Linear Algebra, EE 10810/EECS 205004

Note 4.4 − 5.1

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- Next Quiz on Dec. 2nd, Wednesday.

- **Assignment**:

1. Use Cramer's rule with ratios $det(\overline{\overline{B}}_i)/det(\overline{\overline{A}})$ to solve $\overline{\overline{A}}\,\vec{x} = \vec{b}$. Also find the inverse matrix $(\overline{\overline{A}})^{-1} = \overline{\overline{C}}^t/det(\overline{\overline{A}})$.

$$\overline{\overline{A}}\,\vec{x} = \vec{b} \qquad \text{is} \qquad \begin{pmatrix} 2 & 6 & 2 \\ 1 & 4 & 2 \\ 5 & 9 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{1}$$

2. A box has edges from $(0,0,0)$ to $(3,1,1)$ and $(1,3,1)$ and $(1,1,3)$. Find its volume and the area of each parallelogram face using $|\vec{u} \times \vec{v}|$.

3. For the matrix

$$\overline{\overline{A}} = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix} \tag{2}$$

   (a) Determine all the eigenvalues of $\overline{\overline{A}}$.

   (b) For each eigenvalues λ of $\overline{\overline{A}}$, find the set of eigenvectors corresponding to λ.

   (c) If possible, find a basis for R^3 consisting of eigenvectors of $\overline{\overline{A}}$.

   (d) If successful in finding such a basis, determine an invertible matrix $\overline{\overline{Q}}$ and a diagonal matrix $\overline{\overline{D}}$ such that $\overline{\overline{Q}}^{-1}\overline{\overline{AQ}} = \overline{\overline{D}}$.

From Scratch !!

- Cramer's Rule for solving $\overline{\overline{A}}\,\vec{x} = \vec{b}$:

$$x_i = \frac{det(\overline{\overline{B}}_i)}{det(\overline{\overline{A}})}, \tag{3}$$

where the matrix $\overline{\overline{B}}_i$ has the j-th column of $\overline{\overline{A}}$ replaced by the vector $\vec{b}$.

- Cramer's Rule for finding the inverse of the matrix $\overline{\overline{A}}$:

$$(\overline{\overline{A}})^{-1}_{ij} = \frac{\overline{\overline{C}}_{ji}}{det(\overline{\overline{A}})}, \tag{4}$$

where $\overline{\overline{C}}$ is the cofactor matrix for $\overline{\overline{A}}$.

- Area of triangle: $\frac{1}{2}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$

- Volume of box: $\left| det \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} \right|$

- Cross product: $\vec{u} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

- Triple product: $(\vec{u} \times \vec{v}) \cdot \vec{w} = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$

- n-dimensional volume: $\left| det(\overline{\overline{A}}_{n\times n}) \right|$

- Section 5.1: Eigenvalues and Eigenvectors

$$\hat{T}(\vec{v}) = \lambda\,\vec{v} \qquad \text{or} \qquad \overline{\overline{A}}\,\vec{v} = \lambda\,\vec{v} \tag{5}$$

- Definition: Diagonalizable
- Theorem 5.1: $\hat{T}$ is diagonalizable iff there exists an ordered basis β for $\mathcal{V}$ consisting of eigenvectors of $\hat{T}$.
- Theorem 5.2: The scalar λ is an eigenvalue of $\overline{\overline{A}}$ iff $det(\overline{\overline{A}} - \lambda\overline{\overline{I}}_n) = 0$.
- Definition: characteristic polynomial of $\overline{\overline{A}}$:

$$f(t) = det(\overline{\overline{A}} - t\overline{\overline{I}}_n) \tag{6}$$

- Theorem 5.3:
  1. The characteristic polynomial of $\overline{\overline{A}}$ is a polynomial of degree n with leading coefficient $(-1)^n$.
  2. $\overline{\overline{A}}$ has at most n distinct eigenvalues.
- Theorem 5.4: A vector $\vec{v} \in \mathcal{V}$ is an eigenvector of $\hat{T}$ corresponding to λ iff $\vec{v} \neq 0$ and $\vec{v} \in N(\hat{T} - \lambda\overline{\overline{I}})$.