Linear Algebra, EE 10810/EECS 205004

Note 5.3

Ray-Kuang Lee[1]

[1] *Room 911, Delta Hall, National Tsing Hua University, Hsinchu, Taiwan.*
Tel: +886-3-5742439; E-mail: rklee@ee.nthu.edu.tw

(Dated: Fall, 2020)

- Next Quiz on Dec. 9th, Wednesday.

- **Assignment**:

1. Find the general solution to the system of differential equations:

$$\frac{dx_1}{dt} = x_1 + x_3 \quad (1)$$

$$\frac{dx_2}{dt} = x_2 + 2x_3 \quad (2)$$

$$\frac{dx_3}{dt} = 2x_3 \quad (3)$$

2. Determine whether $\lim_{m\to\infty} \overline{\overline{A}}^m$ exists, and compute the limit if it exists.

$$\begin{pmatrix} -\frac{1}{2} - 2i & 4i & \frac{1}{2} + 5i \\ 1 + 2i & -3i & -1 - 4i \\ -1 - 2i & 4i & 1 + 5i \end{pmatrix} \quad (4)$$

3. Find 2×2 matrices $\overline{\overline{A}}$ and $\overline{\overline{B}}$ having real entries such that $\lim_{m\to\infty} \overline{\overline{A}}^m$, $\lim_{m\to\infty} \overline{\overline{B}}^m$, and $\lim_{m\to\infty} (\overline{\overline{AB}})^m$ all exist, but

$$\lim_{m\to\infty} (\overline{\overline{AB}})^m \neq (\lim_{m\to\infty} \overline{\overline{A}}^m)(\lim_{m\to\infty} \overline{\overline{B}}^m) \quad (5)$$

From Scratch !!

- Section 5.2: Diagonalizability
- Theorem 5.5: Let $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ be distinct eigenvalues of $\hat{T}$. If $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ are the corresponding eigenvectors of $\hat{T}$, then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is linearly independent.
- Corollary: If $\hat{T}$ has n distinct eigenvalues, then $\hat{T}$ is diagonalizable.
- Theorem 5.6: The characteristic polynomial of any diagonalizable linear operator splits.
- Definition: The (algebra) multiplicity of λ is the largest positive integer k for which $(t-\lambda)^k$ is a factor of $f(t)$.
- Definition: The set $E_\lambda = \{\vec{x} \in \mathcal{V} : \hat{T}(\vec{x}) = \lambda\,\vec{x}\} \equiv N(\hat{T} - \lambda\,\hat{I}_v)$ is called the eigenspace of $\hat{T}$ corresponding to the eigenvalue λ.
- Theorem 5.7: Let λ be an eigenvalue of $\hat{T}$ having multiplicity m, then $1 \leq dim(E_\lambda) \leq m$.
- Theorem 5.8: Let S_i, $i = 1, 2, \ldots, k$, be a finite linearly independent subset of the eigenspace E_{λ_i}, then $S = S_1 \cup S_2 \cup \ldots \cup S_k$ is a linearly independent subset of $\mathcal{V}$.
- Theorem 5.9:
  1. $\hat{T}$ is diagonalizable iff the multiplicity of λ_i is equal to $dim(E_{\lambda_i})$ for all i.
  2. If $\hat{T}$ is diagonalizable and β_i is an ordered basis for E_{λ_i}, for each i, then $\beta = \beta_1 \cup \beta_2 \cup \ldots \cup \beta_k$ is an ordered basis for $\mathcal{V}$ consisting of eigenvectors of $\hat{T}$.
- Test of Diagonalization:
- Systems of Differential Equations:

$$\frac{dx_1}{dt} = 3x_1 + x_2 + x_3 \tag{6}$$
$$\frac{dx_2}{dt} = 2x_1 + 4x_2 + 2x_3 \tag{7}$$
$$\frac{dx_3}{dt} = -x_1 - x_2 + x_3 \tag{8}$$

- ~~Direct Sum~~ (skip)
- Section 5.3: Matrix limits
- Definition: The sequence $\{\overline{\overline{A}}_1, \overline{\overline{A}}_2, \ldots\}$ is said to be converge to the matrix $\overline{\overline{L}}$, called the limit of the sequence, if

$$\lim_{m\to\infty} (\overline{\overline{A}}_m)_{ij} = \overline{\overline{L}}_{ij} \tag{9}$$

- Theorem 5.12: For any $\overline{\overline{P}}$ and $\overline{\overline{Q}}$,

$$\lim_{m\to\infty} \overline{\overline{P}}\,\overline{\overline{A}}_m = \overline{\overline{PL}} \qquad \text{and} \qquad \lim_{m\to\infty} \overline{\overline{A}}_m\,\overline{\overline{Q}} = \overline{\overline{LQ}} \tag{10}$$

- Corollary: If $\lim_{m\to\infty} \overline{\overline{A}}^m = \overline{\overline{L}}$, then, for any invertible matrix $\overline{\overline{Q}}$,

$$\lim_{m\to\infty} (\overline{\overline{Q}}\,\overline{\overline{A}}\,\overline{\overline{Q}}^{-1})^m = \overline{\overline{Q}}\,\overline{\overline{L}}\,\overline{\overline{Q}}^{-1} \tag{11}$$

- Theorem 5.13: $\lim_{m\to\infty} \overline{\overline{A}}^m$ exists iff
  1. Every eigenvalue of $\overline{\overline{A}}$ is contained in S.
  2. If 1 is an eigenvalue of $\overline{\overline{A}}$, then the dimension of the eigenspace corresponding 1 equlas the multiplicity of 1 as an eigenvalue of $\overline{\overline{A}}$.
- ~~Markov chain~~ (skip)