Dark solitons in nonlocal media with competing nonlinearities

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We investigate analytically and numerically the propagation properties of dark solitons in nonlocal media with competing nonlinearities. We obtain analytical relations for soliton parameters for an arbitrary degree of nonlocality. In particular, we show that the velocity of dark solitons can be affected by the degree of nonlocality of competing nonlinearities. The analytical results are confirmed by direct numerical simulations of the full model describing propagation of dark solitons in nonlocal media with competing nonlinearities.

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I. INTRODUCTION

There has been growing interest in various aspects of nonlocal response of nonlinear media instigated by the the fact that nonlocality is common to many nonlinear systems including, e.g., in nematic liquid crystals [1], media with thermal nonlinearity [2], atomic vapors [3], plasmas [4], Bose-Einstein condensates [5], etc. In the context of optics nonlocality of nonlinearity means the light-induced refractive index change of a material at a particular location is determined by the light intensity in a certain neighborhood of this location [6]. Nonlocality appears to have a significant effect on propagation of beams and their localization [6]. For instance, nonlocality can promote modulational instability in self-defocusing media, or suppress it in self-focusing media [7,8]. Nonlocality may also suppress transverse instability [9] of optical waves and prevent the catastrophic collapse of self-focusing beams in nonlinear media [10,11]. Moreover, the nonlocal nonlinearity affect the interactions between bright solitons as observed in experiments with lead glasses [2] and nematic liquid crystal [12]. Nonlocality can also support complex solitons states, such as dipole and multipole solitons [13–16], optical lattice solitons [17–19], vortex solitons [20–24], surface solitons [25,26], incoherent solitons [27–29] and vector solitons [13,30–35].

It has been also shown that nonlocality of self-defocusing nonlinearity can stabilize the propagation of spatial optical dark solitons [36,37]. Theoretical and experimental works have shown that the propagation and interactions of dark solitons in nonlocal media exhibit many novel features which do not occur in local media [38–40]. In particular nonlocality can support the self-trapping of polarized vector dark solitons [41] and dark-bright vector soliton pairs [42]. The nonlocality also provides a long-ranged attractive force to balance the repulsive interactions of dark solitons leading to the formation of stationary bound states of dark solitons [43–45].

In recent years spatial optical solitons and their interactions in nonlocal media with competing nonlinearities have been a subject of research efforts [46,47]. Generally competing nonlinearities occur in systems where few different physical processes contribute to the overall nonlinear response. This is, e.g., the case of Bose-Einstein condensate with simultaneous local and long range bosonic interaction [48] and nematic liquid crystals with comparable thermal and orientational nonlinearities [49]. It has been shown that the competing nonlinearities can stabilize many complex soliton structures, e.g., solitons of even and odd parities [50], gap solitons [51], accessible light bullets [52], and higher order vortex solitons [53], which are otherwise unstable in a medium with one type of nonlocal nonlinearity. As far as dark solitons are concerned, the competing nonlocal nonlinearities can, on the one hand, destabilize dark soliton states [54] and, on the other hand, enable coexistence of dark and bright spatial solitons [55]. It was also shown that the competing local quintic contribution to nonlocal cubic nonlinearity has profound effects on the bright and dark solitons in the regime of weak nonlocality [56]. Recently we have revealed that such a system supports unique dark soliton solutions with their width being independent of the degree of nonlocality [57].

In this paper we study analytically and numerically the propagation properties of dark solitons in media with competing nonlocal nonlinearities. By using the phenomenological rectangular model of the nonlocality and the variational approach, we obtain analytical dark soliton solutions for an arbitrary degree of nonlocality. Moreover, we also show how the competition of nonlinearities affects soliton velocity. We confirm analytical results using direct numerical simulations with a split-step Fourier transform method.

II. MODEL AND VARIATIONAL APPROACH

We consider a one-dimensional optical beam with an amplitude $u(x,z)$ propagating along the z axis and diffracting only in the transverse direction x. The evolution of such a beam in nonlocal media with competing defocusing and focusing nonlinearities is governed by the following generic dimensionless nonlocal nonlinear Schrödinger (NLS) equation:

$$i\frac{\partial u}{\partial z}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+\triangle n u=0, \tag{1}$$

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where the nonlinear refractive index change of the medium $\triangle n$ can be represented by the following convolution integral [55]:

$$\begin{aligned}\triangle n(x,I) &= \triangle n_1(x,I) + \triangle n_2(x,I)\\ &= \alpha_1 \int_{-\infty}^{+\infty} R_1(x-\xi)|u(\xi,z)|^2 d\xi\\ &\quad + \alpha_2 \int_{-\infty}^{+\infty} R_2(x-\xi)|u(\xi,z)|^2 d\xi. \end{aligned} \tag{2}$$

Here we used the phenomenological model of the nonlocal nonlinearity. Parameters α_1 and α_2 represent the relative strength and sign of the two nonlinear contributions, respectively. In what follows we will assume $\alpha_1 = -1$, corresponding to the defocusing nonlinearity. Hence the positive parameter α_2 will represent the relative strength of the focusing nonlocal nonlinearity. The nonlocal response function $R_{1,2}(x)$, which is real, symmetric, and normalized $\int_{-\infty}^{+\infty} R(x)dx = 1$, defines the nonlocal character of nonlinearity. Its width determines the degree of nonlocality. In particular, $R(x) = \delta(x)$ for a local Kerr medium. In the limit of weak nonlocality, where the width of nonlocal response function is much narrower than the spatial extent of the beam, the influences of nonlocality can be studied by a single parameter, without considering the specific form of the nonlocal response function [38]. While the propagation of optical beams in a highly nonlocal medium can be treated as linear harmonic oscillation, the incident total power takes the place of the role of a nonlinear term [58].

To analyze analytically the nonlocal NLS equation, first we employ the variational approach. In the case of dark solitons, the renormalized Lagrangian density corresponding to Eq. (1) is given in the following form:

$$\begin{aligned}\mathcal{L} &= \frac{i}{2}\left(u^*\frac{\partial u}{\partial z} - u\frac{\partial u^*}{\partial z}\right)\left(1 - \frac{1}{|u|^2}\right) - \frac{1}{2}\left|\frac{\partial u}{\partial x}\right|^2\\ &\quad + \frac{\alpha_1}{2}(|u|^2-1)\int_{-\infty}^{+\infty} R_1(x-\xi)(|u|^2-1)d\xi\\ &\quad + \frac{\alpha_2}{2}(|u|^2-1)\int_{-\infty}^{+\infty} R_2(x-\xi)(|u|^2-1)d\xi, \end{aligned} \tag{3}$$

with the background intensity of the solitons normalized to unity. In particular, the form of the nonlocal response function is determined by the nonlocal process of actual physical system, e.g., exponential function $R(x) = (2\sigma)^{-1}\exp(-|x|/\sigma)$ describes orientational-type nonlinearities of nematic liquid crystals [1]. To make the problem solvable analytically and without loss of generality, we consider here the nonlocal response function $R_i(x)$ with a rectangle profile [40,57]:

$$R_1(x) = \begin{cases} \frac{1}{2\sigma_1}, & |x| \leqslant \sigma_1,\\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

$$R_2(x) = \begin{cases} \frac{1}{2\sigma_2}, & |x| \leqslant \sigma_2,\\ 0, & \text{otherwise}, \end{cases} \tag{5}$$

here $\sigma_{1,2}$ defines the width of the respective nonlocal response. This particular choice of nonlocal response function is obviously an approximation. However, its power lies in the fact that it enables correct analytical description of soliton properties and even soliton interaction as confirmed by our earlier works [40,45]. It is also worth mentioning that while $R_i(x)$ exhibits sharp jumps in its profile the actual light-induced refractive index profile is a smooth function of spatial coordinates because it is defined by the convolution between $R_i(x)$ and the light intensity distribution. Therefore the slowly varying envelope approximation used to model soliton dynamics is still applicable.

In order to investigate the dark solitons analytically for an arbitrary degree of nonlocality, we consider the following localized dark soliton solution as the ansatz in variational calculations [40,57]:

$$u(x,z) = B\tanh[D(x-x_0)] + iA, \tag{6}$$

where parameters A and B satisfy the normalization condition $A^2 + B^2 = 1$. Here A, B, D, and x_0 are assumed to be functions of the propagation variable z, x_0 gives soliton center location, and the soliton width at the half maximum is $w = \frac{1}{D}\text{arctanh}[\frac{1}{\sqrt{2}}]$ [38].

Substituting Eqs. (4), (5), and (6) into the Lagrangian density in Eq. (3) and integrating over the whole x space, one can get the averaged Lagrangian,

$$\begin{aligned}L &= \int_{-\infty}^{\infty}\mathcal{L}(u)dx = 2\frac{dx_0}{dz}\left[-AB + \tan^{-1}\left(\frac{B}{A}\right)\right] - \frac{2}{3}B^2D\\ &\quad - \frac{\alpha_1 B^4}{D}\left[\text{csch}^2(D\sigma_1) - \frac{\coth(D\sigma_1)}{D\sigma_1}\right]\\ &\quad - \frac{\alpha_2 B^4}{D}\left[\text{csch}^2(D\sigma_2) - \frac{\coth(D\sigma_2)}{D\sigma_2}\right]. \end{aligned} \tag{7}$$

From the corresponding Euler-Lagrangian equations we can find that $B = \text{const}$, and

$$\begin{aligned}\frac{1}{3B^2} &= \frac{\alpha_1\coth(D\sigma_1)}{D\sigma_1}\left[\sigma_1^2\text{csch}^2(D\sigma_1) - \frac{1}{D^2}\right]\\ &\quad + \frac{\alpha_2\coth(D\sigma_2)}{D\sigma_2}\left[\sigma_2^2\text{csch}^2(D\sigma_2) - \frac{1}{D^2}\right], \end{aligned} \tag{8}$$

$$\begin{aligned}\frac{dx_0}{dz} &= \frac{AD}{3B} + \frac{\alpha_1 AB}{D}\left[\text{csch}^2(D\sigma_1) - \frac{\coth(D\sigma_1)}{D\sigma_1}\right]\\ &\quad + \frac{\alpha_2 AB}{D}\left[\text{csch}^2(D\sigma_2) - \frac{\coth(D\sigma_2)}{D\sigma_2}\right]. \end{aligned} \tag{9}$$

III. SOLITONS SOLUTIONS AND THEIR DYNAMICS

The formulas in Eqs. (8) and (9) represent analytical relations among parameters of the dark soliton in a nonlocal medium with competing nonlinearities. In Fig. 1 we show the dependence of soliton width w as a function of the degrees of nonlocality (σ_1 and σ_2) and the relative strength of competing nonlinearity α_2. The analysis of Eq. (8) reveals that the stationary solution always exists if $|\alpha_1| > \alpha_2 > 0$, as shown in Figs. 1(a)–1(c). It is evident that the soliton width decreases first for a small value of σ_1 with a given σ_2, reaches its minimum value, and then monotonically increases, as shown in Fig. 1(b). This behavior can be explained as follows. When the degree of nonlocality increases, the nonlinearity induced refractive index changes advance towards the region of lower intensity leading to narrower self-induced waveguide structure and, consequently, soliton. However, for a large value of σ_1, the nonlinear index change expressed by the convolution

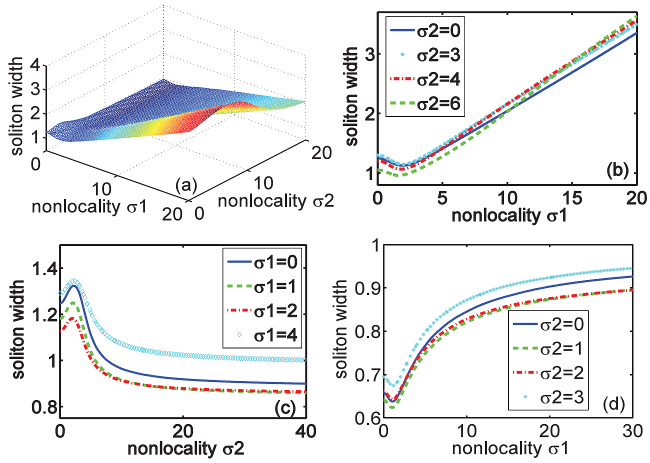

FIG. 1. (Color online) Illustrating the effect of nonlocality and competing nonlinearities on the width of dark soliton w, for the cases of (a)–(c) $|\alpha_1| > \alpha_2 > 0, \alpha_2 = 0.5$; and (d) $\alpha_2 < 0, \alpha_2 = -0.8$. Other parameters used are $B = 1$ and $A = 0$.

integral in Eq. (2) becomes weaker and broader, acquiring a rectangular-shape waveguide and resulting in the increase of soliton width.

On the other hand, when σ_1 is fixed, the width of the dark soliton increases first with σ_2, and then decreases slowly to approach a constant value, as shown in Fig. 1(c). This is due to the fact that the increased σ_2 effectively decreases the self-focusing contribution and removes its deleterious effect on soliton formation. In the highly nonlocal regime the focusing contribution becomes linear, resulting in only the defocusing nonlinearity being responsible for the formation of dark solitons with its width determined by σ_1.

In order to confirm these properties of a single dark soliton in nonlocal media with competing defocusing and focusing nonlinearities, we used the split-step Fourier method to integrate Eq. (1) numerically. Figure 2 illustrates the effect of nonlocality and the competing nonlinearities on soliton propagation and stability. The variational results are applied as the initial conditions, and we also assume the following parameters $\alpha_2 = 0.5$, $B = 1$. The left column in Fig. 2 corresponds to the case of fixed degree of nonlocality of the focusing response ($\sigma_2 = 3$). It is clear that for small σ_1 (weak nonlocality) the nonlinearity enhances the localization of solitons, while the soliton width increases continuously with large σ_1 [Figs. 2(a)–2(d)]. Our numerical results agree well with the variational results, as shown in Fig. 1(b). The existence of the dispersive waves is due to the fact that our initial conditions do not represent the exact stationary soliton profile in the nonlocal regime.

For a given defocusing nonlocal nonlinearity ($\sigma_1 = 2$), we can get a different behavior, as shown in Figs. 2(e)–2(h). With the increase of nonlocality, spatial dark solitons will broaden in the weak nonlocal regime and maintain almost the same width for a large nonlocality, see Fig. 1(c).

When $\alpha_2 < 0$, Eq. (1) represents the propagation of dark solitons in nonlocal media with synthetical self-defocusing nonlinearities. This situation is similar to that discussed in our previous work [57]. It is obvious from Fig. 1(d) when σ_2 (σ_1) is given, the width of the dark solitons will decrease first for a small value σ_1 (σ_2), reach a minimum value, and then increase with σ_1 (σ_2).

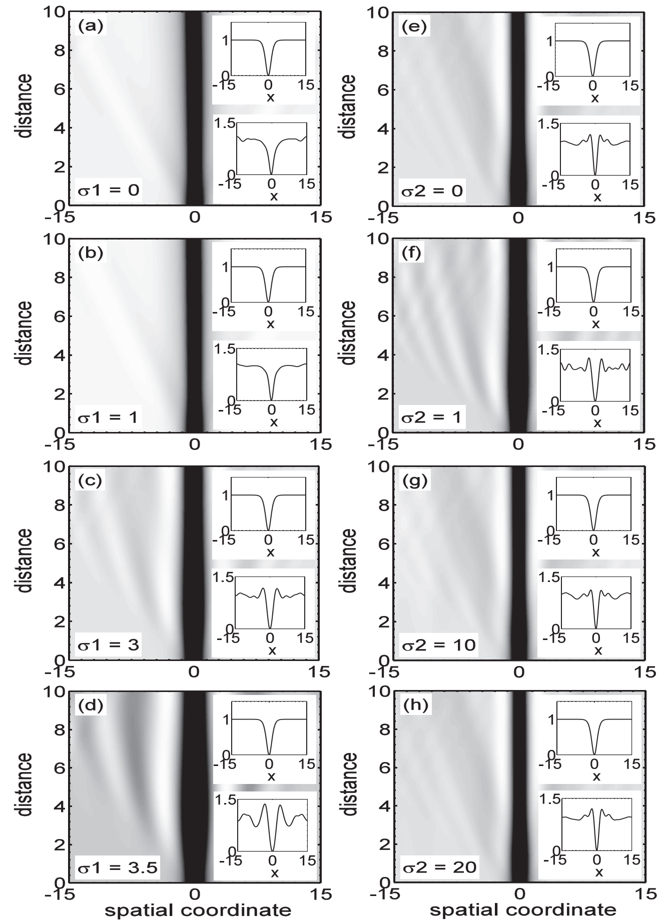

FIG. 2. Propagation of dark solitons for various degrees of nonlocality (a)–(d) σ_1 ($\sigma_2 = 3$, $w = 1.299$) and (e)–(h) σ_2 ($\sigma_1 = 2$, $w = 1.126$). The insets depict the soliton profiles at the beginning (top) and the end (bottom) of the propagation distance. Other parameters used are $\alpha_2 = 0.5$, $B = 1$.

Although we have confirmed our results numerically, to check the validity of the approximately analytical solutions, we could also compare the above variational results with the exact analytical solutions, which can be obtained in the limit of a weak nonlocality [38]. When the response function is much narrower than the soliton width ($\sigma_i \ll 1$), the convolution integral in Eq. (2) can be expanded in a Taylor series, and Eq. (2) turns into

$$\int_{-\infty}^{+\infty} R(x-\xi)|u(\xi,z)|^2 d\xi \approx \alpha |u(x)|^2 + \gamma \frac{\partial^2 |u(x)|^2}{\partial x^2}, \quad (10)$$

with

$$\gamma = \frac{1}{2}\int_{-\infty}^{+\infty} R(x)x^2 dx = \frac{\alpha_1 \sigma_1^2}{6} + \frac{\alpha_2 \sigma_2^2}{6} \quad (11)$$

is the parameter describing the weak nonlocality, here $R = \alpha_1 R_1 + \alpha_2 R_2$, $\alpha = \alpha_1 + \alpha_2$. We also introduce a spatial variable $\zeta = x - Vz$, with V being the soliton transverse velocity. Following the earlier theoretical work [38], we can get the exact solutions of the dark solitons in nonlocal media with competing nonlinearities, which is given by the following

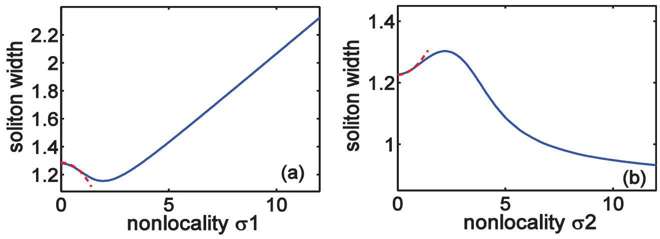

FIG. 3. (Color online) Comparison of the soliton width vs degree of nonlocality in a weakly nonlocal regime obtained by the variational approach (solid curve) and exact analytical solutions (dashed curve). (a) $\alpha_2 = 0.5$, $\sigma_2 = 1$; and (b) $\alpha_2 = 0.5$, $\sigma_1 = 0.5$. Other parameters used are $B = 1$ and $A = 0$.

relation between the soliton intensity $|u|^2$ and the spatial coordinate ζ:

$$\pm\zeta = \frac{1}{\sqrt{-\alpha}\delta_0}\tanh^{-1}\left(\frac{\delta}{\delta_0}\right) + \sqrt{\frac{4\gamma}{\alpha}}\tan^{-1}(\sqrt{-4\gamma}\delta), \quad (12)$$

here $\delta^2(|u|^2) = (|u|^2 - |u_1|^2)/(1 + 4\gamma|u|^2)$, $\delta_0 = \delta(|u_0|^2)$, $|u_0|^2 = 1$ is the soliton background intensity, and $|u_1|^2 = A^2$ is the center intensity. This relation is valid only if the nonlocality parameter $-\gamma$ does not exceed a certain critical value $-\gamma < 1/4|u_0|^2$ (and $|\alpha_1| > \alpha_2$). This explicit analytical solution is shown in Fig. 3 by the dashed curve. It is obvious that the variational solution (the solid line) is in good agreement with the exact solution for a weak nonlocality.

IV. THE ROLE OF NONLOCALITY ON TRANSVERSE VELOCITY OF DARK SOLITONS

As Eq. (9) shows, the transverse velocity V of solitons is related to the amplitude of the solitons (for local Kerr solitons, $dx_0/dz = A$).

It has been already shown earlier that in the case of pure defocusing nonlocal nonlinearity, i.e., $\alpha_2 = 0$, the transverse velocity of the dark soliton monotonically decreases with the degree of nonlocality σ_1 [57,59]. This behavior is depicted in Fig. 4(a) by the dashed curve. It appears that the presence of nonlocal focusing contribution to the nonlinearity leads to even stronger decrease of soliton velocity as represented by the solid line in Fig. 4(a). Even more interesting is the dependence of the soliton transverse velocity V on degree of nonlocality of the focusing nonlinearity. The velocity increases rapidly with σ_2 and then saturates for a large degree of nonlocality. This effect is shown in Fig. 4(d). We confirmed the sensitivity of soliton velocity on the degree of nonlocality numerically using variational solutions [represented by the solid lines in Figs. 4(a) and 4(d)] as initial conditions. Results of numerical simulations are represented as squares in Figs. 4(a) and 4(d), while plots in Figs. 4(b) and 4(c) and Figs. 4(e) and 4(f) depict trajectories of the solitons.

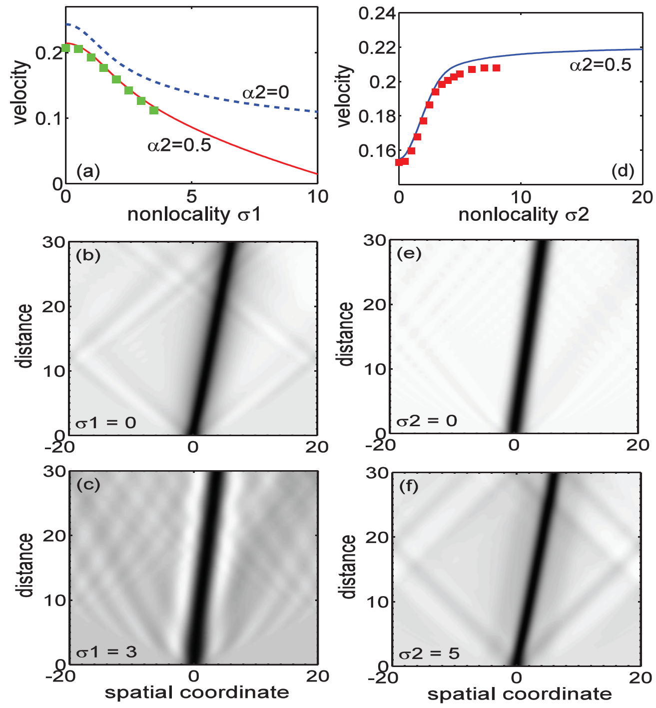

FIG. 4. (Color online) Transverse velocity of dark solitons as a function of (a) the degree of nonlocality σ_1 ($\sigma_2 = 3$) and (d) the degree of nonlocality σ_2 ($\sigma_1 = 1$). Solid and dashed lines represent results of variational calculations; squares depict results of numerical simulations of soliton propagation. Here we set $B = 0.97$. The propagation of dark solitons are shown in (b) and (c) ($\alpha_2 = 0.5$, $\sigma_2 = 3$, $w = 1.345$) and (e) and (f) ($\alpha_2 = 0.5$, $\sigma_1 = 1$, $w = 1.219$).

V. CONCLUSION

In conclusion, we have studied analytically and numerically the effect of competing nonlocal nonlinearities on properties and propagation of dark solitons. By using a phenomenological rectangular model for the nonlocal response functions, we employed the variational technique to derive analytical solutions of dark soliton, for an arbitrary degree of nonlocality. We found that transverse velocity of solitons is strongly affected by competition between nonlinearities. In particular, solitons slow down when the degree of nonlocality of defocusing response increases. On the other hand, the soliton propagates faster when the focusing contribution to the nonlinear response becomes more nonlocal. We confirmed numerically these analytical results.

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