

Hints:

1. Divide the given equation by $2xy$.
2. Define the new variable $u \equiv \frac{y}{x}$,
   then reduce the Eq. into a separable form.

Integrals:

$$\int \frac{1}{a^2+x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}\frac{x}{a},$$

$$\int \frac{x}{a^2+x^2}\,\mathrm{d}x = \frac{1}{2}\ln(a^2+x^2),$$

Solution:

$$x^2 + y^2 = cx$$

Homework #1:

1. (25%) Solve

$$y\,y' = (x-1)e^{-y^2}, \qquad y(0) = 1. \tag{1}$$

2. (25%) Solve

$$y' = \frac{2\sqrt{xy} - y}{x}, \qquad \text{Hint: try} \quad y = ux. \tag{2}$$

3. (25%) Solve

$$(\cos x \sin x - xy^2)\,\mathrm{d}x + y(1-x^2)\,\mathrm{d}y = 0, \qquad y(0) = 2. \tag{3}$$

4. (25%) Show that any equation which is *separable*, that is, of the form:

$$M(x) + N(y)y' = 0,$$

is also **exact**.

Homework #1:

1. Please do the homework Yourself !!

2. Homework is designed for your PRACTICE, Take It Easy ^.^

3. If you have any questions, please write an email to me or come to my office.

5. Please return the Homework by the Deadline:

Deadline Sep. 21 (next Tuesday), 1:00PM before the class!!

First-order ODEs: Exact Eq.

- Explicit form for a 1st-order ODE: $y' = f(x,y) = -\frac{M(x,y)}{N(x,y)}$.
- Re-write 1st-order ODE:

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0.$$

The necessary and sufficient condition to have an *exact differential equation* is

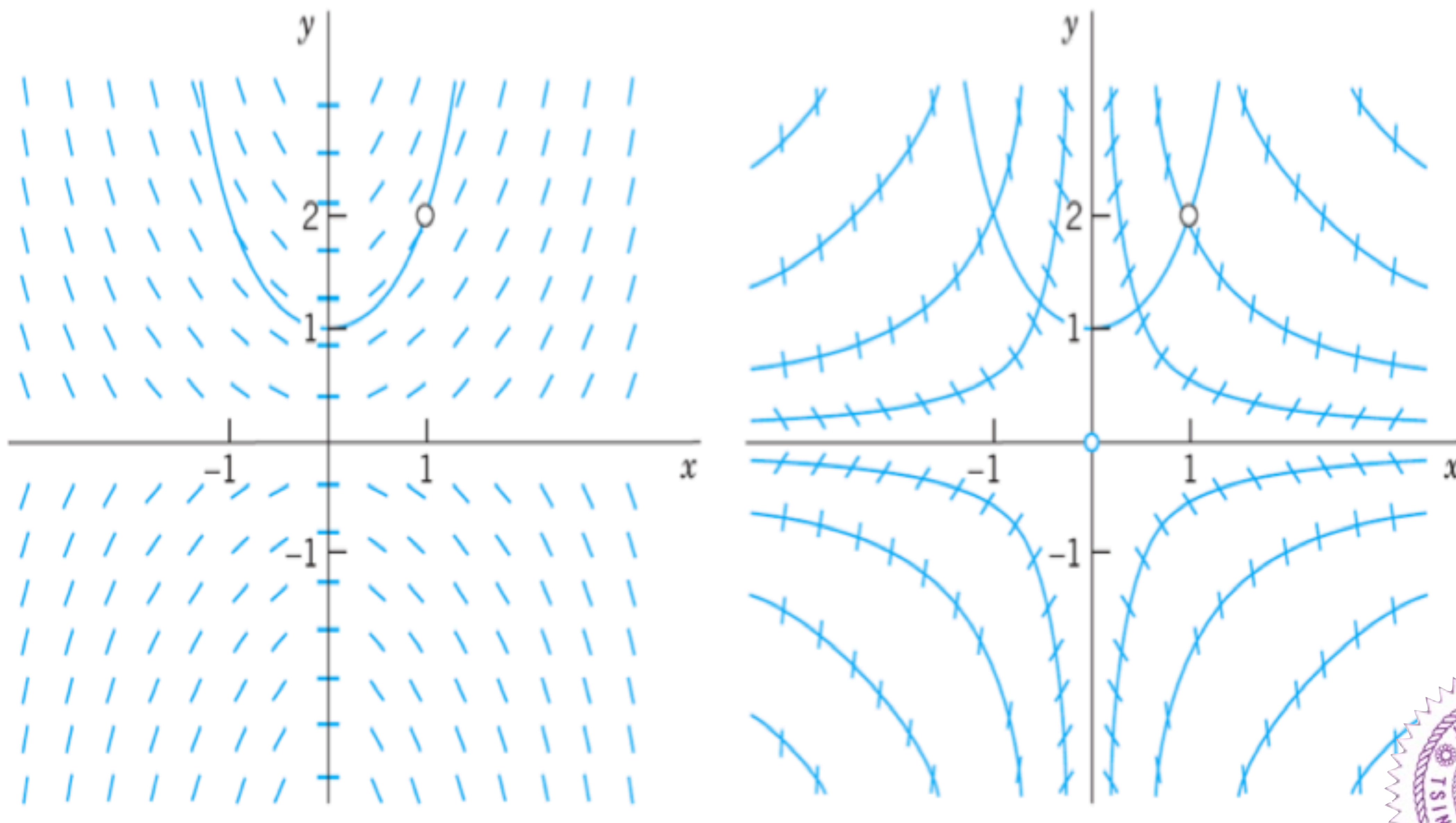
$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}.$$

- If there is a function $u(x,y) = c$, then the total differential of $u(x,y)$ is

$$\mathrm{d}u(x,y) = \frac{\partial u(x,y)}{\partial x}\,\mathrm{d}x + \frac{\partial u(x,y)}{\partial y}\,\mathrm{d}y = 0.$$

Geometric meaning: Direction Field

$$y' = x\,y$$



(a) By a CAS (b) By isoclines

First-order ODEs: Exact Equation

- One can rewrite $y' = f(x,y)$ as

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0.$$

- If there is a function $u(x,y) = c$, then the total differential of $u(x,y)$ is

$$\mathrm{d}u(x,y) = \frac{\partial u(x,y)}{\partial x}\,\mathrm{d}x + \frac{\partial u(x,y)}{\partial y}\,\mathrm{d}y = 0$$

- The necessary and sufficient condition to have an *exact differential equation* is

$$\frac{\partial^2 u(x,y)}{\partial x \partial y} = \boxed{\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}}.$$

- Integrate $M(x,y)$ or $N(x,y)$ to get,

$$u(x,y) = \int M(x,y)\,\mathrm{d}x + k(y), \qquad \text{or}$$
$$u(x,y) = \int N(x,y)\,\mathrm{d}y + h(x).$$

First-order ODEs: Exact Equation, Example

Example:

$$\cos(x+y)\,dx + [3y^2 + 2y + \cos(x+y)]\,dy = 0$$

Hints:

1. Test for exactness.
2. Integrate dx then dy, or Integrate dy then dx.

Solution:

$$u(x,y) = \sin(x+y) + y^3 + y^2 = c$$

First-order ODEs: Non-Exact Equations

- For the explicit form,

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0,$$

- If the test for exactness fails, i.e.,

$$\frac{\partial M(x,y)}{\partial y} \neq \frac{\partial N(x,y)}{\partial x}.$$

QUIZ: How to solve a non-exact equation?

First-order ODEs: Integrating Factor

- Multiply a given non-exact equation by a function $F(x,y)$, *an integrating factor*,

$$F(x,y)M(x,y)\,\mathrm{d}x + F(x,y)N(x,y)\,\mathrm{d}y = 0,$$

- To result in a exact equation:

QUIZ: Is there always an Integrating Factor to find?

- We can choose

$$F(x) = \exp\int\left[\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\right]\mathrm{d}x, \qquad \text{or}$$

$$F(y) = \exp\int\left[\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\right]\mathrm{d}y, \qquad \text{or}$$

$F(x,y)$.

First-order ODEs: Integrating Factor

Example: $(e^{x+y} + ye^y)\,\mathrm{d}x + (xe^y - 1)\,\mathrm{d}y = 0, \quad \text{with} \quad y(0) = -1$

Hints:

- Find the Integrating factor,

$$F(x) = \exp \int [\frac{1}{xe^y - 1}(e^{x+y} + e^y + ye^y - e^y)]\,\mathrm{d}x,$$ fails

$$F(y) = \exp \int [-1]\,\mathrm{d}y = e^{-y}.$$

Solution: $u(x,y) = e^x + xy + e^{-y} = 1 + e.$

First-order ODEs: Linear ODEs

- Linear ODE:

$$y' + p(x)y = 0, \qquad \text{homogeneous}$$
$$y' + p(x)y = r(x), \qquad \text{non-homogeneous}$$

- $y(x) = 0$ is the *trivial solution* for the homogeneous ODEs.

- Non-linear ODE:

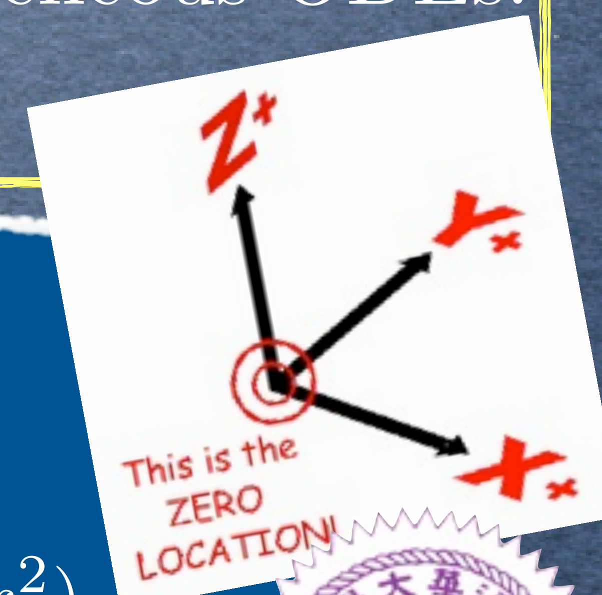
QUIZ: Which one is a linear ODE?

☒ $y' + 3x^2y = 0,$

☒ $y' + 3x^2y = 5\cos(x^2),$

☐ $y'^2 + 3x^2y = 5\cos(x^2),$

☐ $y' + 3x^2\sin(y) = 5\cos(x^2).$



QUIZ: Why Linear Systems are so important ?

1. Basis.

2. Vector space.

3. Matrix.

4. Superposition principle.

- Linear Algebra.

- Signals and Systems.

QUIZ: Does superposition principle apply to?

1. Homogeneous Linear ODEs?

2. Non-homogeneous Linear ODEs?

3. Non-linear ODEs?

$$1 + 1 \neq 2$$

First-order ODEs: Nonhomogeneous & Linear

- Non-homogeneous Linear ODE:

$$y' + p(x)y = r(x),$$

- The *general solution* of the homogeneous ODE is

$$y(x) = ce^{-\int p(x)\,\mathrm{d}\,x} \equiv ce^{-h(x)},$$

- The solution for the non-homogeneous ODE is

$$\begin{aligned} y(x) &= \boxed{e^{-h(x)} \int e^{h(x_1)} r(x_1)\,\mathrm{d}x_1} + \boxed{ce^{-h(x)}}, \\ &= \boxed{\text{non-homogeneous solution}} + \boxed{\text{homogeneous solution}} \end{aligned}$$

Total Output = Response to the Input + Response to the Initial Data .

First-order ODEs: Hormone Level Problem

- Let $y(t)$ be the hormone level at time t.
- The removal rate is $Ky(t)$.
- The input rate is $A + B\cos(2\pi t/24)$, where A is the average input rate and B is the amplitude of a sinusoidal input with a 24-hour period.
- Modeling:

$$y' = -Ky + A + B\cos(\frac{1}{12}\pi t),$$

- The initial condition for a particular solution is given by $y(t=0) = y_0$.

Hormone Level Problem, cont.

- Modeling:

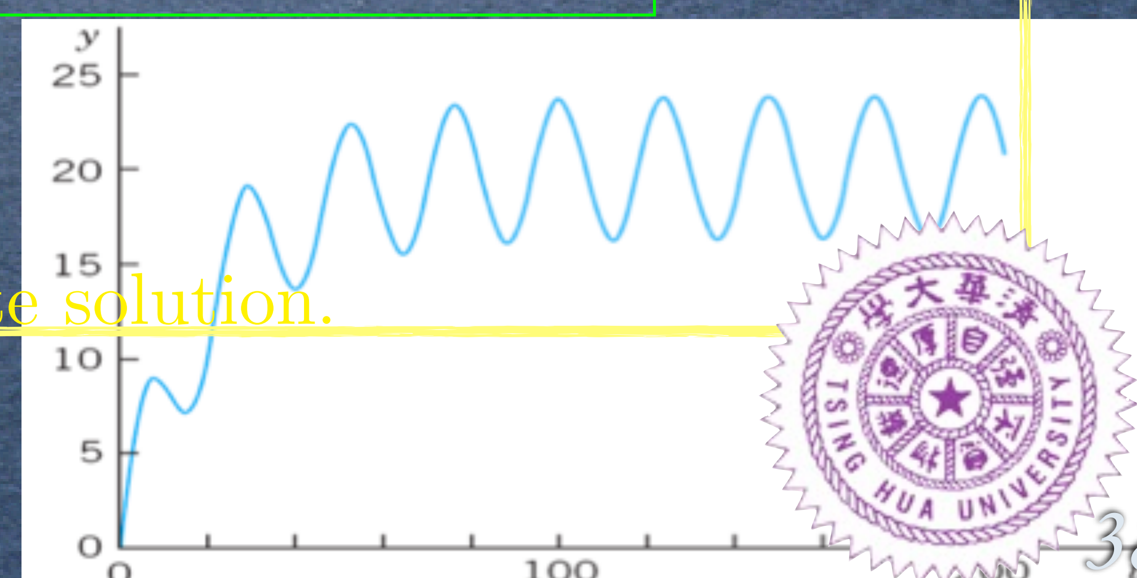
$$y' = -Ky + A + B\cos(\frac{1}{12}\pi t),$$

- Solving:

$$y(t) = e^{-Kt}\int e^{Kt_1}[A + B\cos(\frac{\pi t_1}{12})]\,\mathrm{d}t_1 + ce^{-Kt},$$

$$= \frac{A}{K} + \frac{B}{144K^2+\pi^2}[144K\cos\frac{\pi t}{12} + 12\pi\sin\frac{\pi t}{12}] + ce^{-Kt}.$$

- Steady-State solution.
- Entire solution is called Transient-State solution.



Terminologies:

ODE	PDE	number of independent variable
Homogeneous	Non-homogeneous	RHS
General solution	Particular solution	Special solution
Linear Eq.	Nonlinear	Superposition
Exact Eq.	Non-exact Eq.	Exactness
Steady-state	Transient-state	$t \to \infty$

Homework #2:

1. (25%) Solve the non-exact Eq.:

$$(3xy + y^2) + (x^2 + xy)y' = 0. \tag{1}$$

2. (25%) Solve the non-homogeneous Eq.:

$$x^3 y' + 3x^2 y = 5\sinh(10x), \tag{2}$$

   Problem 17 in the [Textbook], at p.p. 32.

3. (25%) Solve the Bernoulli's Eq.:

$$2y\,y' + y^2 \sin x = \sin x, \qquad y(0) = \sqrt{2}, \tag{3}$$

   Problem 23 in the [Textbook], at p.p. 33.

4. (25%) Solve a 1st-order ODE by using Richard's method of iteration:

$$y' = y - 1, \qquad y(0) = 2,$$

Homework #2: Richard's method

1. To solve an initial-value problem:

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

2. Integrate both sides with respect to x directly, with the initial value, i.e.

$$y_1(x) = y_0 + \int_{x_0}^{x} f(x, y_0)\, \mathrm{d}x,$$

3. Integrate both sides with respect to $x$directly again, but updating the value of $y(x)$

$$y_2(x) = y_0 + \int_{x_0}^{x} f(x, y_1)\, \mathrm{d}x,$$

4. Then you can find a sequence of unctions:

$$y_1(x), y_2(x), \cdots, y_n(x).$$

5. To the limit of $y_n(x)$ as $n \to \infty$, we have the exact solution for the give initial-value problem,

$$y(x) = 1 + e^x.$$

This approach is called the Picard's method of iteration.

Homework #2:

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Deadline Sep. 23 (Thursday), 1:00PM before the class!!

First-order ODEs: Bernoulli Equation

- Some nonlinear ODEs can be transformed to linear ODEs.
- Bernoulli equation:

$$y' + p(x)y = g(x)y^a, \quad a \text{ is a real number.}$$

- If $a = 0$ or $a = 1$, it is linear; otherwise, it is nonlinear.
- *Hint:*

$$u(x) = [y(x)]^{1-a},$$

- The transformed ODE for $u(x)$ is linear,

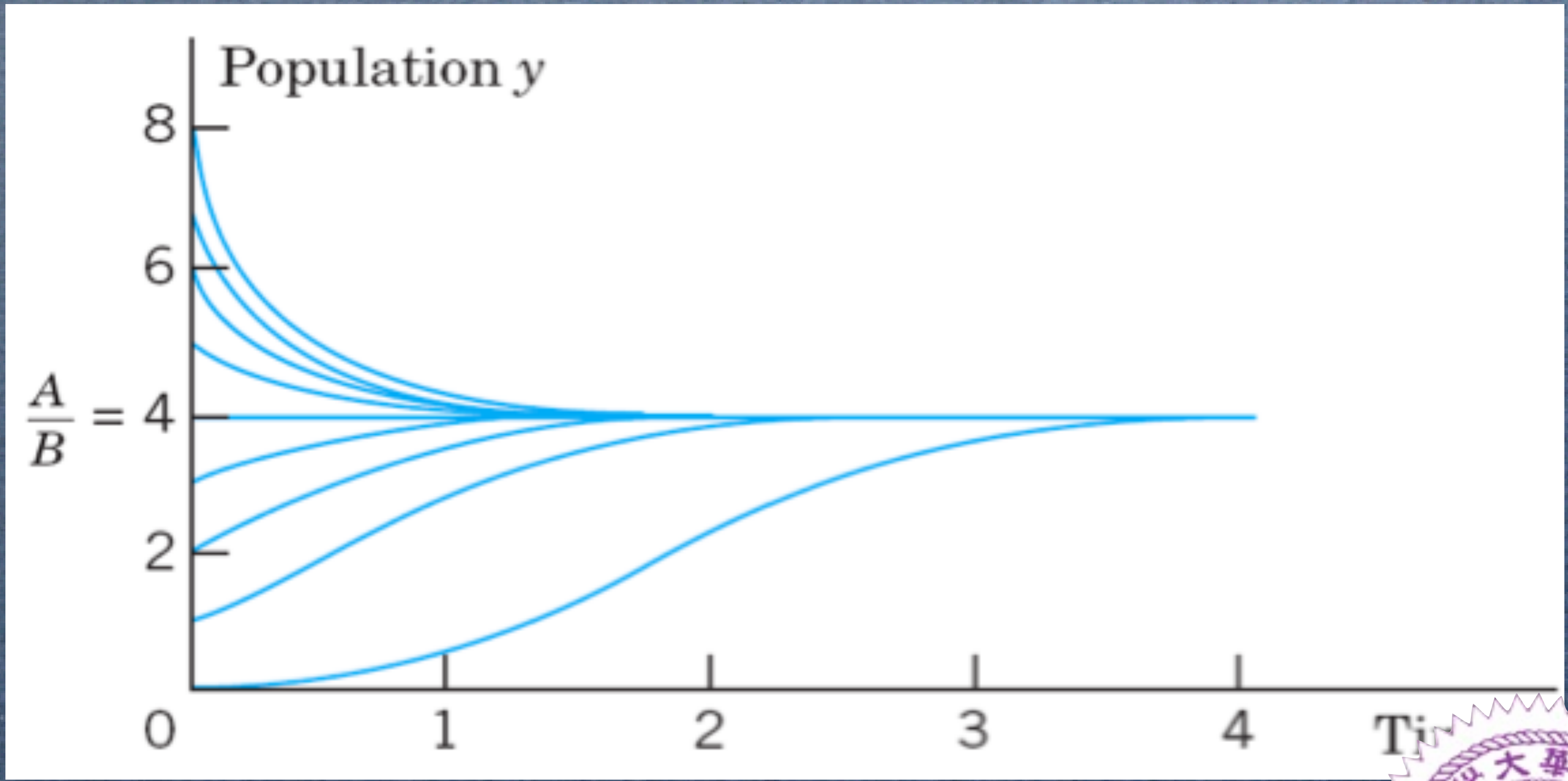
$$u' + (1-a)p(x)u(x) = (1-a)g(x).$$

First-order ODEs: Bernoulli Equation, cont.

Example: Logistic Equation (Verhulst Equation)

$$y' = Ay - By^2,$$

Hints:



Solution:

$$y(t) = \frac{1}{ce^{-At} + \frac{B}{A}}$$

First-order ODEs: Existence Theorem

- For *linear* 1st-order ODEs in the **initial value problem**,

$$y' = f(x, y), \qquad y(x_0) = y_0,$$

- $f(x, y)$ is continuous and bounded at alo pints (x, y) in some rectangle,

$$R : |x - x_0| < a, \quad |y - y_0| < b$$

- there is a number K such that,

$$|f(x, y)| \leq K, \qquad \text{for all } (x, y) \text{ in } R$$

- Then the **initial value problem** has at least one solution $y(x)$ in the sub-interval $|x - x_0| < a(b/K)$.

First-order ODEs: Uniqueness Theorem

- For *linear* 1st-order ODEs in the **initial value problem**,

$$y' = f(x, y), \qquad y(x_0) = y_0,$$

- Let $f(x, y)$ and its partial derivative $f_y = \partial f / \partial y$ is continuous and bounded at alo pints (x, y) in some rectangle,

$$|f(x, y)| \le K,$$
$$|f_y(x, y) \le M, \qquad \text{for all } (x, y) \text{ in } R$$

- Then the **initial value problem, IVP** has at most one solution $y(x)$.
- Combine the Existence and Uniqueness theorems, the IVP has precisely one solution in the sub-interval $|x - x_0| < \alpha$.

Existence and Uniqueness Theorem, Example 1

Example:

$$y' = x - y + 1, \qquad y(1) = 2$$

Hints:

Both $f(x,y) = x - y + 1$ and $f_y(x,y) = -1$

are defined and continuous at all points (x,y),

Solution:

The theorem guarantees a *unique* solution to the ODE exists in some open interval centered at 1.

$$y(x) = x + ce^{-x}.$$

Existence and Uniqueness Theorem, Example 2

Example:

$$y' = 1 + y^2, \qquad y(0) = 0$$

Hints:

Both $f(x, y) = 1 + y^2$ and $f_y(x, y) = 2y$

are defined and continuous at all points (x, y).

Solution:

The theorem guarantees a *unique* solution to the ODE exists in some open interval centered at 0.

$$y(x) = \tan(x + c),$$

which is defined for all $x \neq (2n+1)/\pi$, n is an integer.

Existence and Uniqueness Theorem, Example 3

Example:

$$y' = 2y/x, \qquad y(x_0) = y_0$$

Hints:

Both $f(x, y) = 2y/x$ and $f_y(x, y) = 2/x$

are defined and continuous at all points $x \neq 0$.

Solution:

The theorem guarantees a *unique* solution to the ODE exists in some open interval centered at $x_0 \neq 0$.

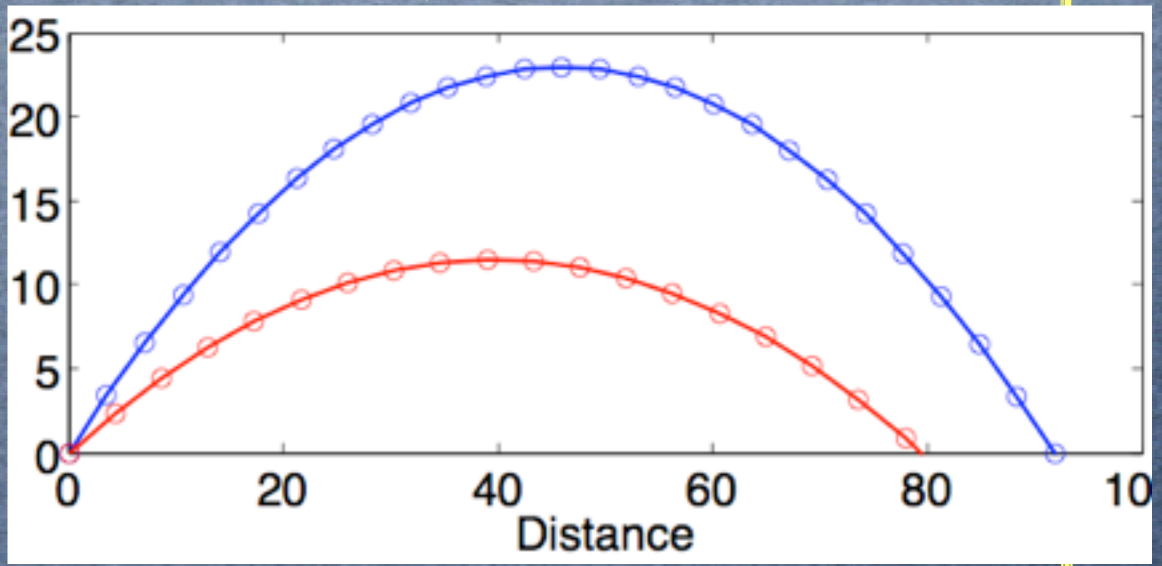
$$y(x) = cx^2,$$

- No solution if $x_0 = 0$ and $y_0 \neq 0$;
- Infinitely many solutions if $x_0 = 0$ and $y_0 = 0$.

Homework #0: Projectile motion without Air Resistance

- Find the analytical solutions for the projectile motion,

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = -g,$$
$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = 0,$$



with the initial velocity $v_x = v_0 \cos\theta$ and $v_y = v_0 \sin\theta$.

- For a constant v_0, find the projectile angle θ that gives the longest projectile distance.

- Based on Finite-Difference method, write a code to test your analytical results.

Finite Difference Approximation

$$\frac{\mathrm{d}}{\mathrm{d}x}y(x)|_{x=x_j} \approx \frac{y(x_j)-y(x_{j-1})}{x_j - x_{j-1}}$$

- Taylor's expansion:

$$\begin{aligned} u(x_{j+1}) &= u(x_j) + u'(x_j)\Delta x + \frac{u''(x_j)}{2}(\Delta x)^2 + \frac{u'''(x_j)}{3!}(\Delta x)^3 + \ldots, \\ u(x_{j-1}) &= u(x_j) - u'(x_j)\Delta x + \frac{u''(x_j)}{2}(\Delta x)^2 - \frac{u'''(x_j)}{3!}(\Delta x)^3 + \ldots, \end{aligned}$$

- Euler's 2nd-order FD approximation:

$$\begin{aligned} u'(x_j) &= \frac{u(x_{j+1})-u(x_{j-1})}{2\Delta x} - \frac{u'''(x_j)}{2*3!}(\Delta x)^2 + \ldots, \\ &\approx \frac{u(x_{j+1})-u(x_{j-1})}{2\Delta x} + \mathbf{O}(\Delta x^2), \end{aligned}$$

- 4th-order FD method:
- Runge-Kutta method:
- Differential matrix:

First-order ODEs: Summary

ODEs

- 1st-order
  - ☒ Modeling, Ch. 1.1
  - ☐ Direction Fields, Ch. 1.2
  - ☒ Separable Eq. Ch. 1.3
  - ☒ Exact Eq. Ch. 1.4
  - ☒ Integrating Factor, Ch. 1.4
  - ☒ Linear ODEs, Ch. 1.5
  - ☒ Non-homogeneous sol., Ch. 1.5
  - ☒ Bernoulli Eq., Ch. 1.5
  - ☐ Orthogonal Trajectory, Ch. 1.6
  - ☒ Existence and Uniqueness, Ch. 1.7
  - ☐ Numeric methods
- 2nd-order
- Higher-order
- Systems of ODEs