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OPTICAL PHYSICS

Quantum phase transition of a finite number of atoms in electromagnetically induced transparency media

R. A. Robles Robles[1] and Ray-Kuang Lee[1,2,3,4,*]

[1]Institute of Photonics Technologies, National Tsing Hua University, Hsinchu 30013, Taiwan
[2]Physics Division, National Center for Theoretical Sciences, Hsinchu 30013, Taiwan
[3]Department of Physics, National Tsing Hua University, Hsinchu 30013, Taiwan
[4]Center for Quantum Technology, Hsinchu 30013, Taiwan
*Corresponding author: rklee@ee.nthu.edu.tw

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Instead of an infinite number of atoms in the thermodynamic limit, we study the ground states of a finite number of three-level atoms in electromagnetically induced transparency media. With the help of a classical control field, critical coupling strengths are derived analytically for the existence of quantum phase transitions in the ground states of this extended Dicke model. Compared to the classical limit, evolution of a finite-sized excitation during the storage and retrieval process is also illustrated, as well as atom–field entanglement. The results derived in this work provide the connection not only to the Dicke model, but also to the Lipkin–Meshkov–Glick model.

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1. INTRODUCTION

As the number of two-level atoms increases, collective effects due to the interactions of atoms among themselves, give rise to intriguing many-body phenomena, known as superradiance phase transition [1–3]. By embedding a single photon stored in a large cloud of atoms, single photon superradiance supported in the Dicke model also provides the directionality of spontaneously emitted photons [4,5]. Instead of studying the ground state phase transition in the thermodynamic limit, $N \to \infty$ with N denoting the number of atoms, the existences of a first-order quantum phase transition (QPT) and a second-order superradiant phase transition in the Dicke model are found with a finite-sized ensemble of atoms [6,7]. Instead of non-zero temperatures, QPTs are singularities that occur only at zero temperature. QPTs have provided a platform for understanding dynamic and thermodynamic properties, in particular at temperatures such that $\hbar\omega_T \ll k_B T$, where ω_T is the typical frequency at which the degrees of freedom fluctuate [8,9]. The QPTs in the ground state of the finite Dicke model have been associated with entanglement between the ensemble and quantum field [10,11] and with bipartite entanglement among qubits due to the effects of finite size [12–15].

With the help of a control field, quantum interference happens for three-level atoms in a Λ-type configuration, known as electromagnetically induced transparency (EIT) [16]. As a promising ingredient for the development of quantum technologies, EIT media have provided a platform to implement quantum memory [17,18], through the experimental realization of storage and retrieval of light both with single photons [19,20] and squeezed states [21,22]. To explore the quantum properties of EIT media, we reveal the condition to preserve quantum entanglement [23], the distortion of quantum noise beyond the adiabatic approximation [24], and the generation of quantum noise squeezing, as well as entanglement [25,26].

As a natural consequence of the quantum optics with three-level atoms, in this paper, we study the ground states of a finite number of three-level atoms in EIT media. Instead of an infinite number of atoms in the thermodynamic limit [27], critical coupling strengths are derived analytically for the existence of QPT in the ground states of this system with the help of a classical control field. Our results for a finite-sized system can be reduced to the known results in the thermodynamic limit for three-level atoms, or to the same scenarios in two-level atoms when the control field is set to zero. As the quantum critical phenomenon has been associated with atom–field entanglement and bipartite atomic entanglement due to the effects of finite size, with the dark state polariton for EIT media, our dynamical results also explore the regions beyond the thermodynamics limit. The main advantage of going beyond the thermodynamic limit is that only with a finite number of atoms does one have the possibility to access a more compact system (in terms of the number

of elements), e.g., to consider a single (or artificial) atom passing through an optical cavity.

2. MODEL

Here, we consider an ensemble of EIT media, which consists of a finite number, N, of three-level atoms in the Λ-type configuration (see Fig. 1). EIT media interact with two electromagnetic fields: one is a strong control field, denoted by its Rabi frequency Ω, and the other a quantum probe field, with its frequency ω_f corresponding to a quantized cavity mode. To derive analytically the critical coupling strengths for QPT, instead of introducing collective momentum operators, we take advantage of Schwinger representation to write the corresponding light–atom Hamiltonian in the following, up to a constant:

$$H = \hbar\omega_f \hat{a}_f^\dagger \hat{a}_f + \hbar\epsilon_b \left(\hat{a}_b^\dagger \hat{a}_b - N\right) + \hbar\epsilon_a \hat{a}_a^\dagger \hat{a}_a + \hbar\epsilon_c \hat{a}_c^\dagger \hat{a}_c + \left[\hbar g \hat{a}_f \hat{a}_a^\dagger \hat{a}_b + \hbar\Omega(t) e^{-i\nu t} \hat{a}_a^\dagger \hat{a}_c + \text{h.c}\right], \quad (1)$$

where $\hat{a}_f^\dagger$ and $\hat{a}_f$ are the usual creation and annihilation operators for the probe field, respectively, and h.c represents a Hermitian conjugate. For atomic parts, ϵ_i $(i = a, b, c)$ denotes the corresponding energy, while $\hat{a}_i^\dagger$ and $\hat{a}_i$ are the corresponding operators for Schwinger bosons in each atomic level denoted by $\sum_{i=1}^{N} \hat{\sigma}_{\eta\nu}^i = \hat{a}_\eta^\dagger \hat{a}_\nu$. The coupling strength between the probe field and atomic transition $|b\rangle \leftrightarrow |a\rangle$ is defined by g, while the transition $|a\rangle \leftrightarrow |c\rangle$ is driven by a time-dependent Rabi frequency $\Omega(t)$ at the angular frequency ν. The convenience of this representation is that the commutation relations of the collective momentum operators are automatically satisfied.

As the number of excitations is conserved in this three-level system, i.e., $\Lambda \equiv \langle\hat{\Lambda}\rangle = \langle\hat{a}_f^\dagger \hat{a}_f + \hat{a}_a^\dagger \hat{a}_a + \hat{a}_c^\dagger \hat{a}_c\rangle$, one can divide the Hilbert space into subspaces with a defined value of Λ. For $\Lambda < N$, the dimension of each subspace is $\dim = \frac{(\Lambda+1)(\Lambda+2)}{2}$. The subspace with no excitation is $|0\rangle_f |0\rangle_a |0\rangle_c |N\rangle_b$, and this state has zero energy. The first non-trivial subspace has an equivalent Hamiltonian in the following form:

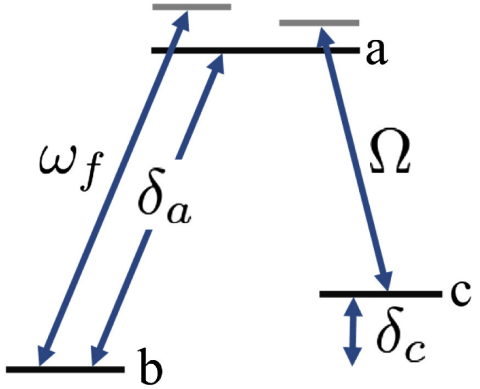

Fig. 1. Schematic of the EIT system considered in a Λ-type configuration, where the allowable transitions $|b\rangle \leftrightarrow |a\rangle$ and $|a\rangle \leftrightarrow |c\rangle$ are driven by a quantized probe field, denoted by its frequency ω_f and a classical control field denoted by its Rabi frequency Ω with the frequency detunings with respect to the ground state $|b\rangle$ marked by δ_a and δ_c, respectively.

$$H_{(1)} = \hbar \begin{pmatrix} \omega_f & g\sqrt{N} & 0 \\ g\sqrt{N} & \delta_a & e^{i\nu t}\Omega \\ 0 & e^{-i\nu t}\Omega & \delta_c \end{pmatrix}, \quad (2)$$

where $\delta_a \equiv \epsilon_a - \epsilon_b$ and $\delta_c \equiv \epsilon_c - \epsilon_b$ are the energy differences of levels $|a\rangle$ and $|c\rangle$ with respect to level $|b\rangle$, respectively. The Hamiltonian given in Eq. (2) acts on the states

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |1\rangle_f |0\rangle_a |0\rangle_c |N\rangle_b; \quad (3)$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = |0\rangle_f |1\rangle_a |0\rangle_c |N-1\rangle_b; \quad (4)$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = |0\rangle_f |0\rangle_a |1\rangle_c |N-1\rangle_b. \quad (5)$$

The eigenvalues for Hamiltonian $H_{(1)}$ can be solved analytically, as it implies finding the roots of a cubic equation. The resulting eigenvalues can have negative values, which means that the critical value exists when the energies of the two subspaces are equal. This transition is referred as QPT. By equating the eigenvalues of two subspaces, the critical coupling coefficient, g, has the solution

$$g = \frac{\sqrt{\omega_f \delta_a \delta_c - \Omega^2 \omega_f}}{\sqrt{N\delta_c}}. \quad (6)$$

Since the eigenvalue for the unexcited state is zero, this expression, Eq. (6), can be found by setting the independent coefficient of the characteristic equation of $H_{(1)}$ equal to zero. For small coupling strengths, the ground state is the normal phase, with $\Lambda = 0$, while for larger coupling strengths, the ground state is in the so called superradiant phases, with $\Lambda > 0$.

When the Rabi frequency of the control field Ω or the coupling strength g increases, a series of QPTs occurs, as shown in Fig. 2. Numerical solutions for the number of photons, number of atoms in states $|a\rangle$ and $|c\rangle$, and number of excitations in the ground state are shown in Figs. 2(a)–2(d), respectively. Here, the number of photons is obtained by calculating $\langle\hat{a}_f^\dagger \hat{a}_f\rangle$ and the number of atoms by $\langle\hat{a}_i^\dagger \hat{a}_i\rangle$, with $i = a, c$ for the corresponding atomic state. As one can see, each phase corresponds to an integer number of excitation. In the same plot, we also depict our analytical formula given in Eq. (6), which perfectly describes the first critical value for QPT (white curve). Explicitly, we have the ground state in the normal phase if the coupling coefficient g is smaller than the critical value when $\Omega < \sqrt{\delta_a \delta_c}$, while the superradiant phase is always supported when $\Omega > \sqrt{\delta_a \delta_c}$. In addition to QPT in ground states, in Fig. 2, borders for QPT in excited states can also be clearly recognized in our numerical solutions. It is worth further studying the analytical formula for these QPTs in excited states.

To gain insight into the series of QPTs, a study of particular regimes is required. Specifically, regimes with one of the coupling coefficients equal to zero can give us the number of transitions on the borders of Fig. 2. On one hand, in the limit $\Omega = 0$, our result can be reduced to the known critical

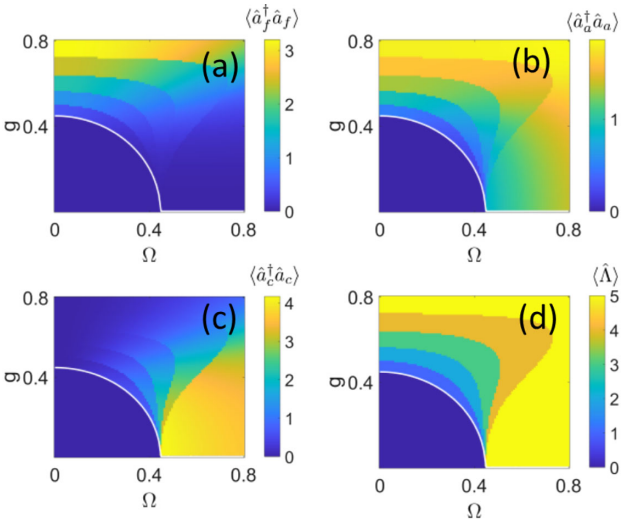

Fig. 2. Numerical solutions for (a) number of photons, (b) number of atoms in state $|a\rangle$, (c) number of atoms in state $|c\rangle$, and (d) number of excitations Λ in the ground state as a function of the Rabi frequency Ω and coupling strength g. A series of QPTs is revealed in which each phase corresponds to an integer number of excitation. The white curve is our analytical formula given in Eq. (6), which exactly matches the first QPT. Here, the other parameters used are $\omega_f = 1$, $\delta_a = 1$, $\delta_c = 0.2$, $t = 0$, and $N = 5$.

strength for QPT in the Dicke model [6,7]. Furthermore, since the $|c\rangle$ state does not appear in this limit, we expect that there will be zero population on such a state. The resulting Hamiltonian is equivalent to a Tavis–Cummings Hamiltonian [2], i.e., an ensemble of effective two-level atoms interacting with a quantum field. Now, the conservation of the number of excitations is still valid, and the eigenstates will exist in a subspace with defined Λ. Therefore, a series of N QPTs is expected, as observed on the vertical axes in Fig. 2. For larger g, the ground state will tend to have more excitations up to $\Lambda \leq N$. For this reason, in the limit $\Omega = 0$, we observe N phase transitions.

Alternatively, if $g = 0$, the ground state has no photons, as they no longer interact with the rest of the system. Additionally there is no interaction from the $|b\rangle$ state, so this state has to be completely populated or completely depleted in order to have a minimal energy state. In this limit, the system is equivalent to a Lipkin–Meshkov–Glick (LMG) Hamiltonian [28–30] because only the atomic part contributes to the energy. It can be shown that the state with the least energy of all the subspaces crosses zero at the critical point $\Omega = \sqrt{\delta_a \delta_c}$. In this limit, the ground state energies of all subspaces cross the zero-energy at the same point; therefore, a singular QPT can be observed on the horizontal axes in Fig. 2. Now, the atoms in the ground state $|b\rangle$ do not interact explicitly but the population in $|b\rangle$ will affect the populations in $|a\rangle$ and $|c\rangle$. The minimum energy states in this limit must have all atoms in $|b\rangle$ with $\Lambda = 0$ or none with $\Lambda = N$. So, for $g = 0$, only two phases exist. On the other hand, in the thermodynamical limit, our result is also consistent with the one known for a large number of atoms [27].

3. DARK STATE POLARITON

In addition to a constant coupling field, a time-dependent Rabi frequency $\Omega(t)$ also supports the dark state polaritons [17]. In the thermodynamic limit, this dark state polariton has eigenstates with zero population in the unstable excited atomic level, which evolves coherently between the probe field and ground state excitation as a key ingredient for quantum state transfer. However, for an arbitrary number of atoms in EIT media, the validation for such a coherent transfer based on dark state polaritons is questionable.

To go beyond the thermodynamic approximation, for a finite number of atoms, one can look for the dark state solution as an eigenstate for the number of excitations Λ, i.e., the conserved quantity in our system. That is,

$$\hat{\Lambda}|\Psi_D^{(\Lambda)}\rangle_I = \Lambda|\Psi_D^{(\Lambda)}\rangle_I, \tag{7}$$

with a non-negative integer Λ. Here, the dark state is denoted as $|\Psi_D^{(\Lambda)}\rangle_I$, with the subindex (I) standing for the interaction picture. The corresponding eigenenergy in the interaction picture equals zero, i.e.,

$$H_I|\Psi_D^{(\Lambda)}\rangle_I = 0. \tag{8}$$

Here, the interaction Hamiltonian is derived by means of a unitary transformation. The interaction Hamiltonian is

$$H_I = \hbar \left[g e^{-i(\omega_f + \epsilon_b - \epsilon_a)t} \hat{a}_f \hat{a}_a^\dagger \hat{a}_b + \Omega(t) e^{-i(\nu - \epsilon_a + \epsilon_c)t} \hat{a}_a^\dagger \hat{a}_c + \text{h.c} \right]. \tag{9}$$

In order to remove the time dependence in the interaction Hamiltonian, apart from $\Omega(t)$, the Raman resonance condition must be satisfied on the two branches of our Λ-type configuration, i.e., $\omega_f = \epsilon_a - \epsilon_b$ and $\nu = \epsilon_a - \epsilon_c$, as can be deduced from Eq. (9). Even though this Raman condition is not fulfilled, we can still search for the eigenstates at any time instance. Furthermore, as the dark states have no population in level $|a\rangle$, one can write the solution in the following form:

$$|\Psi_D^{(\Lambda)}\rangle_I = \sum_{k=0}^{\Lambda} c_k |k\rangle_f |0\rangle_a |\Lambda - k\rangle_c |N - \Lambda + k\rangle_b. \tag{10}$$

By substituting the ansatz given in Eq. (10) into Eq. (8), one can obtain the corresponding coefficient:

$$c_k = \left(\frac{\Omega e^{i(\delta_2 - \delta_1)t}}{g} \right)^k \sqrt{\frac{\Lambda!(N-\Lambda)!}{k!(\Lambda - k)!(N - \Lambda + k)!}} c_0, \tag{11}$$

with the short-handed notations $\delta_1 \equiv -\omega_f - \epsilon_b + \epsilon_a$ and $\delta_2 \equiv -\nu - \epsilon_c + \epsilon_a$, which reflect the detuning between the transition energies and the corresponding fields, respectively. If $\delta_1 = \delta_2$, the time dependence can be eliminated. This result is comparable to the one in [31], where the atomic ensemble is used to generate deterministic Fock states, being the number of photons equal to the number of atoms. This is achieved with the atomic system initially in the ground state; afterwards, the pump power is increased, effectively transferring the atoms to the $|c\rangle$ state and generating N photons in the cavity. However,

the storage and retrieval process is in a sense opposite, since one wants to transfer quantum photon states into the atomic system. For this reason, it is important to highlight that expressions given in Eqs. (10) and (11) also allow for excitation numbers smaller than the number of atoms; this fact implies that any state in the field can be transferred to the atomic system and vice versa, with a limitation on the Hilbert space of the field equal to or smaller than the number of atoms.

It is noted that in Eq. (11), we do not make any approximation regarding the number of atoms, so our solution itself is an expression for the dark state polariton for any arbitrary number of atoms. As an illustration, we give two notable situations to simplify the coefficients. First, for one excitation, $\Lambda = 1$, we can reduce Eq. (11) into two non-zero coefficients, c_0 and c_1:

$$c_0 = \frac{1}{\sqrt{\left(\frac{\Omega}{g\sqrt{N}}\right)^2 + 1}}, \tag{12}$$

$$c_1 = \frac{\Omega \exp[-i(\delta_2 - \delta_1)t]}{g\sqrt{N}\sqrt{\left(\frac{\Omega}{g\sqrt{N}}\right)^2 + 1}} \tag{13}$$

for any arbitrary number of atoms N. Second, for a large number of atoms $N \gg \Lambda$, we note that

$$\frac{(N-\Lambda)!}{(N-\Lambda+k)!} \approx \frac{1}{N^k},$$

and then the normalization condition arrives at

$$|c_0| \approx \left[\left(\frac{\Omega}{g\sqrt{N}}\right)^2 + 1\right]^{-\Lambda/2},$$

$$c_k \approx \left[\left(\frac{\Omega}{g\sqrt{N}}\right)^2 + 1\right]^{-\Lambda/2} \left[\frac{\Omega \exp\{i(\delta_2 - \delta_1)t\}}{g\sqrt{N}}\right]^k \sqrt{\binom{\Lambda}{k}}. \tag{14}$$

The last expression connects the two regimes that are already explored: single excitation and the thermodynamic limit. Note here that the thermodynamic approximation can be obtained by cascading from a finite number of N.

To reveal the capacity to store photons in EIT media with a finite number of atoms, we calculate the expectation values for the number of photons and the population as a function of the coupling Rabi frequency $\Omega(t)$:

$$\langle \hat{a}_c^\dagger \hat{a}_c \rangle \approx \frac{\Lambda g^2 N}{g^2 N + \Omega^2}, \tag{15}$$

$$\langle \hat{a}_f^\dagger \hat{a}_f \rangle \approx \frac{\Lambda \Omega^2}{g^2 N + \Omega^2}. \tag{16}$$

With a comparison to known results obtained by using the thermodynamic limit, Eqs. (15) and (16) also give the same relation proportional to the number of excitations in the system. These two values are bounded between zero and Λ for any Rabi frequency, $\Omega > 0$. Then, with an adiabatic change in the Rabi frequency, $\Omega(t)$, one can drive the excitations in the quantum field into the atoms.

In analogy to the storage and retrieval process for the dark state polariton, in Fig. 3, we show the evolution of expectation values in our Hamiltonian, Eq. (1), with $N = 4$ atoms, all of them initially in the ground state along with two photons ($\Lambda = 2$) as the initial state. Here, the adiabatic evolution is calculated by applying a time-dependent Rabi frequency and restoring it to its initial value, i.e.,

$$\theta(t) = \cos^{-1}\left[\Omega(t)/\sqrt{g^2 N + \Omega(t)^2}\right]. \tag{17}$$

For dark state polaritons, as one can see from the time evolution of the number of photons and atoms in the atomic level $|c\rangle$, in Figs. 3(a) and 3(b), respectively, even with a finite number of atoms, the excitation in the field, $\langle \hat{a}_f^\dagger \hat{a}_f \rangle$, is transferred to the atomic system as that in the thermodynamical limit. In analogy, for an initial state equal to the ground state but with different excitation levels $\Lambda = 1$, 2, and 3, again, the corresponding time evolution of the number of photons, as shown in Fig. 3(c), also illustrates the storage and retrieval process. Nevertheless, the number of atoms in atomic level $|c\rangle$ increases as the number of excitations increases, but never approaches the initial excitation number ($\Lambda = 2$) [see Fig. 3(d)]. Moreover, due to the non-zero value of θ and non-perfectly adiabatic change in Ω, some oscillations occur, too. Note that the initial state applied here belongs to the non-trivial phases in the phase diagram shown in Fig. 2 for $g = 0.45$, 0.5, and 0.55, respectively, and $\Omega = 0.1722$ for the three cases given in the plot.

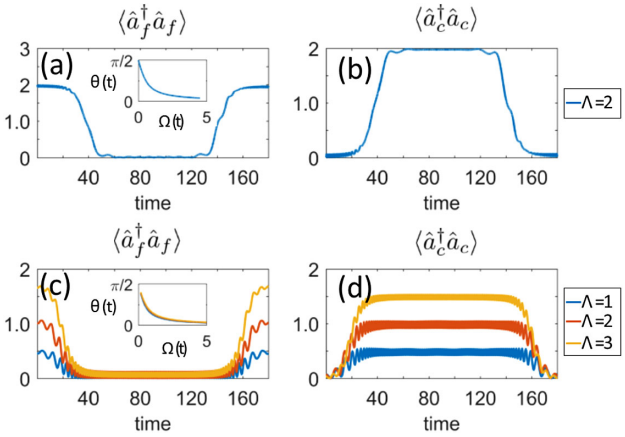

Fig. 3. (a), (b) Storage and retrieval process for the dark state polariton with a finite number ($N = 4$) of atoms, by changing only the control field adiabatically. (a) Evolution of the photon number and (b) population in the atomic level $|c\rangle$ are shown with an initial state equal to two photons. (c), (d) In analogy, storage and retrieval process for an initial state equal to the ground state at different parameter values. (c) Time evolution of number of photons and (d) number of atoms in the atomic level $|c\rangle$ are shown by changing slowly the control field Ω. Note that the initial state belongs to the non-trivial phases in the phase diagram shown in Fig. 2, with the excitation levels $\Lambda = 1$, 2, and 3 for $g = 0.45$, 0.5, and 0.55, respectively, and $\Omega = 0.1722$ for the three cases given in the plot. Insets in (a) and (c) show the corresponding $\theta(t)$, given by Eq. (17).

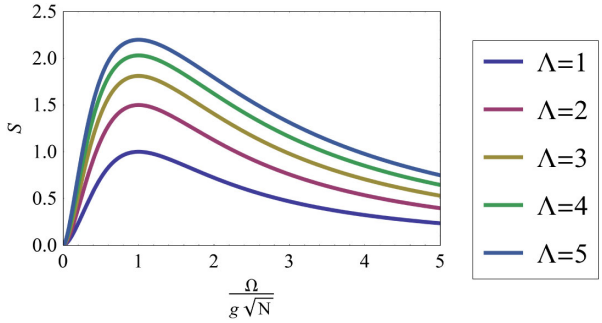

Fig. 4. Entanglement entropy of dark state polaritons, calculated based on Eq. (18), as a function of $\Omega/g\sqrt{N}$ for different excitation numbers.

4. ATOM–FIELD ENTANGLEMENT

With the analytical solutions to a dark state polariton with a finite size of atoms, we can also study the entanglement between atomic ensemble and quantum field, by calculating the corresponding von Neumann entropy of the reduced density matrix of the quantum field [10–15], i.e., $S=-\mathrm{Tr}_f[\rho_f \log_2(\rho_f)]$, with $\rho_f=\mathrm{Tr}_{a,b,c}[|\Psi_D^{(\Lambda)}\rangle_{II}\langle|\Psi_D^{(\Lambda)}|]$. By using Eq. (10), we have $S=-\sum_{k=0}^{\Lambda}|c_k|^2\log_2|c_k|^2$, which gives us the closed form in the limit of a large number of atoms:

$$S=\Lambda\log_2\left\{\left(\frac{\Omega}{g\sqrt{N}}\right)^2+1\right\}-\frac{2\Lambda\left(\frac{\Omega}{g\sqrt{N}}\right)^2\log_2\left\{\frac{\Omega}{g\sqrt{N}}\right\}}{\left(\frac{\Omega}{g\sqrt{N}}\right)^2+1}$$
$$-\frac{\sum_{k=1}^{\Lambda-1}\left(\frac{\Omega}{g\sqrt{N}}\right)^{2k}\binom{\Lambda}{k}\log_2\left\{\binom{\Lambda}{k}\right\}}{\left[\left(\frac{\Omega}{g\sqrt{N}}\right)^2+1\right]^{\Lambda}}. \tag{18}$$

Based on Eq. (18), in Fig. 4 we plot the entanglement entropy of dark state polaritons for different excitations. Here, the maximum value of von Neumann entropy occurs at $\Omega/g\sqrt{N}=1$; at this point, equal populations in the photons and atomic ensemble exist, or equivalently, $\theta=\pi/4$ in Eq. (17). Also, at $\Omega/g\sqrt{N}=0$, the entropy is zero due to the lack of entanglement, and at this point, the excitations are present only in the atoms. In the case of large control field, $\Omega/g\sqrt{N}\to\infty$, the entropy vanishes. In this limit, the excitation lies only in the field. Also, the entanglement entropy increases when Λ increases.

Before the conclusion, we want to remark that such a set of finite numbers of three-level atoms has also been studied to form a dark state polariton, in particular for the generation of deterministic optical Fock states [31]. However, our proposed application is in a totally opposite direction. We studied an arbitrary quantum state in the field, which can be stored in the finite atomic system when the Hilbert space of such a system is smaller than or equal to the number of atoms. In contrast to only a single number state generated, we not only derived the critical value for QPT, but also used the time evolution for the storage and retrieval of a number state. Moreover, our result shows that quantum storage and retrieval do not require an infinite (very large) number of atoms as long as the Hilbert space is smaller than the number of atoms. A simple explanation for this similarity between a finite and infinite number of atoms comes from the dark state in such a Λ-type configuration.

Additionally, access to finite ensembles is nowadays available. For example, schemes for probing dynamics of superradiant QPT have been proposed in trapped ions [32–34]. Here, by means of a single trapped ion that interacts with one of its vibrational modes, one can probe the Rabi model and its dynamics [32,33], as well as the realization of a finite-sized Jaynes–Cummings lattice in trapped ions [34]. A general extension of these results is to consider three-level systems. On the other hand, recent experiments have demonstrated the observations of EIT signatures in a superconducting quantum circuit [35], or through the interaction with two microwave fields [36]. Definitely, the implementation with a single artificial atom is naturally extended to an arbitrary number of atoms.

Physical systems will be limited by decoherence of the atomic ensemble. For example, in circuit quantum electrodynamics systems, the reported coupling strengths have $g/2\pi=1.45$ GHz and $\Omega/2\pi=62$ kHz with field frequencies at resonance conditions $\omega_f/2\pi=5.64$ GHz [35]. As for the possible decoherence channels, the excited energy level, $|a\rangle$, has decay rates $\gamma_{a,b}/2\pi=0.35$ MHz and $\gamma_{a,c}/2\pi=0.47$ MHz. As a comparison, these two parameters should not have great effects on the efficiency of the quantum memory, as this state is not populated in dark state polariton. The main limitation will arise from the decay of the meta-stable state to the ground state, i.e., $\gamma_{c,b}/2\pi=2.74$ KHz, which is much larger than the characteristic frequencies of the system [35]. It is this small decay rate that imposes the limitation in the storage time. As a consequence, our studies on the implementation of quantum memory using a small number of elements are feasible. Taking advantage of QPT can be a tool to aid the preparation of states in such devices with a finite number of atoms.

5. CONCLUSION

In conclusion, beyond the thermodynamical limit, we derive the critical coupling strength for the first QPT occurring in the ground state for a finite size of atoms by dividing the Hilbert space for an EIT Hamiltonian. When the coupling strength is larger than this critical value, non-trivial ground states are supported. In addition to our analytical solution, we also numerically reveal a series of QPTs for stronger interactions, which possesses a defined excitation number. Moreover, we give the analytic solutions for dark state polaritons in such a finite number of atomic ensembles, which converge to the form found in the thermodynamic limit and for small excitations. By means of von Neumann entropy, a maximum entanglement between the atoms and quantized field happens midway through the storage and retrieval process. Our results pave the way to perform quantum memory protocols in a small number of atoms, given that the number of atoms is larger than the quantum state to be stored.

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