Laser Physics

PAPER • OPEN ACCESS

Triple measurements uncertainty and the distinguishment between the separable and entangled states

IOP Publishing Laser Phys. 36 (2026) 025203 https://doi.org/10.1088/1555-6611/ae4197

Laser Physics

Astro Ltd.

OPEN ACCESS

RECEIVED
20 January 2026

ACCEPTED FOR PUBLICATION
26 January 2026

PUBLISHED
26 February 2026

PAPER

Triple measurements uncertainty and the distinguishment between the separable and entangled states

Minyi Huang[1] and **Ray-Kuang Lee**[2,3,*]

[1] Department of Mathematical Sciences, Zhejiang Sci Tech University, Hangzhou 310018, People's Republic of China
[2] Department of Physics, National Tsing Hua University, Hsinchu 300, Taiwan
[3] Institute of Photonics Technologies, National Tsing Hua University, Hsinchu 300, Taiwan
[*] Author to whom any correspondence should be addressed.

E-mail: rklee@ee.nthu.edu.tw, 11335001@zju.edu.cn and hmyzd2011@126.com

Keywords: uncertainty relation, uncertainty constant, variance, entanglement

Abstract

Uncertainty and entanglement are both profound and key concepts in quantum theory. For three observables, recent works have revealed the tightest uncertainty constants for both product and summation forms. In this work, we give an alternative proof of the uncertainty relations, as well as a physical interpretation of the uncertainty constants. Our results show that such relations are intimately connected with the distinguishment between separable and entangled states. The uncertainty constant is a critical point whether such a condition of entanglement detection is valid.

1. Introduction

Uncertainty is one of the most fundamental concepts in quantum mechanics. In 1927, Heisenberg discovered his seminal uncertainty principle which defines the fundamental constraints on quantum measurements [1]. In 1929, Robertson formulated precise uncertainty relations as symmetric quadratic polynomials of the variances of two observables [2]. For any two quantum mechanical observables A and B, Robertson has found the following uncertainty inequality,

$$\Delta A \Delta B \geqslant \frac{1}{2}|\langle [A,B]\rangle|, \tag{1}$$

where $\Delta\Omega \triangleq \sqrt{\langle\Omega^2\rangle - \langle\Omega\rangle^2}$ is the standard deviation and $\langle\Omega\rangle$ is the expected value of the observable Ω with respect to the state ρ. The variance is denoted as $(\Delta\Omega)^2 \triangleq \langle\Omega^2\rangle - \langle\Omega\rangle^2$ and we may use $(\Delta\Omega)^2_\rho$ when the state ρ needs to be emphasized. Since then, there are lots of discussions in this field, generalizing the uncertainty relation to more observables and different metrics [3–16].

In the case of three observables, an investigation of the uncertainty relation reveals the tightest uncertainty constants for both product and summation forms [17, 18]. For three observables H_j, it is showed that

$$\prod_{j=1}^{3}(\Delta H_j)^2 \geqslant (\frac{1}{\sqrt{3}})^3(\prod_{j=1}^{3}|\langle [H_j, H_{j+1}]\rangle|, \tag{2}$$

$$\sum_{j=1}^{3}(\Delta H_j)^2 \geqslant \frac{1}{\sqrt{3}}\sum_{j=1}^{3}|\langle [H_j, H_{j+1}]\rangle|. \tag{3}$$

In addition to the uncertainty, entanglement is also one of the most important concepts in quantum information theory. The discussions of entanglement can be dated back to the Einstein–Podolsky–Rosen paradox [19]. Since then, entanglement has been one of the key issues in quantum theory, which has greatly deepened our understanding of quantum mechanics, with numerous applications in quantum technologies [20]. Entanglement also stimulates the discussions on the constraints of probability distributions, operator structures, game theory and many other problems in the field of mathematical physics [21, 22].

Behind uncertainty and entanglement lies the concept of non-commutativity of operators. In fact, one can see immediately that the uncertainty in equation (1) comes from the non-commutativity of observables. On the other hand, researchers have showed the crucial role of non-commutativity in the discussions of entanglement or nonlocality [21–24]. Thus it is natural that researches have shown close relations between the two concepts of uncertainty and entanglement [25–27]. In this paper, we follow a different route to prove the inequalities of the variance in the sum and product forms. Our proof gives a more natural physical meaning, with a clear physical interpretation to the uncertainty constants. It is showed that they are intimately related to the distinguishment between the separable and entangled states.

The structure of the paper is organized as follows. Section 2 is the proof of the inequalities of the variance, i.e. equations (2) and (3). In section 3, it is shown that these inequalities, as well as the uncertainty constants, can be utilized to give a way to distinguish between separable and entangled states. In section 4, a concrete example is given. Section 5 is some discussions and the summary. The last section is the appendix.

2. The inequalities of variance in the sum and product forms

In order to prove the uncertainty relations in equations (2) and (3), one can introduce an ancillary system and consider the variance of the global system to indicate the local uncertainty relation in the subsystem. Then, the summation form in equation (3) suggests that it may be more suitable for giving a physical interpretation. For this reason, unlike [18], we first prove equation (3) instead of equation (2). As a consequence, equation (2) follows directly from equation (3).

First, one can define a global operator $R=\sum_{j=1}^{3} H_j\otimes\sigma_j$, where σ_j are the Pauli operators. Direct calculations show that

$$R^2=\sum_{j=1}^{3} H_j^2\otimes I+\sum_{j=1}^{3}\left[H_j,H_{j+1}\right]\otimes i\sigma_{j+2} \tag{4}$$

Note that the subscript j modulo by 3, i.e. j mod 3.

Now we want to investigate the variance of R. To this end, we first consider the bound of R and its square. Generally, it depends on the concrete form of R and R^2. Suppose that the state is a product state, i.e. $\mu\otimes\nu$, where μ and ν are states of the subsystems.

Then we have

$$Tr(\mu\otimes\nu)R^2=\sum_{j=1}^{3}\langle H_j^2\rangle+\sum_{j=1}^{3}\langle i\left[H_j,H_{j+1}\right]\rangle\langle\sigma_{j+2}\rangle, \tag{5}$$

$$\left[Tr(\mu\otimes\nu)R\right]^2=\left(\sum_{j=1}^{3}\langle H_j\rangle\langle\sigma_j\rangle\right)^2. \tag{6}$$

Note that by the Schwarz inequality, for any state ρ , one has

$$\left[Tr\rho R\right]^2\leqslant Tr\rho R^2. \tag{7}$$

For a product state, still by the Schwarz inequality, we have

$$\begin{aligned}\left[Tr(\mu\otimes\nu)R\right]^2&=\left(\sum_{j=1}^{3}\langle H_j\rangle\langle\sigma_j\rangle\right)^2\\&\leqslant\left(\sum_{j=1}^{3}\langle H_j\rangle^2\right)\left(\sum_{j=1}^{3}\langle\sigma_j\rangle^2\right).\end{aligned}$$

Note that $\sum_j\langle\sigma_j\rangle^2\leqslant 1$ [25, 28], thus

$$\left[Tr(\mu\otimes\nu)R\right]^2\leqslant\sum_{j=1}^{3}\langle H_j\rangle^2. \tag{8}$$

By equations (5) and (8), one can see that

$$\begin{aligned}(\Delta R)^2_{\mu\otimes\nu}&\triangleq Tr(\mu\otimes\nu)R^2-\left[Tr(\mu\otimes\nu)R\right]^2\\&\geqslant\sum_{j=1}^{3}\Delta H_j^2+\sum_{j=1}^{3}\langle i\left[H_j,H_{j+1}\right]\rangle\langle\sigma_{j+2}\rangle\end{aligned} \tag{9}$$

Note that identity holds in equation (9) only if identity holds in equation (8). According to the Schwarz inequality, it is valid when $\langle H_j\rangle=t\langle\sigma_j\rangle$ and $|\langle\sigma_j\rangle|=\frac{1}{\sqrt{3}}$. Moreover, according to the appendix, actually the state ν can be specifically chosen to satisfy

$$\sum_{j=1}^{3}\langle i\left[H_j,H_{j+1}\right]\rangle\langle\sigma_{j+2}\rangle=-\frac{1}{\sqrt{3}}\sum_{j=1}^{3}|\langle\left[H_j,H_{j+1}\right]\rangle|. \tag{10}$$

Thus when $\langle H_j\rangle=t\langle\sigma_j\rangle$ and $|\langle\sigma_j\rangle|=\frac{1}{\sqrt{3}}$ (in which case the identity in equation (9) is valid), it follows from equation (10) and the positivity of $(\Delta R_{\mu\otimes\nu})^2$ that,

$$\sum_{j=1}^{3}\Delta H_j^2\geqslant\frac{1}{\sqrt{3}}\sum_{j=1}^{3}|\langle\left[H_j,H_{j+1}\right]\rangle|, \tag{11}$$

It should be mentioned that the condition $\langle H_j\rangle=t\langle\sigma_j\rangle$ may fail in general. Hence at the moment, equation (11) is proved only for the H_j satisfying the previous conditions. However, one can see immediately that the condition is satisfied when $\langle H_j\rangle=0$, thus equation (11) is valid. Now for general H_j, one can take $H_j'=H_j-\langle H_j\rangle I$, then $\langle H_j'\rangle=0$. However, since $\Delta H_j'=\Delta H_j$ and $[H_j',H_{j+1}']=[H_j,H_{j+1}]$, equation (11) is valid for any operators H_j. This completes the proof of equation (3).

By equation (11), one can further obtain the triple measurements uncertainty in the product form. In fact, by the Algebraic-Geometric Mean inequality,

$$\frac{1}{3}\sum_{j=1}^{3}\Delta H_j^2 \geqslant \frac{1}{\sqrt{3}}\left(\frac{1}{3}\sum_{j=1}^{3}|\langle[H_j,H_{j+1}]\rangle|\right) \geqslant \frac{1}{\sqrt{3}}\left(\prod_{j=1}^{3}|\langle[H_j,H_{j+1}]\rangle|\right)^{\frac{1}{3}}. \quad (12)$$

It follows that

$$\left(\frac{1}{3}\sum_{j=1}^{3}\Delta H_j^2\right)^3 \geqslant \left(\frac{1}{\sqrt{3}}\right)^3\prod_{j=1}^{3}|\langle[H_j,H_{j+1}]\rangle|. \quad (13)$$

Apparently, when ΔH_j^2 are equal, one can deduce from the above equation that

$$\prod_{j=1}^{3}\Delta H_j^2 \geqslant \left(\frac{1}{\sqrt{3}}\right)^3\prod_{j=1}^{3}|\langle[H_j,H_{j+1}]\rangle|. \quad (14)$$

Now for the case ΔH_j^2 are not equal, one can choose coefficients $\kappa_j = \frac{(\prod_{k=1}^{3}\Delta H_k^2)^{\frac{1}{6}}}{\Delta H_j}$ and $H_j' = \kappa_j H_j$. Now direct calculations show that $\prod_{j=1}^{3}\kappa_j^2 = 1$ and $(\Delta H_j')^2$ are equal. Moreover, $\prod_{j=1}^{3}(\Delta H_j')^2 = \prod_{j=1}^{3}\Delta H_j^2$ and $\prod_{j=1}^{3}|\langle[H_j',H_{j+1}']\rangle| = \prod_{j=1}^{3}|\langle[H_j,H_{j+1}]\rangle|$. Thus one can see that equation (14) is also valid for general H_j, which completes the proof of equation (2).

3. The difference between separable and entangled states

In [25], the researchers propose a method to detect entanglement by considering the variance of separable and entangled states, in which the observable can be written as $\sum_j A_j\otimes I + I\otimes B_j$. Although the operator R in this paper does not have such a good decomposition, it is still to use it for entanglement detection, i.e. the uncertainty relations in section II is related to the distinguishment between separable and entangled states.

Recall that a state is said to be separable if it can be written as a convex combination of the product states, i.e.

$$\rho = \sum_j \lambda_j \mu_j\otimes\nu_j. \quad (15)$$

Direct calculations show that

$$\begin{aligned}(\Delta R)_\rho^2 &= Tr\left[\sum\lambda_j\left(\mu_j\otimes\nu_j\right)R^2\right] - Tr\left[\sum\lambda_j\left(\mu_j\otimes\nu_j\right)R\right]^2\\ &\geqslant \sum\lambda_j Tr\left[\left(\mu_j\otimes\nu_j\right)R^2\right] - \sum\lambda_j\sum\lambda_j\left[Tr\left(\mu_j\otimes\nu_j\right)R\right]^2\\ &= \sum_j\lambda_j(\Delta R)^2_{\mu_j\otimes\nu_j},\end{aligned} \quad (16)$$

where the $\geqslant$ in the above equation is due to the Schwarz inequality. Note that if the variance of any product state is larger than zero, i.e. $(\Delta R)^2_{\mu_j\otimes\nu_j} > 0$, then for any separable state ρ, $(\Delta R)^2_\rho > 0$. Since the separable states form a compact set [29], one can obtain that there exists some positive constant $c>0$ such that $(\Delta R)^2_\rho \geqslant c > 0$. Now note that in the derivation above, we have used product state. If the identity in equations (9) and (11) does not hold for any product state, i.e. $\sum_{j=1}^{3}\Delta H_j^2$ is strictly larger than $\frac{1}{\sqrt{3}}\sum_{j=1}^{3}|\langle[H_j,H_{j+1}]\rangle|$, then the variance of any product state is larger than zero. Thus we know that $(\Delta R)^2_\rho \geqslant c > 0$ for any separable state ρ. On the other hand, the variance of a general state can violate such a bound. In fact, if ρ is an eigenstate of $R=\sum_{j=1}^{3}H_j\otimes\sigma_j$, then its variance $(\Delta R)^2_\rho = 0$. Moreover, since it is not separable, this eigenstate must be an entangled state. Thus one can obtain a sufficient condition of entanglement, if the variance of the state is less than the constant c, it is an entangled state. In this sense, the uncertainty constant is a critical point whether such a condition of entanglement is valid.

4. An example

In this section, we give an example to show the largest bound of R^2 and how it can be used in the distinguishment between separable and entangled states. For convenience, we assume that $H_i^2 = I$. In this case, R^2 reduces to

$$R^2 = 3I + \sum_{j=1}^{3}\left[H_j,H_{j+1}\right]\otimes i\sigma_{j+2}. \quad (17)$$

Now the norm of R^2 is

$$\begin{aligned}\|R^2\| &= \|3I + \sum_{j=1}^{3}\left[H_j,H_{j+1}\right]\otimes i\sigma_{j+2}\|\\ &\leqslant 3 + \sum_{j=1}^{3}\|\left[H_j,H_{j+1}\right]\otimes i\sigma_{j+2}\|\\ &\leqslant 9.\end{aligned} \quad (18)$$

Note that equation (18) is due to the fact that $\|H_j\| = 1$ and thus $\|[H_j,H_{j+1}]\| \leqslant 2$. It follows that

$$Tr\rho R^2 \leqslant 9, \quad (19)$$

$$|Tr\rho R| \leqslant 3. \quad (20)$$

Indeed, there exist some H_j and ρ such that the upper bounds in equations (19) and (20) can be saturated. Note that $\|[H_j, H_{j+1}] \otimes i\sigma_{j+2}\|$ is the maximal absolute eigenvalue $\lambda(max)$ of $[H_j, H_{j+1}] \otimes i\sigma_{j+2}$. Moreover, for any state ϕ, $\langle\phi|\sum_{j=1}^{3}[H_j, H_{j+1}] \otimes i\sigma_{j+2}|\phi\rangle \leqslant \sum_{j=1}^{3}\|[H_j, H_{j+1}] \otimes i\sigma_{j+2}\| = \sum_{j=1}^{3}\lambda(max)_j$. Thus to obtain the maximal value in equation (18) or equation (19), the operators $[H_j, H_{j+1}] \otimes i\sigma_{j+2}$ necessarily have a common eigenstate. To this end, one can assume that the operators $[H_j, H_{j+1}] \otimes i\sigma_{j+2}$ commute. Direct calculations show that this is valid when $[H_j, H_{j+1}]$ anti-commute. Thus they can generate a Clifford algebra. For example, one can take $H_j = \sigma_j$, then

$$R = \sum_{j=1}^{3} \sigma_j \otimes \sigma_j. \tag{21}$$

The maximal eigenvalue of R^2 is 9, with the corresponding eigenvector $\frac{1}{\sqrt{2}}(0, -1, 1, 0)^T$ which saturates the upper bound. Note that $\frac{1}{\sqrt{2}}(0, -1, 1, 0)^T$ is a pure state and can be written as $\frac{1}{\sqrt{2}}(-|01\rangle + |10\rangle)$, whose Schmidt number is two. Thus it is an entangled state.

On the other hand, when $\rho = \mu \otimes \nu$ be a product state, since $\sum_j \langle\sigma_j\rangle^2 \leqslant 1$, then by the Schwarz inequality we have

$$\begin{aligned} Tr(\mu \otimes \nu) R^2 &= 3 + \sum_{j=1}^{3} \langle i\left[H_j, H_{j+1}\right]\rangle\langle\sigma_{j+2}\rangle \\ &\leqslant 3 + \sqrt{\sum_{j=1}^{3} \|\left[H_j, H_{j+1}\right]\|^2} \\ &\leqslant 3 + 2\sqrt{3}. \end{aligned} \tag{22}$$

Hence

$$|Tr(\mu \otimes \nu) R| \leqslant \sqrt{3 + 2\sqrt{3}}. \tag{23}$$

The above bound is also valid for separable states.

Now since we have obtained the values above, the distinguishment between separable and entangled state can be made by comparing the variances. In fact, by taking $H_j = \sigma_j$ as above, the identity in equation (3) does not hold. Thus the variance in equation (9) is strictly larger than zero. Now for the separable state, the variances have some lower bound $c > 0$ which can be violated by the entangled states. That is, if the variance is less than such a constant c, the state is entangled. However, it is usually difficult to calculate such a constant. Instead, one can consider the expected values in equations (20) and (23). According to the discussions above, one can see that if the expected value is larger than $\sqrt{3 + 2\sqrt{3}}$, then the state is entangled.

5. Discussions

In [18], the proof of the uncertainty relations is based on the positivity of the expected values of the observables. First proving the product form of uncertainty, the summation form of uncertainty is obtained similarly with a previous lemma [18]. In this paper, we directly consider the variance and first prove the summation form of uncertainty relation, which has a clear physical meaning. Correspondingly, the way of mathematical deduction is different from [18]. This difference is also partly reflected by the establishment of the product form of uncertainty as a corollary of the summation form.

In [25], the violation of local uncertainty relations can be viewed as a signature of entanglement. The discussions therein mainly focused on the case where the global observable can be written as $\sum_j A_j \otimes I + I \otimes B_j$. The key is the commutativity of $A_j \otimes I$ and $I \otimes B_j$. In such a case, the variance is additive, i.e. $\Delta(\sum_j A_j \otimes I + I \otimes B_j)^2 = \sum_j(\Delta A_j^2 + \Delta B_j^2)$. This good property gives a natural lower bound of the variance of separable states. In the general case, the complicated form of the observable, e.g. $\sum_j A_j \otimes C_j + D_j \otimes B_j$ usually does not give such a neat formula of uncertainty directly, due to the fact that $A_j \otimes C_j$ and $D_j \otimes B_j$ does not commute in general. However, for the observable $R = \sum_{j=1}^{3} H_j \otimes \sigma_j$ considered in this paper, with the fine properties of Pauli matrices, it is still possible to obtain a lower bound of the variance without the commutativity of $H_j \otimes \sigma_j$.

Moreover, it is demonstrated that in section 4 that the variance of states can help in the distinguishment between separable and entangled states. Such a discussion naturally generalize the results in [25] to the case of a observable without enough commutative property. Furthermore, we showed that the coefficient of uncertainty obtained in [18] is essentially related to the distinguishment between separable and entangled states. This gives a physical interpretation to the result in [18]. On the other hand, as was pointed out in [25], it is usually difficult to determine the value of the lower bound of the variance. For this reason, we also showed that the expected value of $R = \sum_{j=1}^{3} H_j \otimes \sigma_j$ can be used in the distinguishment task, which is usually easier in manipulation and calculation.

Our discussions may shed light on the discussions of the relations of entanglement, uncertainty and the algebraic properties of the observables without enough commutativity.

Acknowledgment

This work is partially supported by the National Natural Science Foundation of China (12371135).

Appendix

By choosing an appropriate ν, one can ensure that the identities hold in equations (8)–(10) simultaneously when $\langle H_j\rangle = t\langle\sigma\rangle_j$. Note that since $i[H_j, H_{j+1}]$ are Hermitian, to ensure that $\langle i[H_j, H_{j+1}]\rangle\langle\sigma_{j+1}\rangle$ are negative, we only need to select some state $|\phi\rangle$ such that $\langle\sigma_{j+1}\rangle$ has the opposite sign to $\langle i[H_j, H_{j+1}]\rangle$. There are eight possible cases [18], for which the state are listed as follows. Calculations verify the signs and that $|\langle\sigma_j\rangle| = \frac{1}{\sqrt{3}}$.

1. $\{+,+,+\}, |\phi\rangle = \sqrt{\frac{1}{2}+\frac{\sqrt{3}}{6}}|0\rangle + e^{i\frac{\pi}{4}}\sqrt{\frac{1}{2}-\frac{\sqrt{3}}{6}}|1\rangle.$
2. $\{+,+,-\}, |\phi\rangle = \sqrt{\frac{1}{2}-\frac{\sqrt{3}}{6}}|0\rangle + e^{i\frac{\pi}{4}}\sqrt{\frac{1}{2}+\frac{\sqrt{3}}{6}}|1\rangle.$
3. $\{+,-,+\}, |\phi\rangle = \sqrt{\frac{1}{2}+\frac{\sqrt{3}}{6}}|0\rangle + e^{-i\frac{\pi}{4}}\sqrt{\frac{1}{2}-\frac{\sqrt{3}}{6}}|1\rangle.$
4. $\{-,+,+\}, |\phi\rangle = \sqrt{\frac{1}{2}+\frac{\sqrt{3}}{6}}|0\rangle - e^{-i\frac{\pi}{4}}\sqrt{\frac{1}{2}-\frac{\sqrt{3}}{6}}|1\rangle.$
5. $\{+,-,-\}, |\phi\rangle = \sqrt{\frac{1}{2}-\frac{\sqrt{3}}{6}}|0\rangle + e^{-i\frac{\pi}{4}}\sqrt{\frac{1}{2}+\frac{\sqrt{3}}{6}}|1\rangle.$
6. $\{-,+,-\}, |\phi\rangle = \sqrt{\frac{1}{2}-\frac{\sqrt{3}}{6}}|0\rangle - e^{-i\frac{\pi}{4}}\sqrt{\frac{1}{2}+\frac{\sqrt{3}}{6}}|1\rangle.$
7. $\{-,-,+\}, |\phi\rangle = \sqrt{\frac{1}{2}+\frac{\sqrt{3}}{6}}|0\rangle - e^{i\frac{\pi}{4}}\sqrt{\frac{1}{2}-\frac{\sqrt{3}}{6}}|1\rangle.$
8. $\{-,-,-\}, |\phi\rangle = \sqrt{\frac{1}{2}-\frac{\sqrt{3}}{6}}|0\rangle - e^{i\frac{\pi}{4}}\sqrt{\frac{1}{2}+\frac{\sqrt{3}}{6}}|1\rangle.$

Thus according to the Schwarz inequality, when $\langle H_j\rangle = t\langle\sigma\rangle_j$ simultaneously with some constant t, the identities in the equations are valid.

ORCID iDs

Minyi Huang 0000-0002-7063-5824
Ray-Kuang Lee 0000-0002-7171-7274

References

[1] Heisenberg W 1927 *Z. Phys.* **43** 172
[2] Robertson H P 1929 *Phys. Rev.* **34** 163
[3] Robertson H 1934 *Phys. Rev.* **46** 794
[4] Maassen H and Uffink J B 1988 *Phys. Rev. Lett.* **60** 1103
[5] Arthurs E and Goodman M 1988 *Phys. Rev. Lett.* **60** 2447
[6] Ozawa M 2004 *Ann. Phys., NY* **311** 350
[7] Busch P, Heinonen T and Lahti P 2007 *Phys. Rep.* **452** 155
[8] Wehner S and Winter A 2008 *J. Math. Phys.* **49** 062105
[9] Wehner S and Winter A 2010 *New J. Phys.* **12** 025009
[10] Watanabe Y, Sagawa T and Ueda M 2011 *Phys. Rev. A* **84** 042121
[11] Huang Y 2012 *Phys. Rev. A* **86** 024101
[12] Chen B, Cao N-P, Fei S-M and Long G-L 2016 *Quantum Inf. Process.* **15** 3909
[13] Li J-L and Qiao C-F 2016 *J. Phys. A: Math. Theor.* **50** 03LT01
[14] Coles P J, Berta M, Tomamichel M and Wehner S 2017 *Rev. Mod. Phys.* **89** 015002
[15] Miyadera T and Imai H 2008 *Phys. Rev. A* **78** 052119
[16] Dodonov V 2019 *J. Phys.: Conf. Ser.* **1194** 012028
[17] Song Q-C and Qiao C-F 2016 *Phys. Lett. A* **380** 2925
[18] Liang X-B, Li B and Fei S-M 2024 *Sci. China Phys. Mech. Astron.* **67** 290311
[19] Einstein A, Podolsky B and Rosen N 1935 *Phys. Rev.* **47** 777
[20] Horodecki R, Horodecki P, Horodecki M and Horodecki K 2009 *Rev. Mod. Phys.* **81** 865
[21] Bell J S 1964 *Phys. Phys. Fiz.* **1** 195
[22] Cirel'son B S 1980 *Lett. Math. Phys.* **4** 93
[23] Landau L J 1987 *Lett. Math. Phys.* **14** 33
[24] Landau L J 1987 *Phys. Lett. A* **120** 54
[25] Hofmann H F and Takeuchi S 2003 *Phys. Rev. A* **68** 032103
[26] Wolf M, Perez-Garcia D and Fernandez C 2009 *Phys. Rev. Lett.* **103** 230402
[27] Oppenheim J and Wehner S 2010 *Science* **330** 1072
[28] Ma W, Chen B, Liu Y, Wang M, Ye X, Kong F, Shi F, Fei S-M and Du J 2017 *Phys. Rev. Lett.* **118** 180402
[29] Watrous J 2018 *The Theory of Quantum Information* (Cambridge University Press)