Linear Algebra, EE 10810/EECS 205004

Note 6.2

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From Scratch !!

- Definition: A vector space $\mathcal{V}$ on F endowed with a specific inner product is called an **inner product space**.
- Theorem 6.1: Inner product space
  1. $\langle\vec{x},\vec{y}+\vec{z}\rangle = \langle\vec{x},\vec{y}\rangle + \langle\vec{x},\vec{z}\rangle$.
  2. $\langle\vec{x},c\,\vec{y}\rangle = \overline{c}\,\langle\vec{x},\vec{y}\rangle$.
  3. $\langle\vec{x},\vec{0}\rangle = \langle\vec{0},\vec{x}\rangle = 0$.
  4. $\langle\vec{x},\vec{x}\rangle = 0$ iff $\vec{x}=\vec{0}$.
  5. $\langle\vec{x},\vec{y}\rangle = \langle\vec{x},\vec{z}\rangle$ for all $\vec{x}\in\mathcal{V}$, then $\vec{y}=\vec{z}$.
- Definition: norm or length of $\vec{x}$, denoted as $\|\vec{x}\| \equiv \sqrt{\langle\vec{x},\vec{x}\rangle}$
- Theorem 6.2:
  1. $\|c\,\vec{x}\| = |c|\cdot\|\vec{x}\|$.
  2. $\|\vec{x}\| = 0$ iff $\vec{x}=\vec{0}$.
  3. Cauchy-Schwarz Inequality: $|\langle\vec{x},\vec{y}\rangle| \le \|\vec{x}\|\cdot\|\vec{y}\|$.
  4. Triangle Inequality: $\|\vec{x}+\vec{y}\| \le \|\vec{x}\| + \|\vec{y}\|$.
- Definition: orthogonal if $\langle\vec{x},\vec{y}\rangle = 0$.
- Unite vector if $\|\vec{x}\| = 1$.
- Definition: orthonormal
- Normalizing:
- Section 6.2: Gram-Schmidt orthogonalization process
- Definition: orthonormal basis
- Theorem 6.3 (Gram-Schmidt process): Let $S=\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_k\}$ be an orthogonal subset of $\mathcal{V}$. If $\vec{y}\in\text{span}(S)$, then

$$\vec{y} = \sum_{i=1}^{k}\frac{\langle\vec{y},\vec{v}_i\rangle}{\|\vec{v}_i\|^2}\vec{v}_i \tag{1}$$

- Theorem 6.4: Let $S=\{\vec{w}_1,\vec{w}_2,\ldots,\vec{w}_n\}$ be a linearly independent subset of $\mathcal{V}$. Define $S'=\{\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_k\}$, where $\vec{v}_1=\vec{w}_1$ and

$$\vec{v}_k = \vec{w}_k - \sum_{j=1}^{k-1}\frac{\langle\vec{w}_k,\vec{v}_j\rangle}{\|\vec{v}_j\|^2}\vec{v}_j. \tag{2}$$

  Then S' is an orthogonal set oof nonzero vector such that $\text{span}(S')=\text{span}(S)$.
- Definition: orthogonal complement of S, i.e. $S^\perp = \{\vec{x}\in\mathcal{V} : \langle\vec{x},\vec{y}\rangle = 0 \text{ for all } \vec{y}\in S\}$.

- Projection of $\vec{b}$ onto the line through $\vec{x}$:

$$\vec{p} = \left(\frac{\vec{x}\vec{x}^t}{\vec{x}^t\vec{x}}\right)\vec{b} \tag{3}$$

- Projection of $\vec{b}$ on a subpace $\overline{\overline{A}} = \{\vec{x}_1,\vec{x}_2,\ldots,\vec{x}_n\}$:

$$\vec{p} = \overline{\overline{A}}\left(\overline{\overline{A}}^t\,\overline{\overline{A}}\right)^{-1}\overline{\overline{A}}^t\,\vec{b} \tag{4}$$

- Least squares approximation