Linear Algebra, EE 10810/EECS 205004

Note 2.5 – 2.6

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- 1st Exam on Oct. 30th, (10:10AM - 1:00 PM, Friday), covering Chap. 1 and Chap. 2.

- **Assignment**:

  1. Prove that if $\overline{\overline{A}}$ and $\overline{\overline{B}}$ are *similar* $n \times n$ matrices, then $\mathrm{tr}(\overline{\overline{A}}) = \mathrm{tr}(\overline{\overline{B}})$.

  2. Let $\mathcal{V} = \mathcal{R}^2$, is $f(x, y) = (2x, 4y)$ a **linear functional** on $\mathcal{V}$?

  3. Let $\mathcal{V} = P_2\mathcal{R}$ with the bases $\beta = \{1, x, x^2\}$, find explicit formulas for vectors of the dual basis β^* for $\mathcal{V}^*$.

From Scratch !!

- Theorem 2.14: For each $\vec{u} \in \mathcal{V}$,
$$[\hat{\mathcal{T}}(\vec{u})]_\gamma = [\hat{\mathcal{T}}]_\beta^\gamma \, [\vec{u}]_\beta \tag{1}$$
- Theorem 2.15:
  1. $[\hat{L}_A]_\beta^\gamma = \overline{\overline{A}}$.
  2. $\hat{L}_A = \hat{L}_B$ iff $\overline{\overline{A}} = \overline{\overline{B}}$.
  3. $\hat{L}_{A+B} = \hat{L}_A + \hat{L}_B$ and $\hat{L}_{aA} = a\hat{L}_A$ for all $a \in F$.
  4. If $\hat{\mathcal{T}} : F^n \to F^m$ is linear, $\exists!$ an $m \times n$ matrix $\overline{\overline{C}}$ s.t. $\hat{\mathcal{T}} = \hat{L}_C$. In fact $\overline{\overline{C}} = [\hat{\mathcal{T}}]_\beta^\gamma$.
  5. If $\overline{\overline{E}}$ is an $n \times p$ matrix, then $\hat{L}_{AE} = \hat{L}_A \hat{L}_E$.
  6. If $m = n$, then $\hat{L}_{I_n} = \overline{\overline{I}}_{F^n}$.
- Theorem 2.16: matrix multiplication is associative, $\overline{\overline{A}}(\overline{\overline{BC}}) = (\overline{\overline{AB}})\overline{\overline{C}}$
- Definition: A function $\hat{U} : \mathcal{W} \to \mathcal{V}$ is said to be an **inverse** of $\hat{T} : \mathcal{V} \to \mathcal{W}$ if
$$\hat{T}\hat{U} = \hat{I}_{\mathcal{W}} \quad \text{and} \quad \hat{U}\hat{T} = \hat{I}_{\mathcal{V}} \tag{2}$$
- Definition: If $\hat{T}$ has an inverse, then $\hat{T}$ is *invertible.*
- Theorem 2.17: $\hat{T}^{-1} : \mathcal{W} \to \mathcal{V}$ is linear.
- Definition: $\overline{\overline{A}}_{n\times n}$ is invertible $\exists\, \overline{\overline{B}}_{n\times n}$ s.t. $\overline{\overline{AB}} = \overline{\overline{BA}} = \overline{\overline{I}}$.
- Theorem 2.18: $\hat{T}$ is invertible iff $[\hat{T}]_\beta^\gamma$ is invertible, $[\hat{T}^{-1}]_\gamma^\beta = ([\hat{T}]_\beta^\gamma)^{-1}$.
- Definition: $\mathcal{V}$ is **isomorphic** to $\mathcal{W}$ if there exists a linear transformation $\hat{T} : \mathcal{V} \to \mathcal{W}$ that is invertible.
- Theorem 2.19: $\mathcal{V}$ is isomorphic to $\mathcal{W}$ iff $dim(\mathcal{V}) = dim(\mathcal{W})$.
- Theorem 2.20: The function $\Phi : \mathcal{L}(\mathcal{V}, \mathcal{W}) \to \overline{\overline{M}}_{m\times n}(F)$, defined by $\Phi(\hat{T}) = [\hat{T}]_\beta^\gamma$ for $\hat{T} \in \mathcal{L}(\mathcal{V}, \mathcal{W})$ is an *isomorphism.*
- Definition: standard representation of $\mathcal{V}$ with respect to β is the function $\phi_\beta : \mathcal{V} \to F^n$, defined by $\phi_\beta(\vec{x}) = [\vec{x}]_\beta$ for each $\vec{x} \in \mathcal{V}$.
- Theorem 2.21: For any finite-dimensional vector space $\mathcal{V}$ with ordered basis β, ϕ_β is an isomorphism.

- Theorem 2.22: $\overline{\overline{Q}} = [I_{\mathcal{V}}]_{\beta'}^\beta$ is invertible, for any $\vec{v} \in \mathcal{V}$, $[\vec{v}]_\beta = \overline{\overline{Q}}[\vec{v}]_{\beta'}$, i.e., compared to the bases: $\vec{x}'_j = \sum_{i=1}^n Q_{ij}\vec{x}_i$.
- Theorem 2.23: linear operator $[\hat{T}]_{\beta'} = \overline{\overline{Q}}^{-1}[\hat{T}]_\beta \overline{\overline{Q}}$
- Definition: $\overline{\overline{B}}_{n\times n}$ is **similar** to $\overline{\overline{A}}_{n\times n}$ if $\exists$ an invertible matrix $\overline{\overline{Q}}$ s.t. $\overline{\overline{B}} = \overline{\overline{Q}}^{-1}\overline{\overline{AQ}}$.

- Definition: the **dual space** of $\mathcal{V}$ is the vector space $\mathcal{L}(\mathcal{V}, F)$, denoted by $\mathcal{V}^*$.
- Theorem 2.24: Let $\beta = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ be the ordered basis of $\mathcal{V}$, and we can find $\beta^* = \{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_n\}$ as an ordered basis for $\mathcal{V}^*$, for any $f \in \mathcal{V}^*$, we have
$$f = \sum_{i=1}^n f(\vec{x}_i)\, f_i. \tag{3}$$
- Definition: the ordered bases $\beta^* = \{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_n\}$ of $\mathcal{V}^*$ that satisfies $f_i(\vec{x}_j) = \delta_{ij}$ is called the **dual basis** of β.
- Theorem 2.25: for any linear transformation $\hat{T} : \mathcal{V} \to \mathcal{W}$, the mapping $\hat{T}^t : \mathcal{W}^* \to \mathcal{V}^*$ definted by $\hat{T}^t(g) = g\hat{T}$ for all $g \in \mathcal{W}^*$ is a linear transformation with the property hat
$$[\hat{T}^t]_{\gamma^*}^{\beta^*} = ([\hat{T}]_\beta^\gamma)^t \tag{4}$$
- Definition: for a $\vec{x} \in \mathcal{V}$, the **linear functional** on $\mathcal{V}^*$ is defined as $\hat{x} : \mathcal{V}^* \to F$ by $\hat{x} = f(x)$.
- Lemma: If $\hat{x}(f) = 0$ for all $f \in \mathcal{V}^*$, then $\vec{x} = 0$.
- Theorem 2.26: $\psi : \mathcal{V} \to \mathcal{V}^{**}$ by $\psi(\vec{x}) = \hat{x}$ is an isomorphism.
- Corollary: every ordered basis for $\mathcal{V}^*$ is the dual basis for some basis for $\mathcal{V}$.