Linear Algebra, EE 10810/EECS 205004

Note 2.4 – 2.5

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- 1st Exam on Oct. 30th, (10:10AM - 1:00 PM, Friday), covering Chap. 1 and Chap. 2.

- **Assignment**: for the Quiz on Oct. 28th

1. Let $\mathcal{V}$ be a finite-dimensional vector space, and let $\hat{\mathcal{T}}: \mathcal{V} \to \mathcal{V}$ be linear.
   (a) If $\text{rank}(\hat{\mathcal{T}}) = \text{rank}(\hat{\mathcal{T}}^2)$, prove that $R(\hat{\mathcal{T}}) \cap N(\hat{\mathcal{T}}) = \{\vec{0}\}$.
   (b) Deduce that $\mathcal{V} = R(\hat{\mathcal{T}}) \oplus N(\hat{\mathcal{T}})$.
   (c) Prove that $\mathcal{V} = R(\hat{\mathcal{T}}^k) \oplus N(\hat{\mathcal{T}}^k)$ for some positive integer k.

2. Let $\overline{\overline{A}}$ and $\overline{\overline{B}}$ be $n \times n$ *invertible* matrices. Prove that
   (a) $\overline{\overline{AB}}$ is invertible.
   (b) $(\overline{\overline{AB}})^{-1} = \overline{\overline{B^{-1}A^{-1}}}$.

3. Let

$$\mathcal{V} = \left\{ \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} : a, b, c \in F \right\} \tag{1}$$

   Construct an **isomorphism** from $\mathcal{V}$ to F^3.

4. For each matrix $\overline{\overline{A}}$ and ordered basis β, find $[\hat{L}_A]_\beta$ and an invertible matrix $\overline{\overline{Q}}$ such that $[\hat{L}_A]_\beta = \overline{\overline{Q^{-1}AQ}}$.
   (a)

$$\overline{\overline{A}} = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad \beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} \tag{2}$$

   (b)

$$\overline{\overline{A}} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad \beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\} \tag{3}$$

From Scratch !!

- The **product** of $\overline{\overline{A}}_{m\times n}$ and $\overline{\overline{B}}_{n\times p}$, denoted as $\overline{\overline{AB}}_{m\times p}$, s.t.

$$(\overline{\overline{AB}})_{ij} = \sum_{k=1}^{n} A_{jk}\, B_{kj}, \quad \text{for} \quad 1 \le i \le m, \quad 1 \le j \le p. \tag{4}$$

- Theorem 2.11: $\left[\hat{\mathcal{U}}\hat{\mathcal{T}}\right]_{\alpha}^{\gamma} = \left[\hat{\mathcal{U}}\right]_{\beta}^{\gamma}\left[\hat{\mathcal{T}}\right]_{\alpha}^{\beta}$
- Theorem 2.12:
  1. $\overline{\overline{A}}(\overline{\overline{B}}+\overline{\overline{C}}) = \overline{\overline{AB}} + \overline{\overline{AC}}$
  2. $a(\overline{\overline{AB}}) = (a\overline{\overline{A}})\overline{\overline{B}} = \overline{\overline{A}}(a\overline{\overline{B}})$
  3. $\overline{\overline{I}}_m\overline{\overline{A}}_{m\times n} = \overline{\overline{A}}_{m\times n} = \overline{\overline{A}}_{m\times n}\overline{\overline{I}}_n$
- Theorem 2.13: Column Vectors

$$\vec{u}_j = \begin{pmatrix} (\overline{\overline{AB}})_{1j} \\ (\overline{\overline{AB}})_{2j} \\ \cdot \\ \cdot \\ \cdot \\ (\overline{\overline{AB}})_{mj} \end{pmatrix} = \overline{\overline{A}} \begin{pmatrix} (\overline{\overline{B}})_{1j} \\ (\overline{\overline{B}})_{2j} \\ \cdot \\ \cdot \\ \cdot \\ (\overline{\overline{B}})_{mj} \end{pmatrix} = \overline{\overline{A}}\,\vec{v}_j \tag{5}$$

- Theorem 2.14: For each $\vec{u} \in \mathcal{V}$,

$$[\hat{\mathcal{T}}(\vec{u})]_{\gamma} = [\hat{\mathcal{T}}]_{\beta}^{\gamma}\,[\vec{u}]_{\beta} \tag{6}$$

- Definition: Left-multiplication transformation, $\hat{L}_A : F^n \to F^m$,

$$\hat{L}_A(\vec{x}) = \overline{\overline{A}}\,\vec{x} \tag{7}$$

- Theorem 2.15:
  1. $[\hat{L}_A]_{\beta}^{\gamma} = \overline{\overline{A}}$.
  2. $\hat{L}_A = \hat{L}_B$ iff $\overline{\overline{A}} = \overline{\overline{B}}$.
  3. $\hat{L}_{A+B} = \hat{L}_A + \hat{L}_B$ and $\hat{L}_{aA} = a\hat{L}_A$ for all $a \in F$.
  4. If $\hat{\mathcal{T}} : F^n \to F^m$ is linear, $\exists!$ an $m \times n$ matrix $\overline{\overline{C}}$ s.t. $\hat{\mathcal{T}} = \hat{L}_C$. In fact $\overline{\overline{C}} = [\hat{\mathcal{T}}]_{\beta}^{\gamma}$.
  5. If $\overline{\overline{E}}$ is an $n \times p$ matrix, then $\hat{L}_{AE} = \hat{L}_A\hat{L}_E$.
  6. If $m = n$, then $\hat{L}_{I_n} = \overline{\overline{I}}_{F^n}$.
- Theorem 2.16: matrix multiplication is associative, $\overline{\overline{A}}(\overline{\overline{BC}}) = (\overline{\overline{AB}})\overline{\overline{C}}$
- Definition: A function $\hat{U} : \mathcal{W} \to \mathcal{V}$ is said to be an **inverse** of $\hat{T} : \mathcal{V} \to \mathcal{W}$ if

$$\hat{T}\hat{U} = \hat{I}_{\mathcal{W}} \quad \text{and} \quad \hat{U}\hat{T} = \hat{I}_{\mathcal{V}} \tag{8}$$

- Definition: If $\hat{T}$ has an inverse, then $\hat{T}$ is *invertible.*
- Theorem 2.17: $\hat{T}^{-1} : \mathcal{W} \to \mathcal{V}$ is linear.
- Definition: $\overline{\overline{A}}_{n\times n}$ is invertible $\exists\,\overline{\overline{B}}_{n\times n}$ s.t. $\overline{\overline{AB}} = \overline{\overline{BA}} = \overline{\overline{I}}$.
- Theorem 2.18: $\hat{T}$ is invertible iff $[\hat{T}]_{\beta}^{\gamma}$ is invertible, $[\hat{T}^{-1}]_{\gamma}^{\beta} = ([\hat{T}]_{\beta}^{\gamma})^{-1}$.
- Definition: $\mathcal{V}$ is **isomorphic** to $\mathcal{W}$ if there exists a linear transformation $\hat{T} : \mathcal{V} \to \mathcal{W}$ that is invertible.
- Theorem 2.19: $\mathcal{V}$ is isomorphic to $\mathcal{W}$ iff $dim(\mathcal{V}) = dim(\mathcal{W})$.
- Theorem 2.20: The function $\Phi : \mathcal{L}(\mathcal{V}, \mathcal{W}) \to \overline{\overline{M}}_{m\times n}(F)$, defined by $\Phi(\hat{T}) = [\hat{T}]_{\beta}^{\gamma}$ for $\hat{T} \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ is an *isomorphism.*
- Definition: standard representation of $\mathcal{V}$ with respect to β is the function $\phi_\beta : \mathcal{V} \to F^n$, defined by $\phi_\beta(\vec{x}) = [\vec{x}]_\beta$ for each $\vec{x} \in \mathcal{V}$.
- Theorem 2.21: For any finite-dimensional vector space $\mathcal{V}$ with ordered basis β, ϕ_β is an isomorphism.
- Theorem 2.22: $\overline{\overline{Q}} = [I_{\mathcal{V}}]_{\beta'}^{\beta}$ is invertible, for any $\vec{v} \in \mathcal{V}$, $[\vec{v}]_\beta = \overline{\overline{Q}}[\vec{v}]_{\beta'}$, i.e., compared to the bases: $\vec{x}'_j = \sum_{i=1}^{n} Q_{ij}\vec{x}_i$.
- Theorem 2.23: linear operator $[\hat{T}]_{\beta'} = \overline{\overline{Q}}^{-1}[\hat{T}]_\beta\overline{\overline{Q}}$
- Definition: $\overline{\overline{B}}_{n\times n}$ is **similar** to $\overline{\overline{A}}_{n\times n}$ if $\exists$ an invertible matrix $\overline{\overline{Q}}$ s.t. $\overline{\overline{B}} = \overline{\overline{Q}}^{-1}\overline{\overline{AQ}}$.