Linear Algebra, EE 10810/EECS 205004

Note 2.3

Ray-Kuang Lee[1]

[1] *Room 911, Delta Hall, National Tsing Hua University, Hsinchu, Taiwan.*
Tel: +886-3-5742439; E-mail: rklee@ee.nthu.edu.tw

(Dated: Fall, 2020)

- 1st Exam on Oct. 30th, (10:10AM - 1:00 PM, Friday), covering Chap. 1 and Chap. 2.

- **Assignment**: for the Quiz on Oct. 28th

1. Let $\mathcal{V}$ be a vector space, and let $\hat{\mathcal{T}}: \mathcal{V} \to \mathcal{V}$ be linear. A subspace $\mathcal{W}$ of $\mathcal{V}$ is said to be **T-invariant** if $\hat{\mathcal{T}}(X) \in \mathcal{W}$ for every $x \in \mathcal{W}$, that is $\hat{\mathcal{T}}(\mathcal{W}) \subseteq \mathcal{W}$. Now, $\mathcal{V}$ is a n-dimensional vector space, and $\mathcal{W}$ has dimension k. Show that there is a basis β for $\mathcal{V}$ such that $\left[\hat{\mathcal{T}}\right]_\beta$ has the form

$$\begin{pmatrix} \overline{\overline{A}} & \overline{\overline{B}} \\ \overline{\overline{O}} & \overline{\overline{C}} \end{pmatrix} \tag{1}$$

where $\overline{\overline{A}}$ is a $k \times k$ matrix and $\overline{\overline{O}}$ is the $(n-k) \times k$ zero matrix.

2. Let $\mathcal{V}$ and $\mathcal{W}$ are vector space, and let S be a subset of $\mathcal{V}$. Define $S^0 = \{\hat{\mathcal{T}} \in \mathcal{L}(\mathcal{V}, \mathcal{W}); \hat{\mathcal{T}}(x) = 0 \text{ for all } x \in S\}$. Prove that S^0 is a subspace of $\mathcal{L}(\mathcal{V}, \mathcal{W})$.

3. Leg $g(x) = 3 + x$. Let $\hat{\mathcal{T}}: P_2(\mathcal{R}) \to P_2(\mathcal{R})$ and $\hat{\mathcal{U}}: P_2(\mathcal{R}) \to \mathcal{R}^3$ be the linear transformations respectively defined by

$$\hat{\mathcal{T}}(f(x)) = f'(x)\, g(x) + 2f(x), \tag{2}$$

$$\hat{\mathcal{U}}(a + bx + cx^2) = (a + b, c, a - b). \tag{3}$$

Let β and γ be the standard ordered bases of $P_2(\mathcal{R})$ and $\mathcal{R}^3$, respectively.

(a) Comput $\left[\hat{\mathcal{U}}\right]_\beta^\gamma$, $\left[\hat{\mathcal{T}}\right]_\beta$, and $\left[\hat{\mathcal{U}}\hat{\mathcal{T}}\right]_\beta^\gamma$.

(b) Let $h(x) = 3 - 2x + x^2$. Compute $[h(x)]_\beta$ and $\left[\hat{\mathcal{U}}(h(x))\right]_\gamma$.

4. Let $\overline{\overline{A}}$ and $\overline{\overline{B}}$ be $n \times n$ matrices. Recall the trace of $\overline{\overline{A}}$ is defined by

$$\text{tr}(\overline{\overline{A}}) = \sum_{i=1}^{n} A_{ii}. \tag{4}$$

Prove that $\text{tr}(\overline{\overline{AB}}) = \text{tr}(\overline{\overline{BA}})$ and $\text{tr}(\overline{\overline{A}}) = \text{tr}(\overline{\overline{A^t}})$.

From Scratch !!

- Definition: A function $\hat{\mathcal{T}}:\mathcal{V}\to\mathcal{W}$ is called *Linear Transformation* from $\mathcal{V}$ to $\mathcal{W}$ if, for all $\vec{x},\vec{y}\in\mathcal{V}$ and $c\in F$, we have
$$\hat{\mathcal{T}}(c\,\vec{x}+\vec{y}) = c\,\hat{\mathcal{T}}(\vec{x})+\hat{\mathcal{T}}(\vec{y}). \tag{5}$$
- Theorem 2.3 (Dimension Theorem):
$$\text{nullity}(\hat{\mathcal{T}})+\text{rank}(\hat{\mathcal{T}}) = \dim(\mathcal{V}) \tag{6}$$
- Definition: ordered basis, **standard ordered basis**, coordinate vectors
- $\vec{x}=\sum_{i=1}^{n} a_i, \vec{v}_i$,
$$[\vec{x}]_\beta = |x\rangle_\beta = \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{pmatrix} \tag{7}$$
- Matrix representation of $\hat{\mathcal{T}}$ in the ordered bases β and γ:
$$\left[\hat{\mathcal{T}}\right]_\beta^\gamma \tag{8}$$
- Theorem 2.7: Let $\hat{\mathcal{T}}$,$\hat{\mathcal{U}}:\mathcal{V}\to\mathcal{W}$ be linear, then for all $a\in F$, $a\hat{\mathcal{T}}+\hat{\mathcal{U}}$ is linear.
- Definition: $\mathcal{L}(\mathcal{V},\mathcal{W})$ denotes the vector space of all linear transformation from $\mathcal{V}$ into $\mathcal{W}$
- Theorem 2.8: $\left[\hat{\mathcal{T}}+\hat{\mathcal{U}}\right]_\beta^\gamma = \left[\hat{\mathcal{T}}\right]_\beta^\gamma + \left[\hat{\mathcal{U}}\right]_\beta^\gamma$ and $\left[a\,\hat{\mathcal{T}}\right]_\beta^\gamma = a\ \left[\hat{\mathcal{T}}\right]_\beta^\gamma$

- Section 2.3: Composition of Linear Transformations and Matrix Multipplication
- Theorem 2.9: Let $\hat{\mathcal{T}}:\mathcal{V}\to\mathcal{W}$ and ,$\hat{\mathcal{U}}:\mathcal{W}\to\mathcal{Z}$ be linear, then for $\hat{\mathcal{U}}\hat{\mathcal{T}}:\mathcal{V}\to\mathcal{Z}$ is linear.
- Theorem 2.10: Let $\hat{\mathcal{T}}$,$\hat{\mathcal{U}}_1$,$\hat{\mathcal{U}}_2\in\mathcal{L}(\mathcal{V})$, then
  1. $\hat{\mathcal{T}}(\hat{\mathcal{U}}_1+\hat{\mathcal{U}}_2)=\hat{\mathcal{T}}\hat{\mathcal{U}}_1+\hat{\mathcal{T}}\hat{\mathcal{U}}_2$.
  2. $\hat{\mathcal{T}}(\hat{\mathcal{U}}_1\hat{\mathcal{U}}_2)=(\hat{\mathcal{T}}\hat{\mathcal{U}}_1)\hat{\mathcal{U}}_2$.
  3. $\hat{\mathcal{T}}\hat{\mathcal{I}}=\hat{\mathcal{I}}\hat{\mathcal{T}}=\hat{\mathcal{T}}$.
  4. $a(\hat{\mathcal{U}}_1\hat{\mathcal{U}}_2)=(a\hat{\mathcal{U}}_1)\hat{\mathcal{U}}_2=\hat{\mathcal{U}}_1(a\hat{\mathcal{U}}_2)$.
- Let $\overline{\overline{A}}$ be an $m\times n$ matrix and $\overline{\overline{B}}$ be an $n\times p$ matrix. The **product** of $\overline{\overline{A}}$ and $\overline{\overline{B}}$, denoted as $\overline{\overline{AB}}$, to be the $m\times p$ matrix such that
$$(\overline{\overline{AB}})_{ij} = \sum_{k=1}^{n} A_{jk}\,B_{kj}, \quad \text{for} \quad 1\le i\le m, \quad 1\le j\le p. \tag{9}$$
- Theorem 2.11: $\left[\hat{\mathcal{U}}\hat{\mathcal{T}}\right]_\alpha^\gamma = \left[\hat{\mathcal{U}}\right]_\beta^\gamma \left[\hat{\mathcal{T}}\right]_\alpha^\beta$
- Kronecker delta
$$\delta_{ij} = \begin{cases} 1, & \text{if} \quad i=j \\ 0, & \text{if} \quad i\neq j \end{cases} \tag{10}$$
- Identity matrix $\overline{\overline{I}}_n$, $(\overline{\overline{I}}_n)_{ij}=\delta_{ij}$.
- Theorem 2.12:
  1. $\overline{\overline{A}}(\overline{\overline{B}}+\overline{\overline{C}})=\overline{\overline{AB}}+\overline{\overline{AC}}$
  2. $a(\overline{\overline{AB}})=(a\overline{\overline{A}})\overline{\overline{B}}=\overline{\overline{A}}(a\overline{\overline{B}})$
  3. $\overline{\overline{I}}_m\overline{\overline{A}}=\overline{\overline{A}}_{m\times n}=\overline{\overline{AI}}_n$
- Theorem 2.13: Column Vectors
$$\vec{u}_j = \begin{pmatrix} (\overline{\overline{AB}})_{1j} \\ (\overline{\overline{AB}})_{2j} \\ . \\ . \\ . \\ (\overline{\overline{AB}})_{mj} \end{pmatrix} = \overline{\overline{A}} \begin{pmatrix} (\overline{\overline{B}})_{1j} \\ (\overline{\overline{B}})_{2j} \\ . \\ . \\ . \\ (\overline{\overline{B}})_{mj} \end{pmatrix} = \overline{\overline{A}}\,\vec{v}_j \tag{11}$$