Linear Algebra, EE 10810/EECS 205004

Note 1.6 – 1.7

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- **Office Hours:**
  1. TA time, Every Monday, 6:00-8:00 PM at Delta 217,
     or by appointment to Mr. Chia-Wei Chen; email: weachen34@gmail.com
  2. TA time, Every Wednesday, 6:30-8:30 PM at Delta 217,
     or by appointment to Mr. Raul Robles-Robles; email: raulamauryrobles@hotmail.com.

- **Integrity**: Next Quiz on October 14th, Wednesday, 10:10 AM - 10:30 AM.

- **Assignment**: for the Quiz on Oct. 14th
  1. Prove that
     $$\text{if}\quad \{\overline{\overline{A}}_1, \overline{\overline{A}}_2, \ldots, \overline{\overline{A}}_k\}\quad \text{is a linearly independent subset,}$$
     of $\overline{\overline{M}}_{n\times n}(F)$,
     $$\text{then}\quad \{(\overline{\overline{A}}_1)^t, (\overline{\overline{A}}_2)^t, \ldots, (\overline{\overline{A}}_k)^t\}\quad \text{is also linearly independent.}$$
  2. Do the polynomials $(x^3 - 2x^2 + 1)$, $(4x^2 - x + 3)$, and $(3x - 2)$ generate $P_3(\mathcal{R})$?
  3. Fin bases for the following subspace of F^5:
     $$W_1 = \{(a_1, a_2, a_3, a_4, a_5), \in F^5 : \quad a_1 - a_3 - a_4 = 0\}$$
     and
     $$W_2 = \{(a_1, a_2, a_3, a_4, a_5), \in F^5 : \quad a_2 = a_3 = a_4 \quad \text{and} \quad a_1 + a_5 = 0\}$$
     What are the dimensions of W_1 and W_2?
  4. Use the Lagrange interpolation formula to construct the polynomial of smallest degree whose graph contains the following points:
     $$(-2, -6), \quad (-1, 5), \quad (1, 3).$$

From Scratch !!

- Theorem 1.5: The span of any subset S of a vector space $\mathcal{V}$ is a *subspace* of $\mathcal{V}$.
- Definition: A subset S of a vector space $\mathcal{V}$ generates (or spans) $\mathcal{V}$ if $\text{span}(S) = \mathcal{V}$.
- Definition: A subset S of a vector space $\mathcal{V}$ is called *linearly dependent* if there exist a finite number of distinct vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{u}_n$ in S and scalars $a_1, a_2, \ldots, a_n$, **not all zero**, s.t.,

$$a_1\,\vec{v}_1 + a_2\,\vec{v}_2 + \ldots + a_n\,\vec{v}_n = 0.$$

- Definition: A subset S of a vector space $\mathcal{V}$ is not linearly dependent is called *linearly independent*, i.e., **ONLY trivial solutions** for

$$a_1\,\vec{v}_1 + a_2\,\vec{v}_2 + \ldots + a_n\,\vec{v}_n = 0.$$

- Theorem 1.6: Let $\mathcal{V}$ be a vector space, and let $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \mathcal{V}$. If $\mathcal{S}_1$ is linear dependent, then $\mathcal{S}_2$ is linearly dependent.
- Corollary: Let $\mathcal{V}$ be a vector space, and let $\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \mathcal{V}$. If $\mathcal{S}_2$ is linear independent, then $\mathcal{S}_1$ is linearly independent.
- Theorem 1.7: Let S be a linearly independent subset of a vector space $\mathcal{V}$, and let $\vec{v}$ be a vector in $\mathcal{V}$ that is not in S. Then $S \cup \{\vec{v}\}$ is linearly dependent iff $\vec{v} \in \text{span}(S)$.

- Definition: A *basis* β for a vector space $\mathcal{V}$ is a linearly independent subset of $\mathcal{V}$ that generates $\mathcal{V}$.
- Theorem 1.8: Let $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ be a subset of $\mathcal{V}$. Then β is a basis for $\mathcal{V}$ iff each $\vec{v} \in \mathcal{V}$ can be **uniquely** expressed as a linear combination of vector of β, that is,

$$\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2 + \ldots + \vec{v}_n,$$

  for unique scalars $a_1, a_2, \ldots, a_n$.

- Theorem 1.9: Finite basis (finite dimension), $\dim(\mathcal{V})$
- Theorem 1.10: Replacement Theorem
- Theorem 1.11: $\dim(\mathcal{W}) \leq \dim(\mathcal{V})$

- Lagrange Interpolation Formula:

$$f_i(x) = \Pi_{k=0,k\neq i}^{n} \frac{x - c_k}{c_i - c_k}$$

- Example:

$$(1, 8), \quad (2, 5), \quad (3, -4).$$

- Definition: Let $\mathcal{F}$ be a family of sets. A member $\mathcal{M}$ of $\mathcal{F}$ is called *maximal* if $\mathcal{M}$ is contained in no member of $\mathcal{F}$ other than $\mathcal{M}$ itself.
- Definition: Let S be a subset of a vector space $\mathcal{V}$. A *maximal linearly independent subset* of S is a subset $\mathcal{B}$ of S satisfying both of the following conditions:
  1. $\mathcal{B}$ is linearly independent.
  2. The only linearly independent subset of S that contains $\mathcal{B}$ is $\mathcal{B}$ itself.

- Corollary 1.13: Every vector space has a basis.