Linear Algebra, EE 10810/EECS 205004

Note 6.5

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(Dated: Fall, 2020)

- Next Quiz on Jan. 6th, Wednesday.
- Final-Exam, 10:10-13:10 on Jan. 13th, Wednesday.

- **Assignment**:

1. Let $\hat{T}$ and $\hat{U}$ be a self-adjoint linear operators on an n-dimensional inner product space $\mathcal{V}$, and let $\overline{\overline{A}} = [\hat{T}]_\beta$, where β is an orthonormal basis for $\mathcal{V}$. Prove the following results.
   (a) $\hat{T}$ is positive definite (semi-definite) if an only if all of its eigenvalues are positive (non-negative).
   (b) $\hat{T}$ is positive definite if and only if
$$\sum_{i,j} A_{i,j}\, a_j\, \bar{a}_i > 0 \quad \text{for all nonzero}\, n\text{-tuples}\, (a_1, a_2, \ldots, a_n) \tag{1}$$
   (c) $\hat{T}$ is positive semidefinite if and only if $\overline{\overline{A}} = \overline{\overline{B}}^* \overline{\overline{B}}$ for some square matrix $\overline{\overline{B}}$.
   (d) If $\hat{T}$ and $\hat{U}$ are positive definite operators such that $\hat{T}^2 = \hat{U}^2$, then $\hat{T} = \hat{U}$.

2. For the following matrix $\overline{\overline{A}}$, find an orthogonal or unitary matrix $\overline{\overline{P}}$ and a diagonal matrix $\overline{\overline{D}}$ such that $\overline{\overline{P}}^* \overline{\overline{AP}} = \overline{\overline{D}}$:
   (a)
$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \tag{2}$$
   (b)
$$\begin{pmatrix} 2 & 3-3i \\ 3+3i & 5 \end{pmatrix} \tag{3}$$

3. Which of the following pairs of matrices are unitarily equivalent?
   (a)
$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4}$$
   (b)
$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix} \tag{5}$$

From Scratch !!

Section 6.3: Adjoint of linear operator

- Theorem 6.9: There exists a unique adjoint function $\hat{T}^*$.
- Theorem 6.10: Let β be an orthonormal basis, $[\hat{T}^*]_\beta = [\hat{T}]^*_\beta$

App: Least squares approximation

- Theorem 6.12: There exists $\vec{x}_0 \in F^n$ such that $(\overline{\overline{A}}^* \overline{\overline{A}})\vec{x}_0 = \overline{\overline{A}}^* \vec{y}$ and $||\overline{\overline{A}}\vec{x}_0 - \vec{y}|| \leq ||\overline{\overline{A}}\vec{x}_0 - \vec{y}||$.

App: Minimal solution to systems of linear equations:

- Theorem 6.13: There exists exactly one minimal solution $\vec{s}$ of $\overline{\overline{A}}\vec{x} = \vec{b}$, and $\vec{s} \in R(\hat{L}_{A^*})$, i.e., $\overline{\overline{A}}(\overline{\overline{A}}^* \vec{u}) = \vec{b}$ and $\vec{s} = \overline{\overline{A}}^* \vec{u}$.

Section 6.4: Normal and Self-adjoint operators

- Theorem 6.14 (Schur): There exists an orthonormal basis β for $\mathcal{V}$, such that the matrix $[\hat{T}]_\beta$ is upper triangular.
- Definition: normal $\hat{T}\hat{T}^* = \hat{T}^*\hat{T}$ or $\overline{\overline{AA}}^* = \overline{\overline{A}}^* \overline{\overline{A}}$.
- Theorem 6.15: Let $\hat{T}$ be a normal operator on $\mathcal{V}$,
  1. $||\hat{T}(\vec{x})|| = ||\hat{T}^*(\vec{x})||$
  2. $\hat{T} - c\,\hat{I}$ is normal for every $c \in F$
  3. If $\hat{T}(\vec{x}) = \lambda \vec{x}$, then $\hat{T}^*(\vec{x}) = \bar{\lambda}\vec{x}$
  4. If λ_1 and λ_2 are distinct eigenvectors of $\hat{T}$ with corresponding eigenvectors $\vec{x}_1$ and $\vec{x}_2$, then $\vec{x}_1$ and $\vec{x}_2$ are orthogonal.
- Theorem 6.16: $\hat{T}$ is normal iff there exists an orthonormal basis for $\mathcal{V}$ consisting of eigenvectors of $\hat{T}$.
- Definition: self-adjoint (Hermitian) if $\hat{T} = \hat{T}^*$ or $\overline{\overline{A}} = \overline{\overline{A}}^*$
- Lemma
  1. Every eigenvalue of a self-adjoint operator $\hat{T}$ is real.
  2. Suppose that $\mathcal{V}$ is a real inner product space, then the characteristic polynomial of $\hat{T}$ splits.
- Theorem 6.17: $\hat{T}$ is self-adjoint iff there exists an orthonormal basis β for $\mathcal{V}$ consisting of eigenvectors of $\hat{T}$.
- Definition: positive definite if $\hat{T}$ is self-adjoint and $\langle \hat{T}(\vec{x}), \vec{x}\rangle > 0$ for all $\vec{x} \neq 0$

Section 6.5: Unitary and Orthogonal operators

- Defintion: unitary operator if $||\hat{T}(\vec{x})|| = ||\vec{x}||$ for all $\vec{x} \in \mathcal{V}$ over $F = \mathcal{C}$.
- Defintion: unitary operator if $||\hat{T}(\vec{x})|| = ||\vec{x}||$ for all $\vec{x} \in \mathcal{V}$ over $F = \mathcal{R}$.
- Theorem 6.18: Let $\hat{T}$ be a unitary operator on $\mathcal{V}$,
  1. $\hat{T}\hat{T}^* = \hat{T}^*\hat{T} = \hat{I}$.
  2. $\langle \hat{T}(\vec{x}), \hat{T}(\vec{y})\rangle = \langle \vec{x}, \vec{y}\rangle$, for all $\vec{x}, \vec{y} \in \mathcal{V}$.
  3. If β is an orthonormal basis for $\mathcal{V}$, then $\hat{T}(\beta)$ is an orthonormal basis for $\mathcal{V}$.
  4. There exists an orthonormal basis β for $\mathcal{V}$ such that $\hat{T}(\beta)$ is an orthonormal basis for $\mathcal{V}$.
- Rotation matrix:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{6}$$

- Definition: orthogonal matrix if $\overline{\overline{A}}^t \overline{\overline{A}} = \overline{\overline{AA}}^t = \overline{\overline{I}}$.
- Definition: unitary matrix if $\overline{\overline{A}}^* \overline{\overline{A}} = \overline{\overline{AA}}^* = \overline{\overline{I}}$.
- Definition: $\overline{\overline{A}}$ and $\overline{\overline{B}}$ are unitarily equivalent (orthogonally equivalent) iff there exists a unitary (orthogonal) matrix $\overline{\overline{P}}$ such that $\overline{\overline{A}} = \overline{\overline{P}}^* \overline{\overline{BP}}$.
- Theorem 6.19: $\overline{\overline{A}}$ is normal iff $\overline{\overline{A}}$ is unitarily equivalent to a diagonal matrix.
- Theorem 6.20: $\overline{\overline{A}}$ is symmetry iff $\overline{\overline{A}}$ is orthogonally equivalent to a diagonal matrix.
- Theorem 6.21 (Schur): Let $\overline{\overline{A}} \in \overline{\overline{M}}_{n\times n}(F)$
  1. If $F = \mathcal{C}$, then $\overline{\overline{A}}$ is unitarily equivalent to a complex upper triangular matrix.
  2. If $F = \mathcal{R}$, then $\overline{\overline{A}}$ is orthogonally equivalent to a real upper triangular matrix.

App: Rigid Motions