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Manifestation of superposition and coherence in $\mathcal{PT}$-symmetry through the η-inner product

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Abstract

Through the η-inner product, we investigate $\mathcal{PT}$-symmetric quantum mechanics from the viewpoint of superposition and coherence theory. It is argued that $\mathcal{PT}$-symmetric quantum systems are endowed with stationary superposition or superposition-free properties. A physical interpretation of η-inner product in $\mathbb{C}^2$ is given through the Stokes parameters, showing the difference between broken and unbroken $\mathcal{PT}$-symmetric quantum systems.

Keywords: superposition, coherence, $\mathcal{PT}$-symmetric, η-inner product

1. Introduction

It is known that both coherence and superposition are distinctive features in quantum physics, which make quantum theory depart from classical physics. In different fields of physics, quantum coherence and superposition play an important role. The investigation of coherence has a long history and recently attracts increasing interests in the resource theory [1–8]. A recent notable progress is the generalization of coherence resource theory to superposition resource theory. Now, we know that superposition can be converted to entanglement, with the analogues of free states and free operations in coherence theory [9–11].

Instead of conventional quantum mechanics with Hermitian Hamiltonians, non-Hermitian parity-time ($\mathcal{PT}$) symmetry was initially introduced to generalize quantum mechanics, which was first established by Bender and his colleagues in 1998 [12]. Here, $\mathcal{P}$ is parity operator and $\mathcal{T}$ is time-reversal operator. Since then, lots of work have been done to investigate

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J. Phys. A: Math. Theor. **51** (2018) 414004

M Huang *et al*

$\mathcal{PT}$-symmetric quantum systems. An important theoretic notion is the metric operator and the η-inner product introduced by Mostafazadeh [13–16]. In addition, although special attention should be paid to local $\mathcal{PT}$-symmetric operations on a quantum composite system [17], many useful applications can be found in different branches of physics through the new degree of freedom by $\mathcal{PT}$-symmetry [18–23].

Compared with conventional quantum mechanics, $\mathcal{PT}$-symmetry quantum theory has some distinct features. For example, to discuss the evolution of a $\mathcal{PT}$-symmetric system, we need to choose a preferred basis, which inspires us the scenarios in coherence and superposition theories. In this note, we will discuss the $\mathcal{PT}$-symmetry theory from the perspective of coherence and superposition, revealing some internal connections through the η-inner product. As an example, in $\mathbb{C}^2$, a physical interpretation of this η-inner product is given through the analogues in optical polarizations.

2. Preliminaries

2.1. Some basic notions of coherence and superposition

We first introduce some notions which will be used. In the resource theory of coherence, given a preferred orthonormal basis $|i\rangle_{i=1}^d$, a state ρ is incoherent if $\rho = \sum_{i=1}^d p_i|i\rangle\langle i|$, where p_i is the probability distribution. The set of all the incoherent states is usually denoted by $\mathcal{I}$.

A Kraus operator K_n is said to be incoherent if for all $\rho \in \mathcal{I}$, $\frac{K_n \rho K_n^\dagger}{\mathrm{Tr}[K_n \rho K_n^\dagger]} \in \mathcal{I}$. An incoherent operation is a completely positive trace preserving (CPTP) map having an incoherent Kraus decomposition.

Similarly, there are concepts of free states and operations in the theory of superposition [11]. Let $\{|c_i\rangle_{i=1}^d\}$ be a normalized, linearly independent and not necessarily orthogonal basis of the Hilbert space $\mathbb{C}^d$. A state ρ is superposition-free if $\rho = \sum_{i=1}^d p_i|c_i\rangle\langle c_i|$, where p_i is the probability distribution. The set of all the superposition-free states is usually denote by $\mathcal{F}$.

A Kraus operator K_n is said to be superposition-free if for all $\rho \in \mathcal{F}$, $\frac{K_n \rho K_n^\dagger}{\mathrm{Tr}[K_n \rho K_n^\dagger]} \in \mathcal{F}$. A superposition-free operation is a CPTP map having a superposition-free Kraus decomposition.

As was proved in [11], for a set of superposition-free Kraus operators K_m such that $\sum K_m^\dagger K_m \leqslant I$, there always exist superposition-free Kraus operators F_n such that $\sum K_m^\dagger K_m + \sum F_n^\dagger F_n = I$. Hence the trace-decreasing operations admitting superposition-free Kraus decompositions are also called superposition-free.

2.2. Introduction to $\mathcal{PT}$-symmetry quantum theory

$\mathcal{PT}$-symmetry quantum theory explores the properties of quantum systems, which are governed by a $\mathcal{PT}$-symmetric Hamiltonian $\mathcal{H}$.

A parity operator $\mathcal{P}$ is a linear operator such that $\mathcal{P}^2 = \mathcal{I}_d$, where $\mathcal{I}_d$ is the identity operator on $\mathbb{C}^d$.

A time reversal operator $\mathcal{T}$ is an anti-linear operator such that $\mathcal{T}^2 = \mathcal{I}_d$. Moreover, it is demanded that $\mathcal{PT} = \mathcal{TP}$.

A linear operator $\mathcal{H}$ on $\mathbb{C}^d$ is said to be $\mathcal{PT}$-symmetric if $\mathcal{HPT} = \mathcal{PTH}$.

In finite dimensional case, a linear operator corresponds uniquely to a matrix and an anti-linear operator corresponds to the composition of a matrix and a complex conjugation [24]. Let A be a matrix. Denote $\overline{A}$ the complex conjugation of A and $A^\dagger$ the transpose of $\overline{A}$. Let P, T and H be the matrices of $\mathcal{P}$, $\mathcal{T}$ and $\mathcal{H}$, respectively. Then the definition conditions of $\mathcal{P}$, $\mathcal{T}$, $\mathcal{H}$ are $P^2 = T\overline{T} = I$, $PT = T\overline{P}$ and $HPT = PT\overline{H}$.

In conventional quantum mechanics, the Hamiltonians are Hermitian and thus are unitarily similar to a real diagonal matrix. Analogously, $\mathcal{PT}$-symmetric Hamiltonians have a canonical form.

Lemma 1 ([25]). *A finite dimensional operator $\mathcal{H}$ is $\mathcal{PT}$-symmetric if and only if there exists a matrix Ψ such that $\Psi^{-1}H\Psi = J$,*

$$
J = \begin{pmatrix}
J_{n_1}(\lambda_1, \overline{\lambda}_1) & & & & & \\
& \ddots & & & & \\
& & J_{n_p}(\lambda_p, \overline{\lambda}_p) & & & \\
& & & J_{n_q}(\lambda_q) & & \\
& & & & \ddots & \\
& & & & & J_{n_r}(\lambda_r)
\end{pmatrix}, \tag{1}
$$

where $J_{n_k}(\lambda_{n_k}, \overline{\lambda}_{n_k}) = \begin{pmatrix} J_{n_k}(\lambda_{n_k}) & 0 \\ 0 & J_{n_k}(\overline{\lambda_{n_k}}) \end{pmatrix}$, $J_{n_j}(\lambda_{n_j})$ *is the Jordan block,* $\lambda_{n_1}, \cdots, \lambda_{n_p}$ *are complex (and not real) numbers and* $\lambda_{n_q}, \cdots, \lambda_{n_r}$ *are real numbers. Moreover,* $PT\overline{\Psi} = \Psi K$,

$$
K = \begin{pmatrix}
S_2 \otimes I_{n_1} & & & & & \\
& \ddots & & & & \\
& & S_2 \otimes I_{n_p} & & & \\
& & & I_{n_q} & & \\
& & & & \ddots & \\
& & & & & I_{n_r}
\end{pmatrix}, \tag{2}
$$

where $S_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *and* I_{n_j} *is an identity matrix with the same order as* $J_{n_j}(\lambda_{n_j})$.

In fact, it is an important result in matrix analysis that any matrix, which is similar to a real matrix, has the Jordan form (1). On the other hand, it can be verified that a $\mathcal{PT}$-symmetric matrix is similar to a real matrix, thus having the Jordan form (1). As for the canonical form of PT in (2) and the details of the proof, see [25].

In equation (1), if all the blocks $J_{n_k}(\lambda_{n_k}, \overline{\lambda}_{n_k})$ vanish and all the blocks $J_{n_j}(\lambda_{n_j})$ are of order one, then J reduces to a real diagonal matrix. The Hamiltonians similar to real diagonal matrices are referred to as unbroken $\mathcal{PT}$-symmetric. The others are said to be broken.

In the context of $\mathcal{PT}$-symmetry theory, the evolution of a state ρ is given by

$$
\rho(t) = \mathcal{U}(t)\rho\mathcal{U}^\dagger(t),
$$

where $\mathcal{U}(t) = \mathrm{e}^{-\mathrm{i}t\mathcal{H}}$. According to the probability interpretation of inner product, $\mathcal{U}(t)$ should be inner product preserving. However, it is not true when $\mathcal{H}$ is $\mathcal{PT}$-symmetric since $\mathcal{U}(t)$ is not unitary.

To settle the problem, introduce a Hermitian operator η and redefine an $\eta-$inner product by $\langle \phi_1, \phi_2 \rangle_\eta = \langle \phi_1, \eta\phi_2 \rangle$, where ϕ_1 and ϕ_2 are two states. The evolution $\mathcal{U}(t)$ preserves the $\eta-$inner product if and only if $\mathcal{H}^\dagger \eta = \eta\mathcal{H}$ [13–16, 26–28]. An operator η satisfying this condition is said to be a metric operator of $\mathcal{H}$. Moreover, the following lemma gives the canonical form of a metric operator.

Lemma 2 ([29]). *For each* $\mathcal{PT}$ *-symmetric operator* $\mathcal{H}$*, there exists invertible Hermitian matrix* η *such that* $H^\dagger \eta = \eta H$*. Moreover, there exists a matrix* Ψ *such that* $\Psi^{-1} H \Psi = J$ *in equation (1) and*

$$\Psi^\dagger \eta \Psi = S = \begin{pmatrix} S_{2n_1} & & & & & & \\ & \ddots & & & & & \\ & & S_{2n_p} & & & & \\ & & & \epsilon_{n_q} S_{n_q} & & & \\ & & & & \ddots & & \\ & & & & & \epsilon_{n_r} S_{n_r} \end{pmatrix}, \tag{3}$$

where n_i *are the orders of the Jordan blocks in equation (1),* $S_k = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}_{k \times k}$ *and*

$\epsilon_{n_i} = \pm 1$ *are uniquely determined by* η*.*

Lemma 2 actually gives a standard way to construct metric operators. Note that S_k is positive definite only if $k = 1$. Utilizing this, one can verify that η is positive definite if and only if H is unbroken. Hence when H is broken, the η-inner product of a state can be negative.

2.3. An example in $\mathbb{C}^2$

In $\mathbb{C}^2$, lemmas 1 and 2 will give relatively simple properties of a $\mathcal{PT}$-symmetry quantum system.

By lemma 1, there are three cases of the canonical form $J = \Psi^{-1} H \Psi$ and $PT\overline{\Psi}$.

(i) J is diagonal and has two real eigenvalues $\lambda_1 = a_1$ and $\lambda_2 = a_2$. Let $|\psi_1\rangle$ and $|\psi_2\rangle$ be the two column vectors of Ψ, then we have,

$$\begin{aligned} H|\psi_i\rangle &= a_i|\psi_i\rangle, \\ \mathcal{PT}|\psi_i\rangle &= |\psi_i\rangle (PT\overline{|\psi_i\rangle} = |\psi_i\rangle). \end{aligned} \tag{4}$$

(ii) J has two equal real eigenvalues $\lambda_1 = \lambda_2 = a$ and is not diagonal.

$$\begin{aligned} H|\psi_1\rangle &= a|\psi_1\rangle, \\ H|\psi_2\rangle &= a|\psi_2\rangle + |\psi_1\rangle, \\ \mathcal{PT}|\psi_i\rangle &= |\psi_i\rangle. \end{aligned} \tag{5}$$

(iii) J is diagonal and has two complex conjugate eigenvalues $\lambda_1 = \overline{\lambda_2} = a + ib$.

$$\begin{aligned} H|\psi_i\rangle &= \lambda_i|\psi_i\rangle, \\ \mathcal{PT}|\psi_1\rangle &= |\psi_2\rangle, \\ \mathcal{PT}|\psi_2\rangle &= |\psi_1\rangle. \end{aligned} \tag{6}$$

One can use lemma 2 to construct corresponding metric operators,

(i) $\eta = (\Psi^{-1})^\dagger \Psi^{-1}$. Then we have

$$\langle \psi_i | \psi_j \rangle_\eta = \delta_{ij}. \tag{7}$$

(ii) $\eta = (\Psi^{-1})^\dagger S_2 \Psi^{-1}$,

$$\langle \psi_1 | \psi_2 \rangle_\eta = \langle \psi_2 | \psi_1 \rangle_\eta = 1,$$
$$\langle \psi_1 | \psi_1 \rangle_\eta = \langle \psi_2 | \psi_2 \rangle_\eta = 0. \tag{8}$$

(iii) $\eta = (\Psi^{-1})^\dagger S_2 \Psi^{-1}$, the η-inner product of ψ_i is the same as that in (8).

Similar to the role of standard inner product in conventional quantum physics, the η-inner product determine the mechanism of a quantum system in $\mathcal{PT}$-symmetry quantum mechanics. In conventional quantum physics, the inner product of a state is interpreted as the probability. However, the η-inner product of a state can be negative, which obscures the physical interpretation of it. In spite of this, an understanding from the perspective of superposition and coherence may be a candidate of such an interpretation.

3. Metric operator and its relation with superposition and coherence

In this part, we will show that the η-inner product depicts some superposition property for broken $\mathcal{PT}$-symmetric systems or some superposition-free property for unbroken $\mathcal{PT}$-symmetric systems. To illustrate this, it is sufficient to discuss the $\mathcal{PT}$-symmetric systems in $\mathbb{C}^2$.

3.1. The broken $\mathcal{PT}$-symmetry

Let H be a broken $\mathcal{PT}$-symmetric Hamiltonian and $\rho = \sum \rho_{ij} |\psi_i\rangle\langle\psi_j|$ be a state.

If H has two complex eigenvalues λ and $\bar{\lambda}$, it follows from (6) that
$$U(t)|\psi_1\rangle = e^{-it\lambda}|\psi_1\rangle,$$
$$U(t)|\psi_2\rangle = e^{-it\bar{\lambda}}|\psi_2\rangle.$$
Let $\rho(t) = U(t)\rho U^\dagger(t) = \sum \rho_{ij}(t)|\psi_i\rangle\langle\psi_j|$. Direct calculations show that $\rho_{12}(t) = \rho_{12}$, $\rho_{21}(t) = \rho_{21}$. Since the superposition only concerns the coefficients of $|\psi_i\rangle\langle\psi_j|(i \neq j)$, the superposition can be considered to be stationary during the evolution.

If H cannot be diagonalized, it follows from (5) that

$$H = \Psi \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \Psi^{-1} = aI_2 + \Psi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Psi^{-1},$$

$$U(t) = e^{-itH} = e^{-ita}\left(I - it\Psi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Psi^{-1}\right),$$

$$e^{-itH}|\psi_1\rangle = e^{-ita}|\psi_1\rangle,$$

$$e^{-itH}|\psi_2\rangle = e^{-ita}|\psi_2\rangle - ite^{-ita}|\psi_1\rangle.$$

Direct calculations show that $\rho_{12}(t) \neq \rho_{12}$ and $\rho_{21}(t) \neq \rho_{21}$. However, $\rho_{12}(t) + \rho_{21}(t) = \rho_{12} + \rho_{21}$. Although the superposition is not stationary, the 'sum of superposition' is preserved.

That is, in some sense, the superposition of a state are preserved by the action of $\mathcal{PT}$-symmetric evolution. Furthermore, let $|\xi\rangle = b_1|\psi_1\rangle + b_2|\psi_2\rangle$ be a state. (8) shows that $\langle\xi,\xi\rangle_\eta = b_1\bar{b_2} + b_2\bar{b_1} = \rho_{12} + \rho_{21}$. Note that $\langle\xi,\xi\rangle_\eta = \langle\xi, \eta\xi\rangle = \text{Tr}(\eta|\xi\rangle\langle\xi|)$. Similarly, for a general state ρ, $\text{Tr}(\eta\rho) = \rho_{12} + \rho_{21}$. Hence, η-inner product is actually the 'the sum of superposition'. In addition, for two states $|\xi_1\rangle$ and $|\xi_2\rangle$, $\langle\xi_1,\xi_2\rangle_\eta$ is the 'sum of superposition' of $|\xi_2\rangle\langle\xi_1|$.

Thus, the η-inner product is not the probability in the usual sense but some description of the superposition. This also gives another way to understand why the η-inner product can be negative for broken $\mathcal{PT}$-symmetric systems. Although the positivity of probability is natural,

the positivity of a quantity concerning superposition is unnatural and not necessary, since superposition is a phenomenon in quantum theory, which does not have a sign itself.

Intuitively, a process conserving the superposition for any state may not exist in the framework of conventional quantum mechanics, inferring the impossibility of simulating a broken $\mathcal{PT}$-symmetric Hamiltonian in the usual sense. In fact, it can be rigorously showed that one cannot simulate a broken $\mathcal{PT}$-symmetric Hamiltonian by utilizing a large Hermitian Hamiltonian [25].

3.2. The unbroken $\mathcal{PT}$-symmetry

When H is unbroken, $\rho(t) = U(t)\rho U^\dagger(t) = \sum \rho_{ij}\mathrm{e}^{\mathrm{i}t(\lambda_j - \lambda_i)}|\psi_i\rangle\langle\psi_j|$. Since $\lambda_1 \neq \lambda_2$ in general, it is apparent that $\rho_{12}(t) \neq \rho_{12}$, $\rho_{21}(t) \neq \rho_{21}$ and $\rho_{12}(t) + \rho_{21}(t) \neq \rho_{12} + \rho_{21}$. However, in this case, $\rho_{11}(t) = \rho_{11}$ and $\rho_{22}(t) = \rho_{22}$. Moreover, $\mathrm{Tr}(\eta\rho) = \rho_{11} + \rho_{22} = \mathrm{Tr}(\eta\rho(t)) = \rho_{11}(t) + \rho_{22}(t)$. In particular, for a pure state $|\xi\rangle = b_1|\psi_1\rangle + b_2|\psi_2\rangle$, then $\langle\xi,\xi\rangle_\eta = \mathrm{Tr}(\eta|\xi\rangle\langle\xi|) = |b_1|^2 + |b_2|^2 = \rho_{11} + \rho_{22}$. Compared with the broken case, η-inner product now characterizes the superposition-free properties of a state.

It is also convenient to discuss the effect of the metric operator η when H is unbroken. Note that ψ_1 and ψ_2 are not orthogonal in the standard inner product. However, $\langle\psi_i|\psi_j\rangle_\eta = \delta_{ij}$, which shows the orthogonality. Thus, the metric operator η mathematically transforms the superposition to coherence. If we do not consider the normalization, a superposition-free state $\rho = \sum p_i|\psi_i\rangle\langle\psi_i|$ in the usual sense is incoherent with respect to the η-inner product. Such a change of inner product is not unitary, if it can be realized in the framework of conventional quantum mechanics, the process can only be probabilistically.

In fact, it is possible to simulate the transformation $\rho \mapsto U(t)\rho U^\dagger(t)$ in a subsystem of a large Hermitian system. To see this, note that $U(t) = \mathrm{e}^{-\mathrm{i}tH} = \Psi\mathrm{e}^{-\mathrm{i}t\Lambda}\Psi^{-1}$, where $\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Thus $\|U(t)\| = \|\Psi\mathrm{e}^{-\mathrm{i}t\Lambda}\Psi^{-1}\| \leqslant \|\Psi\|\|\Psi^{-1}\|$. Hence $U(t)$ forms a set of uniformly bounded operators and it is possible to find some constant c such that $c^2 U^\dagger(t)U(t) \leqslant I$. In fact, using the Naimark Dilation method one can always realize a transformation $\rho \mapsto \frac{U(t)\rho U^\dagger(t)}{\mathrm{Tr}[U(t)\rho U^\dagger(t)]}$. For concrete discussions, see [25, 30].

Furthermore, note that $cU(t) = c\Psi\mathrm{e}^{-\mathrm{i}t\Lambda}\Psi^{-1} = \sum c\mathrm{e}^{-\mathrm{i}t\lambda_i}|\psi_i\rangle\langle\psi_i^\perp|$. Define the free states to be $\rho = \sum p_i|\psi_i\rangle\langle\psi_i|$. It is apparent that for any free state ρ, the state $\frac{U(t)\rho U^\dagger(t)}{\mathrm{Tr}[U(t)\rho U^\dagger(t)]}$ is also free. Hence $cU(t)$ will gives a free operation. Such a result is no coincidence. As was mentioned, η-inner product is an invariant superposition-free quantity, which reflects the stationary superposition-free property of the Hamiltonian $\mathcal{H}$. The free property of $cU(t)$ is just another manifestation. This conclusion is also true in any other finite dimensional space.

The different superposition properties of η-inner product and $\mathcal{PT}$-symmetric Hamiltonian $\mathcal{H}$ itself, essentially arises from the $\mathcal{PT}$-symmetry. Intuitively, for an anti-linear operator $\mathcal{PT}$ and the basis vectors $\{|\psi_i\rangle\}$, the condition $\mathcal{PT}|\psi_i\rangle = |\psi_i\rangle$ in (4) reflects the interrelation of a basis vector with itself under the action of $\mathcal{PT}$. Through the condition $[\mathcal{H}, \mathcal{PT}] = 0$, such interrelations are respected and manifested by the property of unbroken $\mathcal{PT}$-symmetric system. However, $\mathcal{PT}|\psi_1\rangle = |\psi_2\rangle$, $\mathcal{PT}|\psi_2\rangle = |\psi_1\rangle$ in (6) is also possible. This can be viewed as the interrelation of a basis vector with other basis vector under the action of $\mathcal{PT}$, just like the coherence or superposition. Similarly, the interrelations are also preserved in the broken $\mathcal{PT}$-symmetric system, in a form of superposition. (5) actually gives a intermediate case between (4) and (6), in which under the action of H, $|\psi_1\rangle$ interrelates with itself while $|\psi_2\rangle$ interrelates with both $|\psi_i\rangle$. A support of this viewpoint is Bender's model, in which

$H = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix}$ is the $\mathcal{PT}$-symmetric Hamiltonian, where r, s, θ are real parameters [31]. When $s^2 - r^2 \sin^2\theta > 0$, H is unbroken, corresponding to (4). When $s^2 - r^2 \sin^2\theta < 0$, H has two complex conjugate eigenvalues, corresponding to (6). The intermediate condition $s^2 - r^2 \sin^2\theta = 0$ correspond to a case in which H cannot be diagonalized in general, namely, the case of (5).

4. Discussions

In this section, we aim to investigate our results from the perspective of optics. We will see a tight connection between the Stokes parameters and the η-inner product.

In $\mathbb{C}^2$, one can easily map the states as the two degree of freedoms in optical polarizations, i.e. $|\psi_i\rangle$ being the polarization in the vertical or horizontal direction, denoted as x and y coordinates. For broken $\mathcal{PT}$-symmetry, the η-inner product gives the quantity $b_1\bar{b}_2 + b_2\bar{b}_1$, which is nothing but the *Stokes parameters*, $S_2 = E_x\overline{E_y} + E_y\overline{E_x}$ [32]. Here, the polarization components in the x and y directions are denoted by E_x and E_y, respectively. It is known that Stokes parameter, S_2 measures the degree of polarization. However, for unbroken $\mathcal{PT}$-symmetry, the η-inner product gives the quantity $\rho_{11} + \rho_{22}$, which is an analogue to another Stokes parameter, $S_0 = |E_x|^2 + |E_y|^2$, corresponding to the total intensity (here the probability) of the field.

Nevertheless, for the non-diagonalizable case, there is only one eigenstate, failing to fulfill the Stokes parameterization of its own. In this case, one more basis vector is needed, which corresponds to the non-diagonalizable Hamiltonian. Even though the Stokes parameters S_0 and S_2 are essentially different, for such a non-diagonalizable case, one can transfer in different Stokes parameters, from S_0 to S_2, or vice versa.

As an example, consider the $\mathcal{PT}$-symmetric Hamiltonian $H = \begin{pmatrix} re^{i\theta} & s \\ s & re^{-i\theta} \end{pmatrix}$. The

two eigenstates of H are $|\tilde{E}_+(\alpha)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{\frac{i\alpha}{2}} \\ e^{-\frac{i\alpha}{2}} \end{pmatrix}$ and $|\tilde{E}_-(\alpha)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} ie^{-\frac{i\alpha}{2}} \\ -ie^{\frac{i\alpha}{2}} \end{pmatrix}$, where $\sin\alpha = \frac{r}{s}\sin\theta$. Since $\langle\tilde{E}_\pm(\alpha)|\tilde{E}_\pm(\alpha)\rangle_\eta = \cos\theta$, consider the two η-inner product normalised states, $|E_\pm(\alpha)\rangle = \frac{1}{\sqrt{\cos\alpha}}|\tilde{E}_\pm(\alpha)\rangle$. For a state $|\xi\rangle = \begin{pmatrix} x \\ y \end{pmatrix} = c_1|E_+(\alpha)\rangle + c_2|E_-(\alpha)\rangle$, we can obtain the coefficients c_i,

$$c_1 = \sqrt{2\cos\alpha}\frac{xe^{i\frac{\alpha}{2}} + ye^{-i\frac{\alpha}{2}}}{e^{i\alpha} + e^{-i\alpha}},$$

$$c_2 = -i\sqrt{2\cos\alpha}\frac{xe^{-i\frac{\alpha}{2}} - ye^{i\frac{\alpha}{2}}}{e^{i\alpha} + e^{-i\alpha}}.$$

Moreover,

$$\begin{aligned} S_0 &= |c_1|^2 + |c_2|^2 \\ &= \frac{1}{\cos\alpha}(|x|^2 + |y|^2 + i(x\bar{y} - y\bar{x})\sin\alpha). \end{aligned} \tag{9}$$

Note that the $\mathcal{PT}$-symmetry breaking condition is $s^2 - r^2\sin^2\theta = 0$. Hence $\alpha = \frac{\pi}{2}$ is a critical point. As $\alpha \to \frac{\pi}{2}$, $S_0 = |c_1|^2 + |c_2|^2 \to \infty$. To see what actually happens at the critical point of H, note that the two eigenstates $|\tilde{E}_\pm(\frac{\pi}{2})\rangle$ coincide. So the eigenstate only gives one direction, failing to fully realize the Stokes parameterization. This calls for one more basis vector, which is the generalized eigenstate given by (5), leading to a non-trivial non-diagonalizable

case. In addition, utilizing the new state and such a non-diagonalizable Hamiltonian makes us transfer from S_0 to S_2, which are essentially different Stokes parameters. And the discussion of S_2 in the broken case is an analogy to the above.

Nevertheless, when we generalize the results to $\mathbb{C}^n$, a simple optical interpretation becomes unclear. As the scenario for Stokes parameters, the optical interpretation for our η-inner product shares the same problem of complications in higher dimensions.

5. Conclusion

In this note, we discuss $\mathcal{PT}$-symmetry theory in the view of superposition and coherence. A more physical interpretation of η-inner product is given. This shows the physical difference between the broken and unbroken $\mathcal{PT}$-symmetric systems. We also argue that the essence of such a interpretation comes from the $\mathcal{PT}$-symmetry of a system, which is natural according to our physical intuitions. Though the discussions are restricted to $\mathbb{C}^2$, it is possible to generalize the idea by utilizing lemmas 1 and 2. In that case, for a state $\rho = \sum \rho_{ij}|\psi_i\rangle\langle\psi_j|$, only part of the $\rho_{ij}(i \neq j)$ will be involved. However, the interpretation of η-inner product as stationary superposition or superposition-free property is still valid.

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J. Phys. A: Math. Theor. **51** (2018) 414004

M Huang *et al*

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