Linear Algebra, EE 10810/EECS 205004

Note 7.1 – 7.2

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- Final-Exam, 10:10-13:10 on Jan. 13th, Wednesday.

- **Assignment**:

  1. Find a Jordan canonical form $\overline{\overline{J}}$ of $\overline{\overline{A}}$:

$$\overline{\overline{A}} = \begin{pmatrix} 11 & -4 & -5 \\ 21 & -8 & -11 \\ 3 & -1 & 0 \end{pmatrix} \tag{1}$$

  2. Find a Jordan canonical form $\overline{\overline{J}}$ of $\overline{\overline{A}}$, and an invertible matrix $\overline{\overline{Q}}$ such that $\overline{\overline{J}} = \overline{\overline{Q}}^{-1}\,\overline{\overline{A}}\,\overline{\overline{Q}}$:

$$\overline{\overline{A}} = \begin{pmatrix} -3 & 3 & -2 \\ -7 & 6 & -3 \\ 1 & -1 & 2 \end{pmatrix} \tag{2}$$

From Scratch !!

Section 6.6: Singular Value Decomposition (SVD)

- Theorem 6.26 (SVD for Linear Transformations): Let $\hat{T}: \mathcal{V} \to \mathcal{W}$ be a linear transformation of rank r, then there exist positive scalars $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r$ such that

$$\hat{T}(\vec{v}_i) = \begin{cases} \sigma_i \, \vec{u}_i, & \text{if } 1 \leq i \leq r \\ 0, & \text{if } i > r \end{cases} \tag{3}$$

- Definition: the eigenvalues of $\hat{T}^*\hat{T}$ is called the *singular values*.
- Theorem 6.27 (SVD Theorem for Matrices): $\overline{\overline{A}}_{m\times n} = \overline{\overline{U}}_{m\times m}\overline{\overline{\Sigma}}_{m\times n}\overline{\overline{V}}^*_{n\times n}$
- Theorem 6.28 (Polar Decomposition): For any square matrix $\overline{\overline{A}}$, there exists a unitary matrix $\overline{\overline{W}}$ and a positive semidefinite matrix $\overline{\overline{P}}$ such that $\overline{\overline{A}} = \overline{\overline{W}}\,\overline{\overline{P}}$. Furthermore, if $\overline{\overline{A}}$ is invertible, then the representation is unique.
- Definition: Pseudoinverse (or Moore-Penrose generalized inverse): Let $\overline{\overline{A}}_{m\times n}$, there exists $\overline{\overline{B}}_{n\times m}$ such that $(\hat{L}_A)^\dagger : F^m \to F^n$, i.e., $\overline{\overline{B}} = \overline{\overline{A}}^\dagger$.
- Theorem 6.29: $\overline{\overline{A}}^\dagger_{n\times m} = \overline{\overline{V}}_{n\times n}\overline{\overline{\Sigma}}^\dagger_{n\times m}\overline{\overline{U}}^*_{m\times m}$, with the singular values $1/\sigma_i$.
- Lemma: $\hat{T}^\dagger\hat{T}$ is the orthogonal projection of $\mathcal{V}$ on $N(\hat{T})^\perp$.
- Lemma: $\hat{T}\hat{T}^\dagger$ is the orthogonal projection of $\mathcal{W}$ on $R(\hat{T})^\perp$.
- Theorem 6.30: Consider $\overline{\overline{A}}\,\vec{x} = \vec{b}$, then $\vec{z} = \overline{\overline{A}}^\dagger\,\vec{b}$ has the following properties.

Section 7.1-7.2: Jordan canonical form

- Jordan block $\overline{\overline{A}}_i$ with the corresponding eigenvalue λ_i: $[\hat{T}]_\beta = \begin{pmatrix} \overline{\overline{A}}_1 & \overline{\overline{O}} & \ldots & \overline{\overline{O}} \\ \overline{\overline{O}} & \overline{\overline{A}}_2 & \ldots & \overline{\overline{O}} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \overline{\overline{O}} & \overline{\overline{O}} & \ldots & \overline{\overline{A}}_n \end{pmatrix}$
- Definition: generalized eigenvector of $\hat{T}$ corresponding to λ if $(\hat{T} - \lambda\hat{I})^p(\vec{x}) = 0$ for some positive integer p
- Definition: generalized eigenspace of $\hat{T}$ corresponding to λ, denoted $K_\lambda = \{\vec{x} \in \mathcal{V} : (\hat{T} - \lambda\hat{I})^p(\vec{x}) = 0\}$
- Theorem 7.1: (a) K_λ is $\hat{T}$-invariant subspace of $\mathcal{V}$ containing E_λ; (b) For any scalar $\mu \neq \lambda$, the restriction of $\hat{T} - \mu\hat{I}$ to K_λ is one-to-one.
- Theorem 7.2: (a) $dim(K_\lambda) \leq m$; (b) $K_\lambda = N((\hat{T} - \lambda\hat{I})^m)$, with multiplicity m.
- Theorem 7.3: Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of $\hat{T}$, then for every $\vec{x} \in \mathcal{V}$, there exists vectors $\vec{v}_i \in K_{\lambda_i}, 1 \leq i \leq k$, such that $\vec{x} = \vec{v}_1 + \vec{v}_2 + \ldots \vec{v}_k$.
- Let β_i be an ordered basis for K_{λ_i}, then
  - (a) $\beta_i \cap \beta_j = \emptyset$ for $i \neq j$
  - (b) $\beta = \beta_1 \cup \beta_2 \cup \ldots \cup \beta_k$ is an ordered basis for $\mathcal{V}$
  - (c) $dim(K_{\lambda_i}) = m_i$ for all i
- Definition: the order set $\{(\hat{T} - \lambda\hat{I})^{p-1}(\vec{x}), (\hat{T} - \lambda\hat{I})^{p-2}(\vec{x}), \ldots, (\hat{T} - \lambda\hat{I})(\vec{x}), \vec{x}\}$ is called a cycle of generalized eigenvectors of $\hat{T}$ corresponding to λ, with the length of the cycle p.
- Example: $\overline{\overline{A}}_i = \begin{pmatrix} \lambda_i & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_i & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_i & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_i & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_i \end{pmatrix}$
- Theorem 7.11: Let $\overline{\overline{A}}$ and $\overline{\overline{B}}$ be $n\times n$ matrices, each having Jordan canonical forms computed according to the conventions. Then, $\overline{\overline{A}}$ and $\overline{\overline{B}}$ are similar iff they have (up to an ordering of their eigenvalues) the same Jordan canonical form.