Linear Algebra, EE 10810/EECS 205004

Note 6.3 – 6.4

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- Next Quiz on Dec. 30th, Wednesday.
- Final-Exam, 10:10-13:10 on Jan. 13th, Wednesday.

- **Assignment**:

1. Let $\overline{\overline{A}}$ be an $n \times n$ matrix. Prove that

$$det(\overline{\overline{A}})^* = \overline{det(\overline{\overline{A}})} \tag{1}$$

2. Find the minimal solution to the following system of linear equations

$$\begin{array}{rcrcrcl} x & + & y & - & z & = & 0 \\ 2x & - & y & + & z & = & 3 \\ x & - & y & + & z & = & 2 \end{array} \tag{2}$$

3. Let $\mathcal{V}$ be a complex inner product space, and let $\hat{T}$ be a linear operator on $\mathcal{V}$. Define

$$\hat{T}_1 \equiv \frac{1}{2}(\hat{T} + \hat{T}^*), \quad \text{and} \quad \hat{T}_2 = \frac{1}{2i}(\hat{T} - \hat{T}^*) \tag{3}$$

   (a) Prove that $\hat{T}_1$ and $\hat{T}_2$ are self-adjoint.
   (b) Suppose also that $\hat{T} = \hat{U}_1 + i\,\hat{U}_2$, where $\hat{U}_1$ and $\hat{U}_2$ are self-adjoint. Prove that $\hat{U}_1 = \hat{T}_1$ and $\hat{U}_2 = \hat{U}_2$.
   (c) Prove that $\hat{T}$ is normal if and only if $\hat{T}_1\hat{T}_2 = \hat{T}_2\hat{T}_1$.

From Scratch !!

- Defintion: adjoint of the operator T, i.e., $\langle \hat{T}(\vec{x}), \vec{y} \rangle = \langle \vec{x}, \hat{T}^*(\vec{y}) \rangle$.
- Theorem 6.9: There exists a unique adjoint function $\hat{T}^*$.
- Theorem 6.10: $[\hat{T}^*]_\beta = [\hat{T}]^*_\beta$
- Theorem 6.11:
  1. $(\hat{T} + \hat{U})^* = \hat{T}^* + \hat{U}^*$
  2. $(c\hat{T})^* = \bar{c}\hat{T}^*$
  3. $(\hat{T}\hat{U})^* = \hat{U}^*\hat{T}^*$
  4. $\hat{T}^{**} = \hat{T}$
  5. $\hat{I}^* = \hat{I}$
- Least squares approximation

- Lemma: If $\hat{T}$ has an eigenvector, then so does $\hat{T}^*$
- Theorem 6.14 (Schur): There exists an orthonormal basis β for $\mathcal{V}$, such that the matrix $[\hat{T}]_\beta$ is upper triangular.
- Definition: normal $\hat{T}\hat{T}^* = \hat{T}^*\hat{T}$ or $\overline{\overline{A}\overline{\overline{A}}}^* = \overline{\overline{A}}^*\overline{\overline{A}}$.
- Theorem 6.15:
  1. $||\hat{T}(\vec{x})|| = ||\hat{T}^*(\vec{x})||$
  2. $\hat{T} - c\hat{I}$ is normal for every $c \in F$
  3. If $\hat{T}(\vec{x}) = \lambda\vec{x}$, then $\hat{T}^*(\vec{x}) = \bar{\lambda}\vec{x}$
  4. If λ_1 and λ_2 are distinct eigenvectors of $\hat{T}$ with corresponding eigenvectors $\vec{x}_1$ and $\vec{x}_2$, then $\vec{x}_1$ and $\vec{x}_2$ are orthogonal.

- Theorem 6.16: $\hat{T}$ is normal iff there exists an orthonormal basis for $\mathcal{V}$ consisting of eigenvectors of $\hat{T}$.
- Definition: self-adjoint (Hermitian) if $\hat{T} = \hat{T}^*$ or $\overline{\overline{A}} = \overline{\overline{A}}^*$
- Lemma
  1. Every eigenvalue of a self-adjoint operator $\hat{T}$ is real.
  2. Suppose that $\mathcal{V}$ is a real inner product space, then the characteristic polynomial of $\hat{T}$ splits.
- Theorem 6.17: $\hat{T}$ is self-adjoint iff there exists an orthonormal basis β for $\mathcal{V}$ consisting of eigenvectors of $\hat{T}$.
- Definition: positive definite if $\hat{T}$ is self-adjoint and $\langle \hat{T}(\vec{x}), \vec{x} \rangle > 0$ for all $\vec{x} \neq 0$