Linear Algebra, EE 10810/EECS 205004

Note 5.2 – 5.3

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- Next Quiz on Dec. 2nd, Wednesday.

- **Assignment**:

1. Prove Theorem 5.5: Let $\hat{T}$ be. a linear operator on a vector space $\mathcal{V}$, and let $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ be distinct eigenvalues of $\hat{T}$. If $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ are the corresponding eigenvectors of $\hat{T}$, then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is linearly independent.

2. For the following linear operators $\hat{T}$ on a vector space $\mathcal{V}$, test $\hat{T}$ for diagonalizability, and if $\hat{T}$ is diagonalizable, find a basis β for $\mathcal{V}$ such that $[\hat{T}]_\beta$ is a diagonal matrix:

   (a) $\mathcal{V} = P_3(\mathcal{R})$ and $\hat{T}$ is defined by $\hat{T}(f(x)) = f'(x) + f''(x)$, respectively.

   (b) $\mathcal{V} = \mathcal{R}^3$ and $\hat{T}$ is defined by

$$\hat{T}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_2 \\ -a_2 \\ 2a_3 \end{pmatrix} \tag{1}$$

3. Let $\overline{\overline{A}}$ be a $n \times n$ matrix that is similar to an upper triangular matrix and has the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ with corresponding multiplicities $m_1, m_2, \ldots, m_k$. Prove that

   (a) $\text{tr}(\overline{\overline{A}}) = \sum_{i=1}^{k} m_i \, \lambda_i$

   (b) $\det(\overline{\overline{A}}) = (\lambda_1)^{m_1} (\lambda_2)^{m_2} \ldots (\lambda_k)^{m_k}$

From Scratch !!

- Section 5.1: Eigenvalues and Eigenvectors

$$\hat{T}(\vec{v}) = \lambda\,\vec{v} \qquad \text{or} \qquad \overline{\overline{A}}\,\vec{v} = \lambda\,\vec{v} \tag{2}$$

- Definition: Diagonalizable
- Theorem 5.1: $\hat{T}$ is diagonalizable iff there exists an ordered basis β for $\mathcal{V}$ consisting of eigenvectors of $\hat{T}$.
- Theorem 5.2: The scalar λ is an eigenvalue of $\overline{\overline{A}}$ iff $det(\overline{\overline{A}} - \lambda\,\overline{\overline{I}}_n) = 0$.
- Definition: characteristic polynomial of $\overline{\overline{A}}$:

$$f(\lambda) = det(\overline{\overline{A}} - \lambda\,\overline{\overline{I}}_n) \tag{3}$$

- Theorem 5.3:
  1. The characteristic polynomial of $\overline{\overline{A}}$ is a polynomial of degree n with leading coefficient $(-1)^n$.
  2. $\overline{\overline{A}}$ has at most n distinct eigenvalues.
- Theorem 5.4: A vector $\vec{v} \in \mathcal{V}$ is an eigenvector of $\hat{T}$ corresponding to λ iff $\vec{v} \neq 0$ and $\vec{v} \in N(\hat{T} - \lambda\overline{\overline{I}})$.

- Section 5.2: Diagonalizability
- Theorem 5.5: Let $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ be distinct eigenvalues of $\hat{T}$. If $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ are the corresponding eigenvectors of $\hat{T}$, then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is linearly independent.
- Corollary: If $\hat{T}$ has n distinct eigenvalues, then $\hat{T}$ is diagonalizable.
- Theorem 5.6: The characteristic polynomial of any diagonalizable linear operator splits.
- Definition: The (algebra) multiplicity of λ is the largest positive integer k for which $(t-\lambda)^k$ is a factor of $f(t)$.
- Definition: The set $E_\lambda = \{\vec{x} \in \mathcal{V} : \hat{T}(\vec{x}) = \lambda\,\vec{x}\} \equiv N(\hat{T} - \lambda\,\hat{I}_v)$ is called the eigenspace of $\hat{T}$ corresponding to the eigenvalue λ.
- Theorem 5.7: Let λ be an eigenvalue of $\hat{T}$ having multiplicity m, then $1 \leq dim(E_\lambda) \leq m$.
- Theorem 5.8: Let S_i, $i = 1, 2, \ldots, k$, be a finite linearly independent subset of the eigenspace E_{λ_i}, then $S = S_1 \cup S_2 \cup \ldots \cup S_k$ is a linearly independent subset of $\mathcal{V}$.
- Theorem 5.9:
  1. $\hat{T}$ is diagonalizable iff the multiplicity of λ_i is equal to $dim(E_{\lambda_i})$ for all i.
  2. If $\hat{T}$ is diagonalizable and β_i is an ordered basis for E_{λ_i}, for each i, then $\beta = \beta_1 \cup \beta_2 \cup \ldots \cup \beta_k$ is an ordered basis for $\mathcal{V}$ consisting of eigenvectors of $\hat{T}$.
- Test of Diagonalization:
- Direct Sum
- Section 5.3: Matrix limits
- Definition: The sequence $\{\overline{\overline{A}}_1, \overline{\overline{A}}_2, \ldots\}$ is said to be converge to the matrix $\overline{\overline{L}}$, called the limit of the sequence, if

$$\lim_{m\to\infty} (\overline{\overline{A}}_m)_{ij} = \overline{\overline{L}}_{ij} \tag{4}$$

- Theorem 5.12: For any $\overline{\overline{P}}$ and $\overline{\overline{Q}}$,

$$\lim_{m\to\infty} \overline{\overline{P}}\,\overline{\overline{A}}_m = \overline{\overline{PL}} \qquad \text{and} \qquad \lim_{m\to\infty} \overline{\overline{A}}_m\,\overline{\overline{Q}} = \overline{\overline{LQ}} \tag{5}$$

- Corollary: If $\lim_{m\to\infty} \overline{\overline{A}}^m = \overline{\overline{L}}$, then, for any invertible matrix $\overline{\overline{Q}}$,

$$\lim_{m\to\infty} (\overline{\overline{Q}}\,\overline{\overline{A}}\,\overline{\overline{Q}}^{-1})^m = \overline{\overline{Q}}\,\overline{\overline{L}}\,\overline{\overline{Q}}^{-1} \tag{6}$$

- Theorem 5.13: $\lim_{m\to\infty} \overline{\overline{A}}^m$ exists iff
  1. Every eigenvalue of $\overline{\overline{A}}$ is contained in S.
  2. If 1 is an eigenvalue of $\overline{\overline{A}}$, then the dimension of the eigenspace corresponding 1 equlas the multiplicity of 1 as an eigenvalue of $\overline{\overline{A}}$.