Stabilization of counter-rotating vortex pairs in nonlocal media

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The dynamics of vector vortex pairs with explicit and hidden vorticity in nonlocal media with an arbitrary degree of nonlocality is investigated analytically and numerically. We show that the stability dynamics of the vortex pairs depends crucially upon their total angular momentum, the topological charge, and the degree of nonlocality. In particular, we show that nonlocality eliminates the splitting of the vortex pairs, improves their propagation stability, and leads to formation of stationary bound states of vortex pairs.

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I. INTRODUCTION

Propagation dynamics of nonlocal solitons has been a subject of intense research efforts [1]. Nonlocality of nonlinearity occurs in many physical settings which include, for instance, nematic liquid crystals [2,3], thermal media [4,5], atomic vapors [6], or Bose-Einstein condensates with a long-range interparticle interaction [7]. Physically, the nonlocal character of the nonlinearity signifies the fact that nonlinearity such as the light-induced refractive-index change of a material at a particular location is determined by the light intensity in a certain neighborhood of this location [1]. So far many works have shown that nonlocal nonlinearity has profound effects on the propagation dynamics of the beams. For instance, nonlocality can prevent the catastrophic collapse of high-dimensional beams [8] and suppress modulation instability [9,10] and transverse instability [11] in self-focusing media. Nonlocality can support a range of novel soliton states including dipole [12–14] and multipole solitons [15,16], discrete solitons in optical lattices with nonlocal nonlinearity [17–20], and incoherent solitons in nonlocal media with both instantaneous and slow nonlinear response [21–25]. Nonlocal nonlinearity also promotes the stability of vector or multicomponent solitons, such as dipole [26] and multipole solitons [27,28], two-color solitons [29,30], dark solitons [31], and vortex [32] and necklace solitons [33]. Moreover, by providing attractive forces between out-of-phase bright [34,35] and dark solitons [36–38] the nonlocality promotes formation of their bound states.

Vortex solitons, i.e., localized donutlike structures with phase singularities in the center [39], were shown to be unstable in local media [40]. The stability of scalar vortex solitons has been established theoretically for media with competing nonlinearity [41]. In recent years, the propagation of vortex beams has also drawn considerable attention in the context of nonlocal nonlinear media [5,42–52]. The nonlocal nonlinearity can suppress the azimuthal instability of beams with angular momentum, forming stationary vortex solitons and azimuthons. Besides the typical vortex solitons and azimuthons with circular symmetry, nonlocality can also stabilize the more general elliptic vortex solitons called ellipticons [53–55].

It has been demonstrated that the azimuthal instability of vortex solitons in pure local media [56] can be suppressed by using two-component (vectorial) vortex solitons. Such vector solitons carry integer vorticities $(m,\pm m)$ in their two components. Many works have shown that the two-component soliton with *hidden vorticity*, i.e., $(m,-m)$ type, exhibits better stability than the soliton with *explicit vorticity*, i.e., (m,m) type. The instability dynamics of such vortex pairs has been studied extensively in models with saturable nonlinearity [57–59], or cubic-quintic [52,60–63] and quadratic-cubic [64] competing nonlinearities. These works have shown that the vector solitons with an explicit vorticity are always unstable against splitting in saturable nonlinear media [59]. In a recent paper the propagation of fundamental charged vortex pairs with hidden and explicit vortices was demonstrated experimentally in nematic liquid crystals [65], which shows that vortex pairs with hidden vorticity can be stable but vortex pairs with explicit vorticity always break up and transform into vector dipole solitons.

In this paper, we investigate both analytically and numerically the stability dynamics of vector vortex pairs with explicit and hidden vorticities in nonlocal media with an arbitrary degree of nonlocality. We consider solitons with fundamental as well as higher charge. We demonstrate the breakup dynamics of the vortex pairs with different topological charges and show how the nonlocality improves their stability. It appears that while nonlocality enhances the stability of all vortex soliton pairs with either explicit or hidden vorticity, the latter is always more stable.

II. MODEL AND BASIC EQUATIONS

We start with a vector model consisting of two mutually incoherently coupled optical beams propagating in a medium with a spatially nonlocal nonlinear response. Here, the two components refer to the case that the vortices of our vector model have different polarizations [66]. In order to have incoherently coupled vector optical beams, their relative phase must vary more quickly than the characteristic response time of the material. This can be easily realized in some slow (inertial) nonlinear media such as photorefractive crystals or liquid crystals. By requiring the frequency difference between constituent components (typically $\approx$1 kHz) to be larger than the inverse of the material response time, the medium responds only to the sum of intensities [67]. Moreover, without loss of generality, we consider that the components of vector vortex pairs have the same parameters of the amplitude distribution and they are mutually incoherent.

The propagation of the slowly varying beam envelopes $E_n(x,y,z)$ can be described by the normalized coupled nonlocal nonlinear Schrödinger equations ($n=1,2$) [26,33]

$$i\frac{\partial\psi_n}{\partial z}+\frac{\partial^2\psi_n}{\partial x^2}+\frac{\partial^2\psi_n}{\partial y^2}+\psi_n\int R(\mathbf{r}-\mathbf{r}')I(\mathbf{r}',z)d^2\mathbf{r}'=0, \tag{1}$$

where the propagation coordinate z is measured in units of the diffraction length L_D and the transverse coordinates (x,y) are measured in units of $(L_D/k)^{1/2}$. $I=|\psi_1|^2+|\psi_2|^2$ is the total beam intensity and $R(r)$ is the normalized nonlocal response function with $\int_{-\infty}^{\infty}R(r)dr=1$.

In this paper, we focus on the vectorial vortex solitons with explicit (m,m) and hidden $(m,-m)$ vorticity, respectively. Thus, we look for solutions of the vector cortex pairs in the form of $\psi_n=U_n\exp(ikz\pm im\varphi)$, where U_n is the amplitude envelope, k is the propagation constant, and $\varphi=\tan^{-1}(y/x)$. In previous works, the amplitude envelope U_n can be obtained numerically, for instance, by the shooting method with two-point boundary conditions [57,59]. However, here we will employ the Lagrangian (or variational) approach to study the dynamics of the vector vortex pairs. It is easy to show that Eq. (1) can be derived from the following Lagrangian density:

$$\mathcal{L}=\sum_{n=1,2}\frac{i}{2}\left(\psi_n^*\frac{\partial\psi_n}{\partial z}-\psi_n\frac{\partial\psi_n^*}{\partial z}\right)-\left(\left|\frac{\partial\psi_n}{\partial x}\right|^2+\left|\frac{\partial\psi_n}{\partial y}\right|^2\right)+\frac{1}{2}|\psi_n|^2\int R(\mathbf{r}-\mathbf{r}')I(\mathbf{r}',z)d^2\mathbf{r}'. \tag{2}$$

$R(\mathbf{r})$ is the normalized nonlocal response function with a characteristic width σ which represents the degree of nonlocality. In our variational approach, we consider here the case of a Gaussian nonlocal response [13,21]

$$R(\mathbf{r})=(\pi\sigma^2)^{-1}\exp(-\mathbf{r}^2/\sigma^2). \tag{3}$$

The most important point of the variational method is the choice of an appropriate and physical ansatz representing the optical beam. For convenience, we choose a typical single-ring vortex form for the amplitude of a scalar vortex,

$$U=Ar^m\exp(-r^2/2w^2)\exp(ikz+im\varphi), \tag{4}$$

with the amplitude A and the beamwidth w [42]. The power of the scalar vortex beam can be obtained by $P=\iint|U|^2dxdy$; for instance, the total power is $P=\pi A^2w^4$ for a singly charged ($m=1$) vortex beam and $P=720\pi A^2w^{14}$ for a higher-order vortex beam with $m=6$. When considering the vector vortex pairs, we assume that they share the same envelope of the scalar vortex and carry the equal powers $P_1=P_2=P/2$ [56,57],

$$\psi_1=\frac{\sqrt{2}}{2}Ar^m\exp(-r^2/2w^2)\exp(ikz+im\varphi), \tag{5}$$

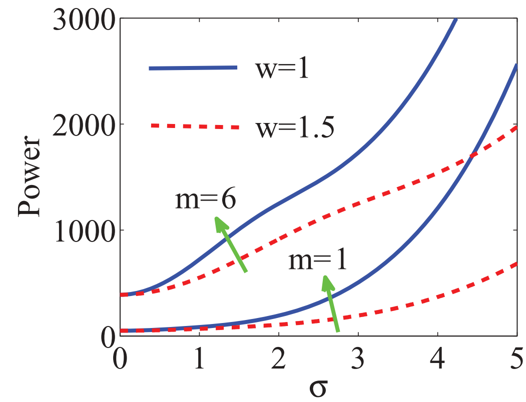

FIG. 1. (Color online) Total power of the vector dipole solitons versus degree of nonlocality for different beamwidths.

and

$$\psi_2=\frac{\sqrt{2}}{2}Ar^m\exp(-r^2/2w^2)\exp(ikz\pm im\varphi). \tag{6}$$

The sign $\pm$ in Eq. (6) corresponds to explicit and hidden vorticity vortex pairs, respectively.

Using the ansatz Eqs. (5) and (6) in the Lagrangian density $\mathcal{L}$ one can evaluate the reduced (or effective) Lagrangian $L=\int_{-\infty}^{\infty}\mathcal{L}dxdy$ which depends only on the parameters A and w. From the corresponding Euler-Lagrange equations we then obtain the approximately analytical solution for the amplitude A of the vortex pairs. In Fig. 1, we show the total power of singly charged ($m=1$) and higher-order charged ($m=6$) vector vortex beams as a function of the degree of nonlocality with different beamwidths. We also emphasize that all quantities plotted in the figures are dimensionless in this paper. It is obvious that, for both singly and higher-order charged vector vortex solitons, the total power of the vector vortex solitons will decrease when the beamwidth increases, whereas the total power will increase when the degree of the nonlocality increases. It is also obvious that the power of higher-order vortex solitons is always larger than the power of singly charged vortex solitons.

III. VORTEX PAIRS WITH FUNDAMENTAL TOPOLOGICAL CHARGE

In the following, we will study the dynamics of the vortex pairs by direct numerical simulations by using the split-step Fourier-transform method. It should be noted that the dynamics of the vortex pairs can also be investigated by a linear stability analysis with a perturbed solution [58,59]. The variational results shown in Fig. 1 will be used as initial conditions in our two-dimensional numerical codes.

In this section, we consider the dynamics of the vortex pairs with fundamental topological charge $m=1$. In Figs. 2–4 we show the stability dynamics and interactions of the vortex pairs with explicit $(1,1)$ and hidden $(1,-1)$ vorticity in nonlocal media with weak, intermediate, and strong degrees of nonlocality. In the simulations throughout the paper we will use solutions with the initial beamwidth $w=1$. For comparison we also present the evolution of a scalar vortex [Eq. (4)] in nonlocal nonlinear media.

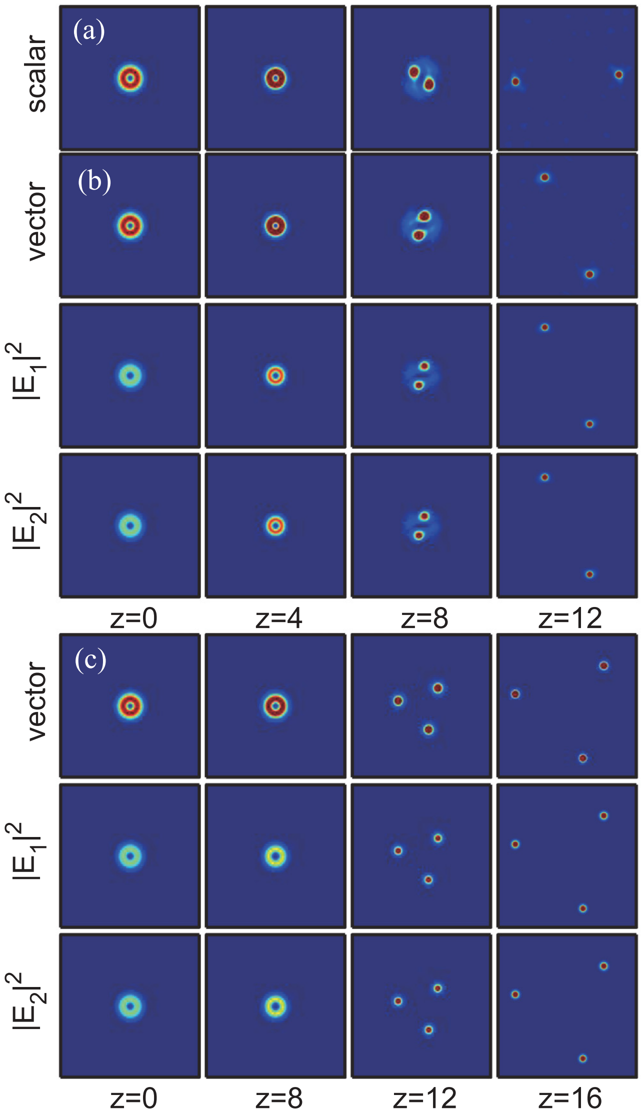

FIG. 2. (Color online) Azimuthal instability of singly charged ($m = 1$) vortex solitons in nonlinear media with weak nonlocality $\sigma = 0.2$. Symmetric breaking of (a) scalar vortex solitons, (b) vector vortex solitons with explicit vorticity (1,1), and (c) vector vortex solitons with hidden vorticity $(1,-1)$. $|E_1|^2$ and $|E_2|^2$ represent the intensity of constituent components of the vector vortex solitons.

From Figs. 2–4, we find that the dynamics of vortex pairs with explicit vorticity (1,1) [graphs (b) in Figs. 2–4] are essentially equivalent to the dynamics of the scalar vortex [graphs (a) in Figs. 2–4]. This result was also previously obtained in local media. Because the nonlocality cannot prevent the azimuthal instability of the vortex beams, the vortex pairs will break up and split into several fundamental filaments in local (not shown) and weakly nonlocal media. However, the breakup dynamics of the vortex pairs with explicit and hidden vorticity are very different. The scalar vortex beam [Fig. 2(a)] and the vortex pair with explicit vorticity [Fig. 2(b)], decay into a pair of scalar solitons with equal power in the case of weak nonlocality with $\sigma = 0.2$. However, the vortex pair with hidden vorticity will split into three fundamental scalar filaments in the case of weak nonlocality with $\sigma = 0.2$ [Fig. 2(c)]. We also plot in Fig. 1 the dynamics of the components of the vector pairs. In this case of weak nonlocality the soliton dynamics (splitting into two or three fundamental solitons) of the vortex pairs with explicit (1,1) and hidden vorticity $(1,-1)$ is basically the same as in local media.

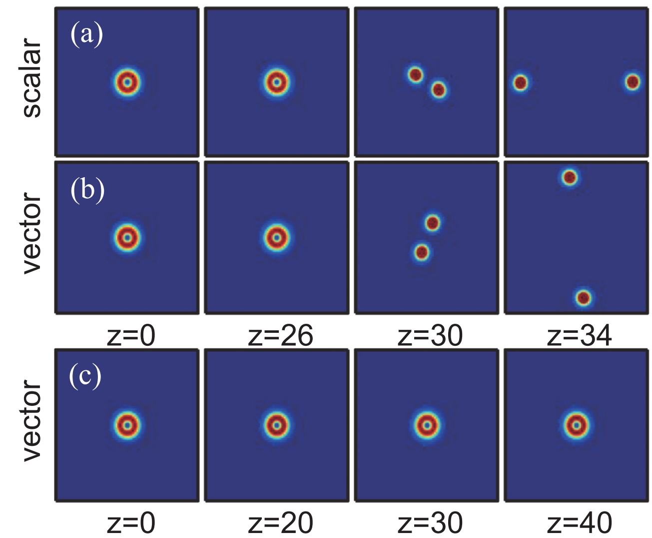

FIG. 3. (Color online) Azimuthal instability of singly charged ($m = 1$) vortex solitons in nonlinear media with moderate degree of nonlocality $\sigma = 1.2$. Symmetric breaking of (a) scalar vortex solitons, (b) vector vortex solitons with explicit vorticity (1,1), and (c) vector vortex solitons with hidden vorticity $(1,-1)$.

As shown in Fig. 3, increase of the degree of nonlocality from weak to moderate with $\sigma = 1.2$ can effectively improve the stability of the vortex pairs with both explicit and hidden vorticity. For example, the vortex pair with explicit vorticity can be stable at the propagation distance $z = 26$ in the case of moderate nonlocality [Fig. 3(b)], whereas it can be stable

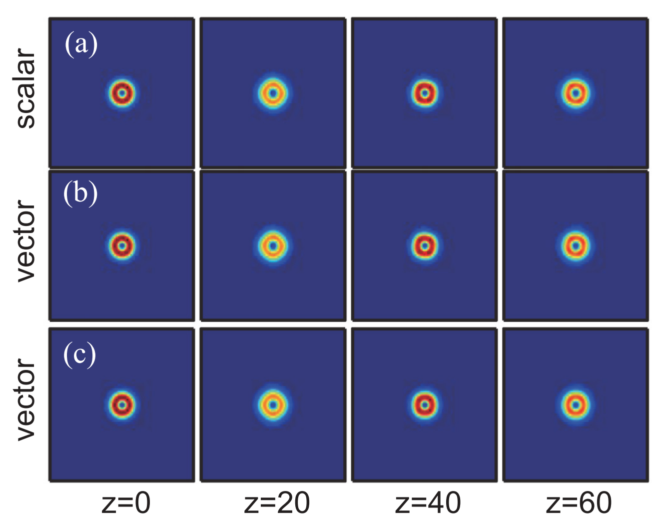

FIG. 4. (Color online) Stationary self-trapping of singly charged ($m = 1$) vortex solitons in nonlinear media with strong degree of nonlocality $\sigma = 10$: (a) scalar vortex solitons, (b) vector vortex solitons with explicit vorticity (1,1), and (c) vector vortex solitons with hidden vorticity $(1,-1)$.

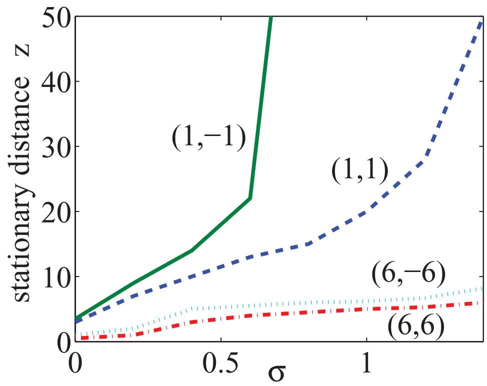

FIG. 5. (Color online) Stationary propagation distance of singly charged vortex solitons (1,±1) and higher-order charged vortex solitons (6,±6) versus degree of nonlocality. The increasing of the stationary propagation distance indicates a better stability.

only at the propagation distance $z = 4$ in weakly nonlocal media [Fig. 3(b)]. This is also the case for the vortex pair with hidden vorticity, i.e., it broke into three fundamental solitons at propagation distance $z = 12$ in the case of weak nonlocality [Fig. 2(c)], but it can be still stable at the propagation distance $z = 40$ with a moderate degree of nonlocality [Fig. 3(c)]. Because the dynamics of the components are similar to those of the vector vortex pairs, we do not show their dynamics in Figs. 3 and 4.

Figures 2 and 3 also show that the vortex pair with hidden vorticity is more stable than the vortex pair with explicit vorticity in weakly and moderately nonlocal media. It is obvious that the stable propagation distances for the vortex pair with explicit vorticity are $z = 4$ and $z = 26$ when the degrees of nonlocality are weak and moderate, respectively. The vortex pair with hidden vorticity exhibits a larger stable propagation distance than the vortex pair with explicit vorticity, i.e., $z = 8$ and more than $z = 40$ in weakly and moderately nonlocal media. This result of increased stability of vortex pairs with hidden vorticity compared to that of explicit vorticity was also found earlier in local media using a linear stability analysis [58].

In Fig. 4, we display completely stable vortex pairs with both explicit and hidden vorticity in strongly nonlocal media with the degree of nonlocality $\sigma = 10$. In contrast to the result that the vortex pair with explicit vorticity is always unstable in local media and the vortex pair with hidden vorticity can only be stable within a small propagation constant interval, the strong nonlocality induces a broad trapping potential well, which can completely suppress the azimuthal instability, leading to the formation of stable scalar vortex solitons [Fig. 4(a)], and vortex pairs with explicit [Fig. 4(b)] and hidden [Fig. 4(c)] vorticity.

To illustrate the stabilizing effect of nonlocality on solitons in Fig. 5 we plot the stationary propagation distance for both the singly charged vortex soliton (1,±1) and the higher-order charged vortex soliton (6,±6) as a function of the degree of nonlocality. It is evident that the stationary propagation distance of solitons with hidden vorticity is always larger than for solitons with explicit vorticity for a singly charged vortex pair.

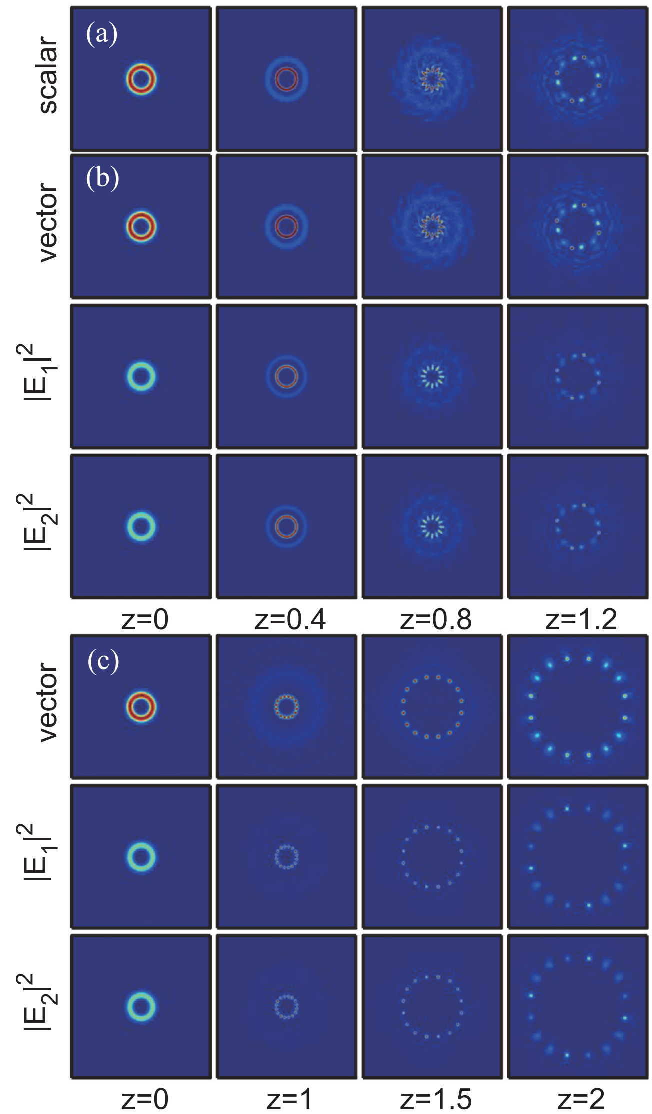

FIG. 6. (Color online) Azimuthal instability of higher-order charged ($m = 6$) vortex solitons in local nonlinear media ($\sigma = 0$). Symmetric breaking of (a) scalar vortex solitons, (b) vector vortex solitons with explicit vorticity (6,6), and (c) vector vortex solitons with hidden vorticity (6,−6). $|E_1|^2$ and $|E_2|^2$ represent the two-component intensity of the vector vortex solitons.

IV. VORTEX PAIRS WITH HIGHER-ORDER TOPOLOGICAL CHARGE

In this section we investigate the dynamics of explicit and hidden vorticity vortex pairs with higher-order topological charge m in nonlocal media. As an example, we demonstrate the dynamics of explicit and hidden vorticity vortex pairs with topological charge $m = 6$. In Figs. 6–9, the stability of the vortex pairs is displayed for zero, weak, intermediate, and strong degrees of nonlocality. It should be stressed that even though we consider here only vortex pairs with a particular value of topological charge $m = 6$, our results are applicable to arbitrary values of m.

It is seen that the vortex pairs with explicit and hidden vorticity will always split into two and three fundamental

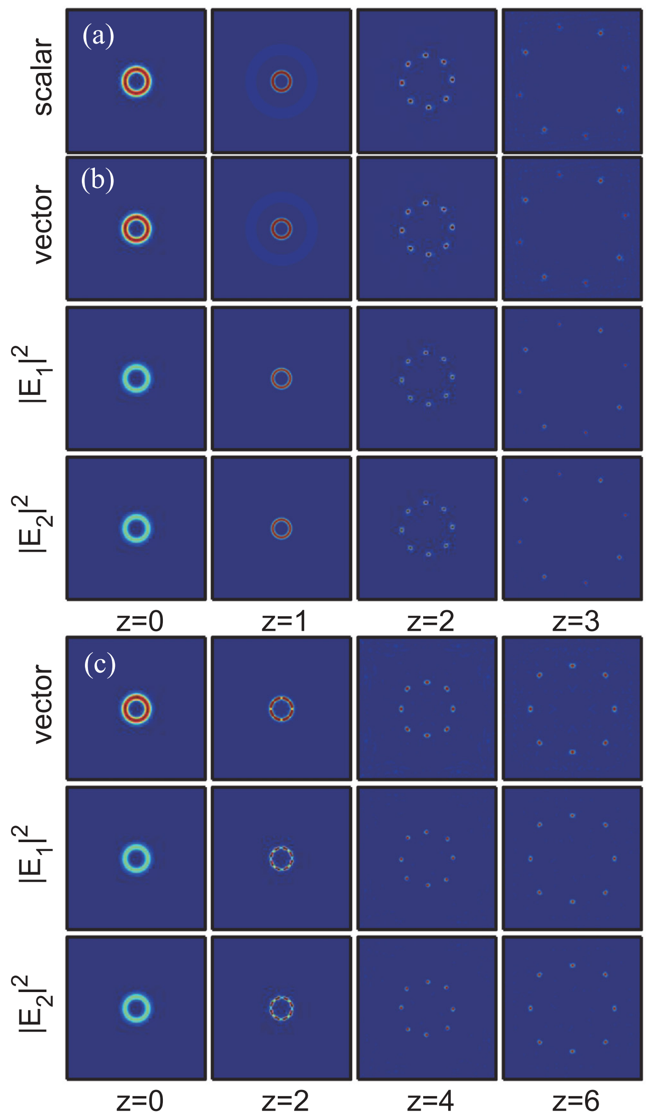

FIG. 7. (Color online) Azimuthal instability of higher-order charged ($m = 6$) vortex solitons in nonlinear media with weak degree of nonlocality $\sigma = 0.2$. Symmetric breaking of (a) scalar vortex solitons, (b) vector vortex solitons with explicit vorticity (6,6), and (c) vector vortex solitons with hidden vorticity (6,−6). $|E_1|^2$ and $|E_2|^2$ represent the intensities of the two constituent components of the vector vortex solitons.

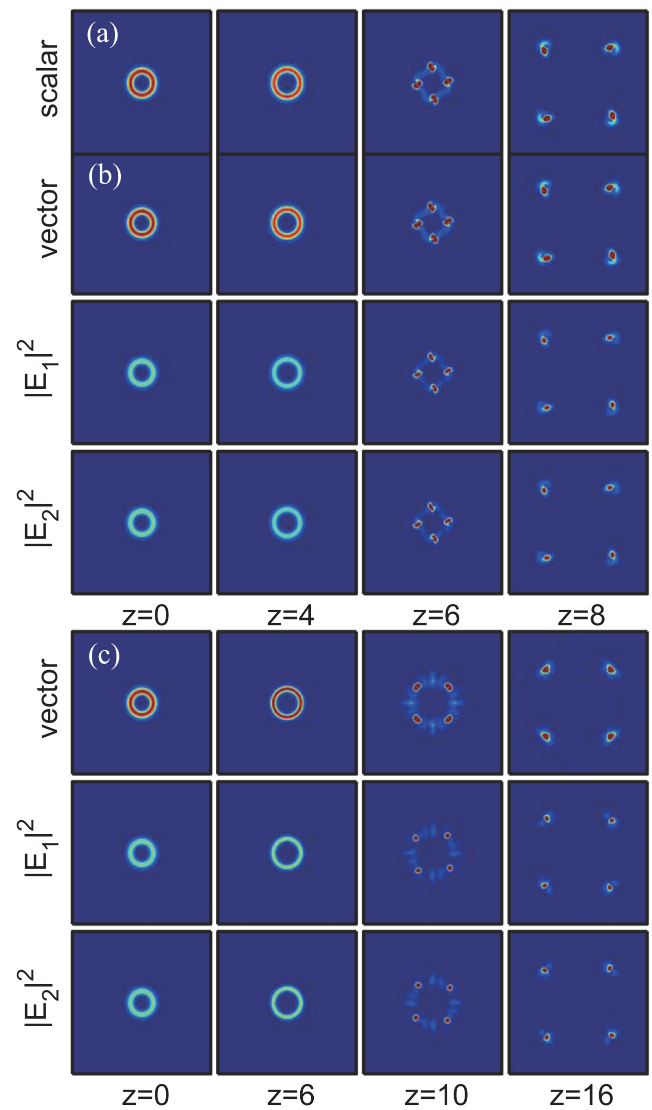

FIG. 8. (Color online) Azimuthal instability of higher-order charged ($m = 6$) vortex solitons in nonlinear media with moderate degree of nonlocality $\sigma = 1.2$. Symmetric breaking of (a) scalar vortex solitons, (b) vector vortex solitons with explicit vorticity (6,6), and (c) vector vortex solitons with hidden vorticity (6,−6). $|E_1|^2$ and $|E_2|^2$ represent the two-component intensity of the vector vortex solitons.

filaments when the vector vortex pairs suffer from azimuthal instability; the breaking dynamics of such vortex pairs with higher-order topological charge exhibit more complexity. As for the vortex pairs with topological charge $m = 1$, the propagation of explicit vorticity vortex pairs is also the same as that for the scalar vortex beam when the charge is $m = 6$, as shown in Figs. 6–9. We show in Fig. 6 the breaking dynamics of the vortex pairs in local media with $\sigma = 0$. For the scalar vortex and the explicit vorticity vortex pairs with topological charge m, the beam will split into $2m$ fundamental filaments in local media. It is clear that both the scalar vortex beam [Fig. 6(a)] and the explicit vorticity vortex pair [Fig. 6(b)] break up into 12 filament beams when the charge is $m = 6$. However, the hidden vorticity vortex pair will split into 16 fundamental beams in local media [Fig. 6(c)].

Increasing the degree of nonlocality will also improve the stability of the vortex pairs, resulting in larger stable propagation distances and smaller numbers of breakup filaments. In local media, the stable propagation distance for the scalar vortex beam and the explicit vorticity vortex pair is $z = 0.4$. When the degree of the nonlocality is weak ($\sigma = 0.2$) and intermediate ($\sigma = 1.2$), the stable propagation distances are $z = 1$ [Figs. 7(a) and 7(b)] and $z = 4$ [Figs. 8(a) and 8(b)], respectively. This also happened to the hidden vorticity vortex pair, for which the stable propagation distances are $z = 1$, $z = 2$ [Fig. 5(c)], and $z = 6$ [Fig. 6(c)] when the degrees of nonlocality are local, weak, and intermediate, respectively. The nonlocality can also reduce the number of breaking filament beams. In local media, the explicit and hidden vorticity vortex pairs will split into 12 and 16 fundamental beams (Fig. 6), and

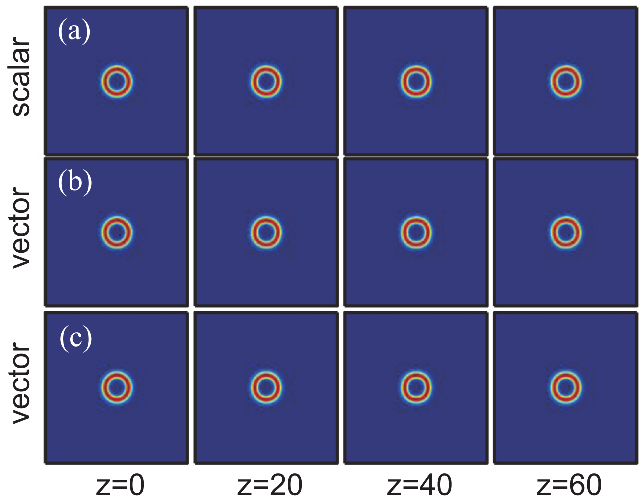

FIG. 9. (Color online) Stationary self-trapping of higher-order charge ($m = 6$) vortex solitons in nonlinear media with high degree of nonlocality $\sigma = 10$: (a) scalar vortex solitons, (b) vector vortex solitons with explicit vorticity (6,6), and (c) vector vortex solitons with hidden vorticity (6,−6).

they break up into only eight (Fig. 7) and four (Fig. 8) filament beams in weakly and moderately nonlocal media. This result clearly indicate that nonlocality can effectively suppress the azimuthal instability of the vortex pairs and improve their stability.

Comparing graphs (b) and (c) in Figs. 6–8, we can see that the stable propagation distance of the vortex pair with hidden vorticity is always larger than that of the vortex pair with explicit vorticity when the degree of the nonlocality is fixed. The result is also evident in Fig. 5 where we show the stationary propagation distance for higher-charged ($m = 6$) vector vortex pair with both explicit and hidden vorticities. As discussed above, we know that the vortex pair with hidden vorticity is also more stable than the vortex pair with explicit vorticity even when the vortex beams carry higher-order topological charge. We also find that the vortex pairs with fundamental charge $m = 1$ have a more stationary dynamics than those with higher-order charge $m = 6$ (also check Fig. 5). As shown in Figs. 2 and 3, the stationary propagation distances of the explicit vorticity vortex pair with fundamental charge (1,1) are $z = 4$ [Fig. 2(b)] and $z = 26$ [Fig. 3(b)] in weakly and moderately nonlocal media, which are larger than those for higher-order charge (6,6) $z = 1$ [Fig. 7(b)] and $z = 4$ [Fig. 8(b)]. This result is also applicable to the case of vortex pairs with hidden vorticity. In local media, the conclusion has been explained as a smaller maximum instability growth rate of the vortex pairs with fundamental charge ($1,\pm1$) than of those with higher-order charge ($m, \pm m$) [58].

Although the vortex pairs with higher charge are less stable than those with fundamental charge, strong nonlocality can also completely stabilize scalar vortex solitons and vortex pairs with explicit and hidden vorticity. In Fig. 9, we show the stable propagation of vortex pairs with both explicit and hidden vorticity in strongly nonlocal media with the degree of nonlocality $\sigma = 10$.

V. CONCLUSION

In conclusion, we have investigated the dynamics of vector vortex pairs with explicit and hidden vorticity in nonlocal media with an arbitrary degree of nonlocality. We showed that a nonlocality-induced trapping potential (or waveguide) eliminates the splitting of the vortex pairs and leads to the formation of bound states of vortex pairs which are otherwise always unstable in local media. Our results also demonstrate that the vortex pair with hidden vorticity is always more robust than that with explicit vorticity although this difference in stability disappears for high charge. Finally, we show that the solitons with fundamental constituent charges exhibit the best stability properties, and the stability worsens with the higher charge. Our general results are in agreement with recent experimental observations of the dynamics of vector solitons with hidden and explicit vorticity in nematic liquid crystals [65].

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