Linear Algebra, EE 10810/EECS 205004

Note 1.3 – 1.4

Ray-Kuang Lee[1]

[1] *Room 911, Delta Hall, National Tsing Hua University, Hsinchu, Taiwan.*
Tel: +886-3-5742439; E-mail: rklee@ee.nthu.edu.tw

(Dated: Fall, 2020)

- **Office hours:**
  1. RK: Every Wednesday, 1:00-5:00 PM at R 911, Delta Hall, or by appointment
  2. Raul Robles-Robles (last year PhD student); email: raulamauryrobles@hotmail.com, or by appointment.
  3. TA time, Every Wednesday, 6:30-8:30 PM at Delta 217.

- **Integrity**: Next Quiz on September 30th, Wednesday, 10:10 AM - 10:30 AM.

- **Assignment**: for the Quiz on Sep. 30th
  1. Let $\mathcal{V} = \{(a_1, a_2) : a_1, a_2 \in \mathcal{R}\}$. For $(a_1, a_2), (b_1, b_2) \in \mathcal{V}$ and $c \in \mathcal{R}$, define

     $$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2), \quad \text{and} \quad c(a_1, a_2) = (ca_1, ca_2).$$

     Is $\mathcal{V}$ a vector space over $\mathcal{R}$ with operations?

  2. Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be subspaces of a vector space $\mathcal{V}$. Prove that $\mathcal{W}_1 \cup \mathcal{W}_2$ is a subspace of $\mathcal{V}$ if and only if $\mathcal{W}_1 \subseteq \mathcal{W}_2$ or $\mathcal{W}_2 \subseteq \mathcal{W}_1$.

  3. Prove that if $\mathcal{W}$ is a subspace of a vector space $\mathcal{V}$ and $w_1, w_2, \ldots, w_n$ are in $\mathcal{W}$, then $a_1w_1 + a_2w_2 + \ldots a_nw_n \in \mathcal{W}$ for any scalars $a_1, a_2, \ldots, a_n$.

  4. In $M_{m\times n}\{F\}$ define $\mathcal{W}_1 = \{A \in M_{m\times n}\{F\} : A_{ij} = 0$ whenever $i > j\}$ and $\mathcal{W}_2 = \{A \in M_{m\times n}\{F\} : A_{ij} = 0$ whenever $i \leq j\}$. Show that $M_{m\times n}\{F\} = \mathcal{W}_1 \oplus \mathcal{W}_2$, where $\oplus$ denotes the direct sum of $\mathcal{W}_1$ and $\mathcal{W}_2$.

  5. Let $\mathcal{W}$ be a subspace of a vector space $\mathcal{V}$ over a field $\mathcal{F}$. For any $\nu \in \mathcal{V}$, the set $\nu + \mathcal{W} = \{\nu + w : w \in \mathcal{W}\}$ is called the *coset* of $\mathcal{W}$ containing ν.
     (a) Prove that $\nu + \mathcal{W}$ is a subspace of $\mathcal{V}$ if and only if $\nu \in \mathcal{W}$.
     (b) Prove that $\nu_1 + \mathcal{W} = \nu_2 + \mathcal{W}$ iff $\nu_1 - \nu_2 \in \mathcal{W}$.

From Scratch !!

- Definition: A *vector space* (or *linear space)* V over a field F consists of a **set** on which two operations are defined, such that:
  1. addition: for each pair of elements $\vec{x}$, $\vec{y}$ in V there is a *unique* element $\vec{x}+\vec{y}$ in V
  2. scalar multiplication: for each element a in F and each element $\vec{x}$ in V there is a *unique* element $a\,\vec{x}$ in V,

  such that the conditions VS1 - VS8 are all hold.
- Theorem 1.1 (Cancellation Law for Vector Addition): $\vec{x}+\vec{z}=\vec{y}+\vec{z} \quad \Leftrightarrow \quad \vec{x}=\vec{y}$.
  - Corollary 1: $\vec{0}$ is unique.
  - Corollary 2: The inverse vector $\vec{x}+\vec{y}=\vec{0}$ is unique.
- Theorem 1.2:
  1. $0\,\vec{x}=0$ for each $x\in\mathcal{V}$.
  2. $(-a)\vec{x}=-(a\vec{x})=a(-\vec{x})$ for each $a\in\mathcal{F}$ and each $\vec{x}$ in $\mathcal{V}$.
  3. $a\vec{0}=0$ for each $a\in\mathcal{F}$.

- Subspace:

- Theorem 1.3:
  1.
  2.
  3.

- Matrix in Subspace: Symmetric Matrix, Diagonal Matrix, Upper (Down)-Triangle Matrix, Trace

- Polynomials in Subspace:

- Theorem 1.4: Any intersection of subspaces of a vector space $\mathcal{V}$ is a subspace of $\mathcal{V}$.

- Question: Is the Union of subspaces of a vector space $\mathcal{V}$ also a subspace of $\mathcal{V}$?

- Systems of Linear Equations:

$$\begin{array}{rrrrrcl} a_1 & -2a_2 & & +2a_4 & -3a_5 & = & 2 \\ 2a_1 & -4a_2 & +2a_3 & & +8a_5 & = & 6 \\ a_1 & -2a_2 & +3a_3 & -3a_4 & +16a_5 & = & 8 \end{array} \tag{1}$$

$$\begin{pmatrix} +2x^3 \\ -2x^2 \\ +12x \\ -6 \end{pmatrix} = a\begin{pmatrix} +x^3 \\ -2x^2 \\ -5x \\ -3 \end{pmatrix} + b\begin{pmatrix} +3x^3 \\ -5x^2 \\ -4x \\ -9 \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} +3x^3 \\ -2x^2 \\ +7x \\ +8 \end{pmatrix} = a\begin{pmatrix} +x^3 \\ -2x^2 \\ -5x \\ -3 \end{pmatrix} + b\begin{pmatrix} +3x^3 \\ -5x^2 \\ -4x \\ -9 \end{pmatrix} \tag{3}$$

- Rules for systems of linear equations:

- pivot