Linear Algebra, EE 10810/EECS 205004

Note 1.5 – 1.6

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- **Office Hours:**
  1. Raul Robles-Robles (last year PhD student); email: raulamauryrobles@hotmail.com.
  2. TA time, Every Wednesday, 6:30-8:30 PM at Delta 217, or by appointment.

- **Integrity**: Next Quiz on October 7th, Wednesday, 10:10 AM - 10:30 AM.

- **Assignment**: for the Quiz on Oct. 7th
  1. Determine whether the following sets are linearly dependent or linearly independent

     (a)
     $$\left\{ \begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}, \quad \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} \right\} \quad \text{in} \quad \overline{\overline{M}}_{2\times 2}(\mathcal{R}) \tag{1}$$

     (b)
     $$\left\{ \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} \right\} \quad \text{in} \quad \overline{\overline{M}}_{2\times 2}(\mathcal{R}) \tag{2}$$

     (c) $\left\{x^3 + 2x^2, -x^2 + 3x + 1, x^3 - x^2 + 2x - 1\right\}$ in $P_3(\mathcal{R})$

  2. Prove that, let $\mathcal{V}$ be a vector space, and let $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \mathcal{V}$. If $\mathcal{S}_1$ is linear dependent, then $\mathcal{S}_2$ is linearly dependent.

  3. Prove that, a set S is linearly dependent iff $S = \{\vec{0}\}$ or there exist distinct vectors $\vec{v}, \vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ in S s.t. $\vec{v}$ is a linear combination of $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$.

From Scratch !!

- **Systems of Linear Equations**:

$$\begin{array}{rrrcr} 2a_1 & +4a_2 & -2a_3 & = & 2 \\ 4a_1 & +9a_2 & -3a_3 & = & 8 \\ -2a_1 & -3a_2 & +7a_3 & = & 10 \end{array} \tag{3}$$

- Alternative: Geometric interpretation of Systems of Linear Equations: 2D and 3D (intersections of plans)

- Matrix form: $\overline{\overline{M}}\,\vec{a} = \vec{b}$

$$\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix} \tag{4}$$

- Augmented Matrix, $\overline{\overline{M}}\vec{a} = \vec{b} \quad \Rightarrow \quad \left[\overline{\overline{M}}\,|\,\vec{b}\right]$.

$$\left(\begin{array}{rrr|r} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{array}\right) \tag{5}$$

- Rules for systems of linear equations: $\overline{\overline{P}}\,\overline{\overline{D}}\,\overline{\overline{E}}$
  1. Scalar Multiplication, Diagonalized Matrix, ex, $\overline{\overline{D}} = \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,
  2. Elimination, Elimination Matrix, ex, $\overline{\overline{E}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l & 0 & 1 \end{bmatrix}$
  3. Row Exchange, Permutation Matrix, ex, $\overline{\overline{P}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

- Gauss-Jordan Elimination: $\overline{\overline{P}}\,\overline{\overline{D}}\,\overline{\overline{E}}\,\overline{\overline{M}}$

$$\left(\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right) \tag{6}$$

- Echelon (Trapezoid) Matrix Form:
- Pivot: the **First non-zero** in the row that does the elimination, p_i.
- Breakdown of Elimination: pivot is zero, $\exists p_i = 0$
- Number of Pivots: # called as the *rank* of a Matrix

- Inverse Matrix, $\overline{\overline{M}}^{-1}$
- LU Decomposition:

- Matrix Operations:
  - Commutative: $\overline{\overline{A}} + \overline{\overline{B}}$; $\overline{\overline{A}} \cdot \overline{\overline{B}}$
  - Associative: $\overline{\overline{A}} + (\overline{\overline{B}} + \overline{\overline{C}})$; $\overline{\overline{A}} \cdot (\overline{\overline{B}} \cdot \overline{\overline{C}})$
  - distributive: $c(\overline{\overline{A}} + \overline{\overline{B}})$; $\overline{\overline{C}}\,(\overline{\overline{A}} + \overline{\overline{B}})$; $(\overline{\overline{A}} + \overline{\overline{B}})\,\overline{\overline{C}}$
  - Commutator: $[\overline{\overline{A}}, \overline{\overline{B}}]$

- Span: $\text{span}(S)$
- Generating Set: $\text{span}(S) = \mathcal{V}$

- Linearly Dependent :
- Linearly Independent : $a_1\,\vec{v}_1 + a_2\,\vec{v}_2 + \ldots + a_n\,\vec{v}_n = 0$ iff $a_1 = a_2 = \ldots = a_n = 0$ (trivial solution).

- Basis: Linearly Independent $\cap$ Generating Set