# Linear Algebra, EE 10810/EECS 205004

**Note** 6.6 – 6.7

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- Final-Exam, 10:10-13:10 on Jan. 13th, Wednesday.

- **Assignment**:

1. For the following matrices $\overline{\overline{A}}$:

   (a)

$$\begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} \tag{1}$$

   (b)

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \tag{2}$$

   (1) Verify that $\hat{L}_A$ possesses a spectral decomposition.

   (2) For each eigenvalue of $\hat{L}_A$, explicitly define the orthogonal projection on the corresponding eigenspace.

   (3) Verify your results using the spectral theorem.

2. Find a singular value decomposition for the following matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix} \tag{3}$$

3. Find a polar decomposition for the following matrix:

$$\begin{pmatrix} 20 & 4 & 0 \\ 0 & 0 & 1 \\ 4 & 20 & 0 \end{pmatrix} \tag{4}$$

**From Scratch !!**

Section 6.5: Unitary and Orthogonal operators

- Defintion: unitary (orthogonal) operator if $||\hat{T}(\vec{x})|| = ||\vec{x}||$ for all $\vec{x} \in \mathcal{V}$ over $F = \mathcal{C}(\mathcal{R})$.
- Theorem 6.18: Let $\hat{T}$ be a unitary operator on $\mathcal{V}$,
  1. $\hat{T}\hat{T}^* = \hat{T}^*\hat{T} = \hat{I}$.
  2. $\langle \hat{T}(\vec{x}), \hat{T}(\vec{y}) \rangle = \langle \vec{x}, \vec{y} \rangle$, for all $\vec{x}, \vec{y} \in \mathcal{V}$.
  3. If $\beta$ is an orthonormal basis for $\mathcal{V}$, then $\hat{T}(\beta)$ is an orthonormal basis for $\mathcal{V}$.
  4. There exists an orthonormal basis $\beta$ for $\mathcal{V}$ such that $\hat{T}(\beta)$ is an orthonormal basis for $\mathcal{V}$.
- Rotation matrix: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
- Definition: orthogonal matrix if $\overline{\overline{A}}^t \overline{\overline{A}} = \overline{\overline{AA}}^t = \overline{\overline{I}}$; and unitary matrix if $\overline{\overline{A}}^* \overline{\overline{A}} = \overline{\overline{AA}}^* = \overline{\overline{I}}$.
- Definition: $\overline{\overline{A}}$ and $\overline{\overline{B}}$ are unitarily equivalent (orthogonally equivalent) iff there exists a unitary (orthogonal) matrix $\overline{\overline{P}}$ such that $\overline{\overline{A}} = \overline{\overline{P}}^* \overline{\overline{BP}}$.
- Theorem 6.19: $\overline{\overline{A}}$ is normal iff $\overline{\overline{A}}$ is unitarily equivalent to a diagonal matrix.
- Theorem 6.20: $\overline{\overline{A}}$ is symmetry iff $\overline{\overline{A}}$ is orthogonally equivalent to a diagonal matrix.
- Theorem 6.21 (Schur): Let $\overline{\overline{A}} \in \overline{\overline{M}}_{n\times n}(F)$
  1. If $F = \mathcal{C}$, then $\overline{\overline{A}}$ is unitarily equivalent to a complex upper triangular matrix.
  2. If $F = \mathcal{R}$, then $\overline{\overline{A}}$ is orthogonally equivalent to a real upper triangular matrix.

App: ~~Rigid Motions~~

Section 6.5: Orthogonal Projection and the Spectral Theorem

- Definition: $\hat{T}$ is an orthogonal projection if $R(\hat{T})^\perp = N(\hat{T})$ and $N(\hat{T})^\perp = R(\hat{T})$
- Theorem 6.24: $\hat{T}$ is an orthogonal projection iff $\hat{T}$ has an adjoint $\hat{T}^*$ and $\hat{T}^2 = \hat{T} = \hat{T}^*$
- Theorem 6.25 (The Spectral Theorem) Assume $\hat{T}$ is normal if $F = \mathcal{C}$ and that $\hat{T}$ is self-adjoint if $F = \mathcal{R}$. For each distinct eigenvalues $\lambda_i$, with the corresponding eigenspace of $\hat{T}$, $W_i$, and let $\hat{T}_i$ be the orthogonal projection of $\mathcal{V}$ on $W_i$.
  (a) $\mathcal{V} = W_1 \oplus W_2 \oplus \ldots \oplus W_k$
  (b) If $W_i'$ denotes the direct sum of the subspace $W_j$, $j \neq i$, then $W_i^\perp = W_i'$
  (c) $\hat{T}_i\hat{T}_j = \delta_{ij}\hat{T}_i$
  (d) $\hat{I} = \hat{T}_1 + \hat{T}_2 + \ldots + \hat{T}_k$
  (e) $\hat{T} = \lambda_1\hat{T}_1 + \lambda_2\hat{T}_2 + \ldots + \lambda_k\hat{T}_k$

Section 6.6: Singular Value Decomposition (SVD)

- Theorem 6.26 (SVD for Linear Transformations): Let $\hat{T} : \mathcal{V} \to \mathcal{W}$ be a linear transformation of rank $r$, then there exist positive scalars $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r$ such that

$$\hat{T}(\vec{v}_i) = \begin{cases} \sigma_i\,\vec{u}_i, & \text{if } 1 \leq i \leq r \\ 0, & \text{if } i > r \end{cases} \tag{5}$$

- Definition: the eigenvalues of $\hat{T}^*\hat{T}$ is called the *singular values*.
- Theorem 6.27 (SVD Theorem for Matrices): $\overline{\overline{A}}_{m\times n} = \overline{\overline{U}}_{m\times m}\overline{\overline{\Sigma}}_{m\times n}\overline{\overline{V}}^*_{n\times n}$
- Theorem 6.28 (Polar Decomposition): For any square matrix $\overline{\overline{A}}$, there exists a unitary matrix $\overline{\overline{W}}$ and a positive semidefinite matrix $\overline{\overline{P}}$ such that $\overline{\overline{A}} = \overline{\overline{W}}\,\overline{\overline{P}}$. Furthermore, if $\overline{\overline{A}}$ is invertible, then the representation is unique.
- Definition: Pseudoinverse (or Moore-Penrose generalized inverse): Let $\overline{\overline{A}}_{m\times n}$, there exists $\overline{\overline{B}}_{n\times m}$ such that $(\hat{L}_A)^\dagger : F^m \to F^n$, i.e., $\overline{\overline{B}} = \overline{\overline{A}}^\dagger$.
- Theorem 6.29: $\overline{\overline{A}}^\dagger_{m\times n} = \overline{\overline{V}}_{n\times n}\overline{\overline{\Sigma}}^\dagger_{n\times m}\overline{\overline{U}}^*_{m\times m}$, with the singular values $1/\sigma_i$.
- Lemma: $\hat{T}^\dagger\hat{T}$ is the orthogonal projection of $\mathcal{V}$ on $N(\hat{T})^\perp$.
- Lemma: $\hat{T}\hat{T}^\dagger$ is the orthogonal projection of $\mathcal{W}$ on $R(\hat{T})^\perp$.
- Theorem 6.30: Consider $\overline{\overline{A}}\,\vec{x} = \vec{b}$, then $\vec{z} = \overline{\overline{A}}^\dagger\,\vec{b}$ has the following properties.