Linear Algebra, EE 10810/EECS 205004

Note 6.1 – 6.2

Ray-Kuang Lee[1]

[1] *Room 911, Delta Hall, National Tsing Hua University, Hsinchu, Taiwan.*
Tel: +886-3-5742439; E-mail: rklee@ee.nthu.edu.tw
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- Next Quiz on Dec. 16th, Wednesday.
- 2nd-Exam, 10:10-13:10 on Dec. 18th, Friday.

- **Assignment**:

1. Provide the reasons why each of thee following is not an inner product on the given vector space.
   (a) $\langle (a,b),(c,d)\rangle = ac - bd$ on $\mathcal{R}^2$.
   (b) $\langle \overline{\overline{A}}, \overline{\overline{B}}\rangle = \mathrm{tr}(\overline{\overline{A}} + \overline{\overline{B}})$ on $\overline{\overline{M}}_{2\times 2}(\mathcal{R})$.

2. Apply the Gram-Schmidt process to the given subset S of the inner product space $\mathcal{V}$ to obtain an orthonormal basis for span(S)
   (a) $\mathcal{V} = \mathcal{R}^3$, $S = \{(1,0,1),(0,1,1),(1,3,3)\}$
   (b) $\mathcal{V} = \overline{\overline{M}}_{2\times 2}(\mathcal{R})$, $S = \{\begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix}, \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix}\}$

3. Let $\mathcal{V}$ be an inner product space, and let $\mathcal{W}$ be a finite-dimensional subspace of $\mathcal{V}$. If $\vec{x} \notin \mathcal{W}$, prove that there exists $\vec{y} \in \mathcal{V}$ such that $\vec{y} \in \mathcal{W}^{\perp}$, but $\langle \vec{x}, \vec{y}\rangle \neq 0$.

From Scratch !!

- Section 5.4: Invariant Subspace and the Cayley-Hamilton Theorem
- Theorem 5.23 (**Cayley-Hamilton theorem**): Let $f(\lambda)$ be the characteristic polynomial of $\hat{T}$. Then $f(\hat{T}) = \hat{T}_0$, the zero transformation. That is, $\hat{T}$ *satisfies* its characteristic equation.
- Corollary: Let $\overline{\overline{A}}$ be an $n \times n$ matrix and let $f(\lambda)$ bee the characteristic polynomial of $\overline{\overline{A}}$. Then $f(\overline{\overline{A}}) = \overline{\overline{O}}$, the $n \times n$ zero matrix.

- Section 6.1: Inner product space
- Definition: An inner product on $\mathcal{V}$ is a function that assigns, to every ordered pair of vectors $\vec{x}$ and $\vec{y}$ in $\mathcal{V}$, a scalar in F, denoted $\langle \vec{x}, \vec{y} \rangle$, such that for all $\vec{x}$, $\vec{y}$, and $\vec{z}$ in $\mathcal{V}$ and all c in F, the following hold:
  1. $\langle \vec{x} + \vec{z}, \vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{z}, \vec{y} \rangle$.
  2. $\langle c\,\vec{x}, \vec{y} \rangle = c\,\langle \vec{x}, \vec{y} \rangle$.
  3. $\overline{\langle \vec{x}, \vec{y} \rangle} = \langle \vec{y}, \vec{x} \rangle$, where the bar denotes complex conjugation.
  4. $\langle \vec{x}, \vec{x} \rangle > 0$ if $\vec{x} \neq 0$.
- Definiton: Conjugate transpose or adjoint of $\overline{\overline{A}}$, i.e., $(A^*)_{ij} = \overline{A_{ji}}$.
- Frobenius inner product
- Definition: A vector space $\mathcal{V}$ on F endowed with a specific inner product is called an **inner product space**.
- Theorem 6.1: Inner product space
  1. $\langle \vec{x}, \vec{y} + \vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$.
  2. $\langle \vec{x}, c\,\vec{y} \rangle = \bar{c}\,\langle \vec{x}, \vec{y} \rangle$.
  3. $\langle \vec{x}, \vec{0} \rangle = \langle \vec{0}, \vec{x} \rangle = 0$.
  4. $\langle \vec{x}, \vec{x} \rangle = 0$ iff $\vec{x} = \vec{0}$.
  5. $\langle \vec{x}, \vec{y} \rangle = \langle \vec{x}, \vec{z} \rangle$ for all $\vec{x} \in \mathcal{V}$, then $\vec{y} = \vec{z}$.

- Definition: norm or length of $\vec{x}$, denoted as $\|\vec{x}\| \equiv \sqrt{\langle \vec{x}, \vec{x} \rangle}$
- Theorem 6.2:
  1. $\|c\,\vec{x}\| = |c| \cdot \|\vec{x}\|$.
  2. $\|\vec{x}\| = 0$ iff $\vec{x} = \vec{0}$.
  3. Cauchy-Schwarz Inequality: $|\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\| \cdot \|\vec{y}\|$.
  4. Triangle Inequality: $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$.

- Definition: orthogonal if $\langle \vec{x}, \vec{y} \rangle = 0$.
- Unite vector if $\|\vec{x}\| = 1$.
- Definition: orthonormal
- Normalizing:

- Section 6.2: Gram-Schmidt orthogonalization process
- Definition: orthonormal basis
- Theorem 6.3 (Gram-Schmidt process): Let $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be an orthogonal subset of $\mathcal{V}$. If $\vec{y} \in \text{span}(S)$, then

$$\vec{y} = \sum_{i=1}^{k} \frac{\langle \vec{y}, \vec{v}_i \rangle}{\|\vec{v}_i\|^2} \vec{v}_i \tag{1}$$

- Theorem 6.4: Let $S = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n\}$ be a linearly independent subset of $\mathcal{V}$. Define $S' = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$, where $\vec{v}_1 = \vec{w}_1$ and

$$\vec{v}_k = \vec{w}_k - \sum_{j=1}^{k-1} \frac{\langle \vec{w}_k, \vec{v}_j \rangle}{\|\vec{v}_j\|^2} \vec{v}_j. \tag{2}$$

  Then S' is an orthogonal set oof nonzero vector such that $\text{span}(S') = \text{span}(S)$.
- Definition: orthogonal complement of S, i.e. $S^{\perp} = \{\vec{x} \in \mathcal{V} : \langle \vec{x}, \vec{y} \rangle = 0 \text{ for all } \vec{y} \in S\}$.