Linear Algebra, EE 10810/EECS 205004

Note 2.2

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- **Office Hours:**
  1. 1st Exam on Oct. 30th, (10:10AM - 1:00 PM, Friday), covering Chap. 1 and Chap. 2.

- **Assignment**: for the Quiz on Oct. 21st
  1. Let $\mathcal{V}$ be a vector space of sequences $\{a_n\}$ in F. Define the function $\hat{\mathcal{T}}, \hat{\mathcal{U}} : \mathcal{V} \to \mathcal{V}$ by

$$\hat{\mathcal{T}}(a_1, a_2, \ldots, a_n) = (a_2, a_3, \ldots), \tag{1}$$
$$\hat{\mathcal{U}}(a_1, a_2, \ldots, a_n) = (0, a_1, a_2, \ldots). \tag{2}$$

Here, $\hat{\mathcal{T}}$ and $\hat{\mathcal{U}}$ are called the **left shift** and **right shift** operators on $\mathcal{V}$, respectively.

(a) Prove that $\hat{\mathcal{T}}$ and $\hat{\mathcal{U}}$ are linear.

(b) Prove that $\hat{\mathcal{T}}$ is onto, but not one-to-one.

(c) Prove that $\hat{\mathcal{U}}$ is one-to-one, but not onto.

2. Let $\hat{\mathcal{T}} : \mathcal{R}^2 \to \mathcal{R}^3$ be defined by $\hat{\mathcal{T}}(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let β be the standard ordered basis for $\mathcal{R}^2$ and $\gamma = \{(1,1,0), (0,1,1), (2,2,3)\}$.

(a) Computer $\left[\hat{\mathcal{T}}\right]_\beta^\gamma$.

(b) If $\alpha = \{(1,2), (2,3)\}$, compute $\left[\hat{\mathcal{T}}\right]_\alpha^\gamma$.

3. Define

$$\hat{\mathcal{T}} : \overline{\overline{M}}_{2\times 2}(\mathcal{R}) \to P_2(\mathcal{R}), \quad \text{by} \quad \hat{\mathcal{T}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + (2d)\,x + b\,x^2.$$

Let

$$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} \quad \text{and} \quad \gamma = \left\{1, x, x^2\right\}.$$

Computer $\left[\hat{\mathcal{T}}\right]_\beta^\gamma$.

4. Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces and let $\hat{\mathcal{T}}$ and $\hat{\mathcal{U}}$ be nonzero lienar transformations from $\mathcal{V}$ to $\mathcal{W}$. If $R(\hat{\mathcal{T}}) \cap R(\hat{\mathcal{U}}) = \{\vec{0}\}$, prove that $\{\hat{\mathcal{T}}, \hat{\mathcal{U}}\}$ is a linearly independent subset of $\hat{\mathcal{L}}(\mathcal{V}, \mathcal{W})$.

From Scratch !!

- Definition: A function $\hat{\mathcal{T}}: \mathcal{V} \to \mathcal{W}$ is called *Linear Transformation* from $\mathcal{V}$ to $\mathcal{W}$ if, for all $\vec{x}, \vec{y} \in \mathcal{V}$ and $c \in F$, we have
$$\hat{\mathcal{T}}(c\,\vec{x} + \vec{y}) = c\,\hat{\mathcal{T}}(\vec{x}) + \hat{\mathcal{T}}(\vec{y}). \tag{3}$$
- Identity Transformation $\hat{\mathcal{I}}$
- Zero Transformation $\hat{\mathcal{T}}_0$
- Definition: Null space (kernel) $\mathcal{N}(\hat{\mathcal{T}}) = \{\vec{x} \in \mathcal{V} : \hat{\mathcal{T}}(\vec{x}) = \vec{0}\}$.
- Definition: Range (image) $R(\mathcal{T}) = \{\hat{\mathcal{T}}(\vec{x}), \vec{x} \in \mathcal{V}\}$.
- If $f : A \to B$, then A is called the *domain* of f; B is called the *codomain* of f, and the set $\{f(x), x \in A\}$ is called the **range** of f.
- If $x \in A$, then $f(x)$ is called the *image* of x, and x is called the *preimage* of $f(x)$.
- If $f : A \to B$ is a function with **range** B, that is, if $f(A) = B$, then f is called **onto**.
- If each element of the range has a unique preimage, $f : A \to B$ is called **on-to-one**; that is
$$f : A \to B \qquad \text{if} \quad f(x) = f(y) \quad \text{implies} \quad x = y, \quad \text{or} \tag{4}$$
$$\text{if} \quad x \neq y \quad \text{implies} \quad f(x) \neq f(y). \tag{5}$$
- Theorem 2.1: $N(\hat{\mathcal{T}})$ and $R(\hat{\mathcal{T}})$ are subspaces of $\mathcal{V}$ and $\mathcal{W}$, respectively.
- Theorem 2.2: If $\beta = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis of $\mathcal{V}$, then
$$R(\hat{\mathcal{T}}) = \text{span}(\hat{\mathcal{T}}(\beta)) = \text{span}(\{\hat{\mathcal{T}}(\vec{v}_1), \hat{\mathcal{T}}(\vec{v}_2), \ldots, \hat{\mathcal{T}}(\vec{v}_n)\}) \tag{6}$$
- Definition: Nullity$(\hat{\mathcal{T}}) = \dim(N(\hat{\mathcal{T}}))$
- Definition: Rank$(\hat{\mathcal{T}}) = \dim(R(\hat{\mathcal{T}}))$
- Theorem 2.3 (Dimension Theorem):
$$\text{nullity}(\hat{\mathcal{T}}) + \text{rank}(\hat{\mathcal{T}}) = \dim(\mathcal{V}) \tag{7}$$
- Theorem 2.4: $\hat{\mathcal{T}}$ is *one-to-one* iff $N(\hat{\mathcal{T}}) = \{\vec{0}\}$
- Theorem 2.5: The following are equivalent,
  1. $\hat{\mathcal{T}}$ is one-to-one.
  2. $\hat{\mathcal{T}}$ is onto.
  3. rank$(\hat{\mathcal{T}}) = \dim(\mathcal{V})$.
- Theorem 2.6: Suppose $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ is a basis for $\mathcal{V}$, there exists exactly one linear transformation $\hat{\mathcal{T}} : \mathcal{V} \to \mathcal{W}$ such that $\hat{\mathcal{T}}(\vec{v}_i) = \vec{w}_i$, for $i = 1, 2, \ldots, n$ and $\vec{w}_j \in \mathcal{W}$.

- Matrix representation
- Definition: ordered basis, **standard ordered basis**, coordinate vectors
- $\vec{x} = \sum_{i=1}^{n} a_i, \vec{v}_i,$
$$[\vec{x}]_\beta = |x\rangle_\beta = \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{pmatrix} \tag{8}$$

- Matrix representation of $\hat{\mathcal{T}}$ in the ordered bases β and γ:
$$\left[\hat{\mathcal{T}}\right]_\beta^\gamma \tag{9}$$

- Theorem 2.7: Let $\hat{\mathcal{T}}, \hat{\mathcal{U}} : \mathcal{V} \to \mathcal{W}$ be linear, then for all $a \in F$, $a\hat{\mathcal{T}} + \hat{\mathcal{U}}$ is linear.
- Definition: $\mathcal{L}(\mathcal{V}, \mathcal{W})$ denotes the vector space of all linear transformation from $\mathcal{V}$ into $\mathcal{W}$
- Theorem 2.8: $\left[\hat{\mathcal{T}} + \hat{\mathcal{U}}\right]_\beta^\gamma = \left[\hat{\mathcal{T}}\right]_\beta^\gamma + \left[\hat{\mathcal{U}}\right]_\beta^\gamma$ and $\left[a\,\hat{\mathcal{T}}\right]_\beta^\gamma = a\,\left[\hat{\mathcal{T}}\right]_\beta^\gamma$