Linear Algebra, EE 10810/EECS 205004

Note 4.2 – 4.4

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- Next Quiz on Nov. 25th, Wednesday.

- **Assignment**:

1. Compute the determinants of the following matrix in $\overline{\overline{M}}_{4\times 4}(R)$:

$$\begin{pmatrix} 1 & 0 & -2 & 3 \\ -3 & 1 & 1 & 2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{pmatrix} \tag{1}$$

2. Use row operations to simplify and compute these determinants.

$$\det\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix}, \quad \det\begin{bmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 103 & 203 & 303 \end{bmatrix}, \quad \det\begin{bmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{bmatrix}. \tag{2}$$

3. A matrix $\overline{\overline{Q}} \in \overline{\overline{M}}_{n\times n}(R)$ is called *orthogonal* if $\overline{\overline{Q}}\,\overline{\overline{Q}}^t = \overline{\overline{I}}$. Prove that if $\overline{\overline{Q}}$ is orthogonal, then $det(\overline{\overline{Q}}) = \pm 1$.

4. Find the determinant of the symmetric Pascal matrices

$$\det = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}, \quad \det = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & \mathbf{19} \end{bmatrix}. \tag{3}$$

From Scratch !!

- Properties of Determinant
  - The determinant changes sign when two rows (or two columns) are exchanged.
  - $det(\overline{\overline{A}}\,\overline{\overline{B}}) = det(\overline{\overline{A}}) \cdot det(\overline{\overline{B}})$
  - $det(\overline{\overline{A}}) = det(\overline{\overline{A}}^t)$
  - $det(\overline{\overline{I}}) = 1$
  - The determinants equal *Volumes*.
  - If two rows of $\overline{\overline{A}}$ are equal, then $det(\overline{\overline{A}}) = 0$.
  - Subtracting a multiple of one row from another row leaves $det(\overline{\overline{A}})$ unchanged.

$$\begin{vmatrix} a & b \\ c - m\,a & d - m\,b \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \tag{4}$$

  - A matrix with a row of zeros has $det(\overline{\overline{A}}) = 0$.
  - If $\overline{\overline{A}}$ is *singular* then $det(\overline{\overline{A}}) = 0$.

- **Section 4.2: Determinants of Order n**
  1. Pivot formula: multiplication of n pivots (times 1 or -1)
  2. Big formula: add up $n!$ terms (times 1 or -1)
  3. Cofactor formula: combine n smaller determinants (times 1 or -1)
- Theorem 4.4: cofactor expansion

$$det(\overline{\overline{A}}) = \sum_{j=1}^{n} (-1)^{i+j} \overline{\overline{A}}_{ij} \cdot det(\tilde{A}_{ij}), \tag{5}$$

  for any integer $1 \leq i \leq n$.
- Theorem 4.3: n-linear function

$$det \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_{r-1} \\ u + k\,v \\ a_{r+1} \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix} = det \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_{r-1} \\ u \\ a_{r+1} \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix} + k\,det \begin{pmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_{r-1} \\ v \\ a_{r+1} \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{pmatrix} \tag{6}$$

- Theorem 4.5: $\overline{\overline{B}}$ is obtained by interchanging any row of $\overline{\overline{A}}$, then $det(\overline{\overline{B}}) = -det(\overline{\overline{A}})$.
- Theorem 4.6: $\overline{\overline{B}}$ is obtained by adding a multiple of one row of $\overline{\overline{A}}$ to another row of $\overline{\overline{A}}$, then $det(\overline{\overline{B}}) = det(\overline{\overline{A}})$.

- **Section 4.3-4.4: Properties and Summary of Determinants**
- Cramer's Rule:
- n-dimensional volume: