Linear Algebra, EE 10810/EECS 205004

Note 3.2 – 3.4

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- Next Quiz on Nov. 13th, Friday.

- **Assignment**:

1. Express the following invertible matrix, as a product of elementary matrices:

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} \tag{1}$$

2. Let $\overline{\overline{A}}$ be an $m \times n$ matrix with *rank* m and $\overline{\overline{B}}$ be an $n \times p$ matrix with *rank* n. Determine the *rank* of $\overline{\overline{AB}}$.

3. Determine which of the following systems of linear equations has solution(s), and if yes, find the solution(s).

(a)

$$\begin{array}{rcrcrcrcl} x_1 & + & x_2 & + & 3x_3 & - & x_4 & = & 0 \\ x_1 & + & x_2 & + & x_3 & + & x_4 & = & 1 \\ x_1 & - & 2x_2 & + & x_3 & - & x_4 & = & 1 \\ 4x_1 & + & x_2 & + & 8x_3 & - & x_4 & = & 0 \end{array} \tag{2}$$

(b)

$$\begin{array}{rcrcrcrcrl} 3x_1 & - & x_2 & + & x_3 & - & x_4 & + & 2x_5 = & 5 \\ x_1 & - & x_2 & - & x_3 & - & 2x_4 & - & x_5 = & 2 \\ 5x_1 & - & 2x_2 & + & x_3 & - & 3x_4 & + & 3x_5 = & 10 \\ 2x_1 & - & x_2 & & & - & 2x_4 & + & x_5 = & 5 \end{array} \tag{3}$$

From Scratch !!

- **Section 3.1-3.2**
- Theorem 3.1: There exists an $m \times m$ ($n \times n$) elementary matrix $\overline{\overline{E}}$, such that $\overline{\overline{B}} = \overline{\overline{E A}}_{m\times n}$ (or $\overline{\overline{B}} = \overline{\overline{A}}_{m\times n}\overline{\overline{E}}$)
- Theorem 3.2: Elementary matrices are invertible, and the inverse oof an elementary matrix is an elementary matrix of the same type.
- Definition: If $\overline{\overline{A}} \in \overline{\overline{M}}_{m\times n}(F)$, the **rank** of $\overline{\overline{A}}$, denoted $rank(\overline{\overline{A}})$, is the rank of the linear transformation $\hat{L}_A : F^n \to F^m$.
- Corollary of Theorem 2.18: an $n \times n$ matrix is invertible if and only if its rank is n.
- Theorem 3.3: $rank(\hat{T}) = rank([\hat{T}]_\beta^\gamma)$
- Theorem 3.4: If $\overline{\overline{P}}_{m\times m}$ and $\overline{\overline{Q}}_{n\times n}$ are invertible matrices, then
  1. $rank(\overline{\overline{A}}_{m\times n}\overline{\overline{Q}}) = rank(\overline{\overline{A}})$,
  2. $rank(\overline{\overline{P A}}_{m\times n}) = rank(\overline{\overline{A}})$,
  3. $rank(\overline{\overline{P A}}_{m\times n}\overline{\overline{Q}}) = rank(\overline{\overline{A}})$,
- Corollary: Elementary row and column operation on a matrix are *rank-preserving*.
- Theorem 3.5: The rank of any matrix equals the maximum number of its linearly independent columns;
- Theorem 3.5: The rank of a matrix is the dimension of the subspace generated by its columns.
- Theorem 3.6: Let $\overline{\overline{A}}_{m\times n}$ has the rank r. Then $r \le m$, $r \le n$, and $\overline{\overline{A}}$ can be transformed into

$$\overline{\overline{D}} = \begin{pmatrix} \overline{\overline{I}}_r & \overline{\overline{O}}_1 \\ \overline{\overline{O}}_2 & \overline{\overline{O}}_3 \end{pmatrix} \tag{4}$$

- Corollary 2: $rank(\overline{\overline{A}}^t) = rank(\overline{\overline{A}})$
- Corollary 3: Every invertible matrix is a product of elementary matrices.
- Theorem 3.7: Let $\hat{T} : \mathcal{V} \to \mathcal{W}$ and $\hat{U} : \mathcal{W} \to \mathcal{Z}$ be linear transformation:
  1. $rank(\hat{U}\hat{T}) \le rank(\hat{U})$
  2. $rank(\hat{U}\hat{T}) \le rank(\hat{T})$
  3. $rank(\overline{\overline{AB}}) \le rank(\overline{\overline{A}})$; $\quad rank(\overline{\overline{AB}}) \le rank(\overline{\overline{B}})$
- Augmented Matrix:
- **Section 3.3**
- Systems of Linear Equations: $\overline{\overline{A}}\,\vec{x} = \vec{b}$
- the solution set S is called *consistent* if its solution set is *non-empty*.
- Definition: Homogeneous if $\vec{b} = \vec{0}$.
- Theorem 3.8: Let K denote the set of all solutions to $\overline{\overline{A}}\,\vec{x} = \vec{0}$. Then, $K = N(\hat{L}_A)$, a subspace of F^n of dimension $n - rank(\overline{\overline{A}})$
- Theorem 3.9: Let K_H be the solution set of the homogenous system $\overline{\overline{A}}\,\vec{x} = \vec{0}$, then for any solution s to $\overline{\overline{A}}\,\vec{x} = \vec{b}$

$$K = \{s\} + K_H = \{s + k : k \in K_H\} \tag{5}$$

- Theorem 3.10: If $\overline{\overline{A}}_{n\times n}\,\vec{x} = \vec{b}$ is invertible, then the system has **exactly one solution**.
- Theorem 3.11: The system is consistent iff $rank(\overline{\overline{A}}) = rank(\overline{\overline{A}}|\vec{b})$.
- **Section 3.4**
- Definition: equivalent
- Theorem 3.13: $\overline{\overline{A}}_{m\times n}\,\vec{x} = \vec{b}$ is equivalent to $(\overline{\overline{C}}_{m\times m}\overline{\overline{A}}_{m\times n})\,\vec{x} = \overline{\overline{C}}_{m\times m}\vec{b}$ with an invertible matrix $\overline{\overline{C}}_{m\times m}$.
- Definition: Reduced Row Echelon form
- Theorem 3.14: **Gaussian elimination transforms any matrix into its reduced row echelon form.**
- Theorem 3.15: $rank(\overline{\overline{A}}) = rank(\overline{\overline{A}}|\vec{b})$ and the general solution has the form

$$s = s_0 + t_1\,u_1 + t_2\,u_2 + \ldots + t_{n-r}\,u_{n-r}, \tag{6}$$

  where $rank(\overline{\overline{A}}) = r$, $\{u_1, u_2, \ldots, u_{n-r}\}$ is a basis for the solution set of the corresponding homogeneous system, and s_0 is a solution to the original system.
- Theorem 3.16: The reduced row echelon form of a matrix is *unique*.