Linear Algebra, EE 10810/EECS 205004

Note 5.4

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(Dated: Fall, 2020)

- Next Quiz on Dec. 9th, Wednesday.

- **Assignment**:

1. For a three-state Markov chain with the initial probability vector $\begin{pmatrix} 0.3 \\ 0.3 \\ 0.4 \end{pmatrix}$, compute the proportions of objects in each state after two stages and the eventual proportions of objections in each state by determining the fixed probability vector:

$$\begin{pmatrix} 0.6 & 0.1 & 0.1 \\ 0.1 & 0.9 & 0.2 \\ 0.3 & 0 & 0.7 \end{pmatrix} \tag{1}$$

2. For a matrix $\overline{\overline{A}} = \begin{pmatrix} 3 & -10 \\ 1 & -4 \end{pmatrix}$, find the solutions of $\vec{x}$ to the following system of differential equations:

$$\frac{d}{dt}\vec{x} = \overline{\overline{A}}\vec{x}. \tag{2}$$

3. Let $\hat{T}$ be a linear operator on a vector space $\mathcal{V}$, and $\vec{v}$ be a nonzero vector in $\mathcal{V}$, and let $\hat{T}$ be the $\hat{T}$-cyclic subspace of $\mathcal{V}$ generated by $\vec{v}$. Prove that
   (a) $\mathcal{W}$ is $\hat{T}$-invariant.
   (b) Any $\hat{T}$-invariant subspace of $\mathcal{V}$ containing $\vec{v}$ also contains $\mathcal{W}$, i.e., $\mathcal{W}$ is the "smallest" $\hat{T}$-invariant subspace of $\mathcal{V}$ containing $\vec{v}$.

4. Let $\overline{\overline{A}}$ denote a $k \times k$ matrix

$$\begin{pmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \dots & 0 & -a_2 \\ . & . & \dots & . & \\ . & . & \dots & . & \\ . & . & \dots & . & \\ 0 & 0 & \dots & 0 & -a_{k-1} \\ 0 & 0 & \dots & 1 & -a_{k-1} \end{pmatrix} \tag{3}$$

where $a_0, a_1, \dots a_{k-1}$ are arbitrary scalars. Prove that the characteristic polynomial of $\overline{\overline{A}}$ is

$$(-1)^k(a_0 + a_1\lambda + \dots + a_{k-1}\lambda^{k-1} + \lambda^k) \tag{4}$$

Hint: Use mathematical induction on k, expanding the determinant along the first row.

From Scratch !!

- Section 5.3: Matrix limits
- Definition: The sequence $\{\overline{\overline{A}}_1, \overline{\overline{A}}_2, \ldots\}$ is said to be converge to the matrix $\overline{\overline{L}}$, called the limit of the sequence, if

$$\lim_{m\to\infty} (\overline{\overline{A}}_m)_{ij} = \overline{\overline{L}}_{ij} \tag{5}$$

- Theorem 5.13: $\lim_{m\to\infty} \overline{\overline{A}}^m$ exists iff
  1. Every eigenvalue of $\overline{\overline{A}}$ is contained in S.
  2. If 1 is an eigenvalue of $\overline{\overline{A}}$, then the dimension of the eigenspace corresponding 1 equlas the multiplicity of 1 as an eigenvalue of $\overline{\overline{A}}$.
- ~~Markov chain~~ (skip)
- Section 5.4: Invariant Subspace and the Cayley-Hamilton Theorem
- Definition: A subspace $\mathcal{W}$ of $\mathcal{V}$ is called a $\hat{T}$-invariant subspace of $\mathcal{V}$ if $\hat{T}(\mathcal{W}) \subseteq \mathcal{W}$, that is, if $\hat{T}(\vec{v}) \in \mathcal{W}$ for all $\vec{v} \in \mathcal{W}$.
- Examples: $\{\vec{0}\}$, $\mathcal{V}$, $R(\hat{T})$, $N(\hat{T})$, E_λ
- Definition: $\hat{T}$-cyclic subspace of $\mathcal{V}$ generated by $\vec{x}$: $\mathcal{W} = \text{span}(\{\vec{x}, \hat{T}(\vec{x}), \hat{T}^2(\vec{x}), \ldots\})$
- Definition: restriction $T_{\mathcal{W}}$
- Theorem 5.21: Let $\mathcal{W}$ be a $\hat{T}$-invariant subspace of $\mathcal{V}$, then the characteristic polynomial of $\hat{T}_{\mathcal{W}}$ divides the characteristic polynomial of $\hat{T}$.
- Theorem 5.22: Let $\mathcal{W}$ be a $\hat{T}$-cyclic subspace of $\mathcal{V}$ generated by a nonzero vector $\vec{v} \in \mathcal{V}$. Let $dim(\mathcal{W}) = k$, then
  1. $\{\vec{v}, \hat{T}(\vec{v}), \hat{T}^2(\vec{v}), \ldots \hat{T}^{k-1}(\vec{v})\}$ is a basis for $\mathcal{W}$.
  2. If $a_0\vec{v} + a_1\hat{T}(\vec{v}) + a_2\hat{T}^2(\vec{v}) + \ldots + a_{k-1}\hat{T}^{k-1}(\vec{v}) + \hat{T}^k(\vec{v}) = 0$, then the characteristic polynomial of $\hat{T}_{\mathcal{W}}$ is

$$f(\lambda) = (-1)^k (a_0 + a_1\lambda + a_2\lambda^2 + \ldots + a_{k-1}\lambda^{k-1} + \lambda^k) \tag{6}$$

- Theorem 5.23 (**Cayley-Hamilton theorem**): Let $f(\lambda)$ be the characteristic polynomial of $\hat{T}$. Then $f(\hat{T}) = \hat{T}_0$, the zero transformation. That is, $\hat{T}$ *satisfies* its characteristic equation.
- Corollary: Let $\overline{\overline{A}}$ be an $n \times n$ matrix and let $f(\lambda)$ bee the characteristic polynomial of $\overline{\overline{A}}$. Then $f(\overline{\overline{A}}) = \overline{\overline{O}}$, the $n \times n$ zero matrix.

- Invariant Subspace and Direct Sum
- Definition: Sum, $\mathcal{W}_1 + \mathcal{W}_2 + \ldots + \mathcal{W}_k$ or $\sum_{i=1}^k \mathcal{W}_i$
- Definition: Direct Sum, $\mathcal{V} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \ldots \oplus \mathcal{W}_k$ if $\mathcal{V} = \sum_{i=1}^k \mathcal{W}_i$ and $\mathcal{W}_j \cap \sum_{i\neq j}^k \mathcal{W}_i = \{\vec{0}\}$ for each $j (1 \le j \le k)$.
- Theorem 5.10: $\mathcal{V} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \ldots \oplus \mathcal{W}_k$ is equivalent to
  1. $\mathcal{V} = \sum_{i=1}^k \mathcal{W}_i$ and for any vectors $\vec{v}_i \in \mathcal{W}_i$.
  2. Each vector $\vec{v} \in \mathcal{V}$ can be uniquely written as $\vec{v} = \vec{v}_1 + \vec{v}_2 + \ldots \vec{v}_k$, where $\vec{v}_i \in \mathcal{W}_i$.
  3. If γ_i is an ordered basis for $\mathcal{W}_i$, then $\gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_k$ is an ordered basis for $\mathcal{V}$.
  4. For each $i = 1, 2, \ldots k$, there exists an order basis γ_i for $\mathcal{W}_i$ such that $\gamma_1 \cup \gamma_2 \cup \ldots \cup \gamma_k$ is an ordered basis for $\mathcal{V}$.
- Theorem 5.11: A linear operator $\hat{T}$ on a finite-dimensional vector space $\mathcal{V}$ is diagonalizable iff $\mathcal{V}$ is the direct sum of the eigenspace of $\hat{T}$.
- Theorem 5.24: Suppose $\mathcal{V} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \ldots \oplus \mathcal{W}_k$, where $\mathcal{W}_i$ is an $\hat{T}$-invariant subspace of $\mathcal{V}$ for each $i (1 \le i \le k)$, and $f_i(\lambda)$ is the characteristic polynomial of $\hat{T}_{\mathcal{W}_i}$, then

$$f_1(\lambda) \cdot f_2(\lambda) \cdot \cdots \cdot f_k(\lambda), \tag{7}$$

  is the characteristic polynomial of $\hat{T}$.
- Direct sum of matrix: $\overline{\overline{A}} = \begin{pmatrix} \overline{\overline{B}}_1 & \overline{\overline{O}} & \ldots & \overline{\overline{O}} \\ \overline{\overline{O}} & \overline{\overline{B}}_2 & \ldots & \overline{\overline{O}} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \overline{\overline{O}} & \overline{\overline{O}} & \ldots & \overline{\overline{B}}_k \end{pmatrix}$
- Theorem 5.25: If $\mathcal{V} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \ldots \oplus \mathcal{W}_k$, then $\overline{\overline{A}} = \overline{\overline{B}}_1 \oplus \overline{\overline{B}}_2 \oplus \ldots \oplus \overline{\overline{B}}_k$.