Stabilization of vortex solitons by combining competing cubic-quintic nonlinearities with a finite degree of nonlocality

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In contrast to an infinite degree of nonlocality, we demonstrate that vortex solitons in nonlinear media under competing self-focusing cubic and self-defocusing quintic nonlocal nonlinearities can be stabilized with a finite degree of nonlocality. Stable vortex solitons in the upper branch, bifurcated from the competing cubic-quintic nonlinearities, are found to be supported when the original double-ring refractive index change is transferred into a single-ring configuration due to the balance between diffusive nonlocality and defocusing quintic nonlinearity. The dynamics and stabilities of the vortex solitons are studied analytically and numerically.

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Solitons, localized waves without changing the profile during propagation and collision, form in nonlinear media when linear diffraction and nonlinear self-trapping balance each other [1]. In higher dimensions, these particlelike wave packets suffer from the instability to collapse [2]. With a long-ranged interaction, it is proved that nonlocal nonlinearity can arrest soliton collapse in arbitrary dimensions [3]. Suppression of soliton instability [4,5] as well as generation of novel soliton states [6–9] have been proposed with applications from nematic liquid crystals [10], thermal media [11], atomic vapors [12], to Bose-Einstein condensates [13]. In two dimensions, it is known that vortex solitons and azimuthons are naturally unstable and break into multipole scalar fundamental solitons in local media [14,15]. Even though the nonlocal nonlinearities can suppress the azimuthal instability of vortex beams [16–23], only highly nonlocal media with a huge degree of nonlocality [18,20,24] support vortex solitons without any restriction on the topological charge. This was confirmed experimentally, e.g., the observation of vortex solitons in thermal media with infinite degrees of nonlocality [19] or in a nematic liquid crystal with strongly nonlocal nonlinearity [25,26].

In addition to the nonlocal nonlinearity, saturable or competing nonlinear responses can also support multistable solitons in multidimensions [27]. By combining competing and nonlocal nonlinearities, higher-order vortex solitons are shown to be stable [28]. Moreover, solitons with even and odd parities [29], gap solitons [30], accessible light bullets [31], and dark solitons [32–34] are studied in nematic liquid crystals [35] or Bose-Einstein condensates [36] in which thermal and orientational effects or simultaneous contact and long-ranged dipolar interactions act as competing nonlocal nonlinearities.

In this Brief Report, we study, analytically and numerically, vortex solitons under competing self-focusing cubic and self-defocusing quintic nonlocal nonlinearities. We demonstrate that two branches for the bifurcated vortex solitons stemmed by the competing effect experience different dynamics and related instability. Similar to the case with single nonlocal nonlinearity only, the lower branch solutions are found to always be stable when the degree of nonlocality is sufficiently strong. Instead, the upper branch solutions are revealed to be conditionally stable with a suitable parameter set. When the double-ring structure in the induced nonlinear refractive index change is transferred into a single ring by the balance between the diffusive nonlocality and the defocusing quintic nonlinearity, a stable vortex soliton is found with only a finite degree of nonlocality.

We consider a vortex beam propagating in a nonlinear medium with the slowly varying envelope $\psi(x,y,z)$, described by the normalized nonlocal nonlinear Schrödinger equation [9,17],

$$i\frac{\partial\psi}{\partial z}+\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\delta n(I)\psi=0, \tag{1}$$

where the coordinates x, y, and z are measured in the unit. The nonlinear refractive index change in the media with competing focusing cubic and defocusing quintic nonlocal nonlinearities is represented by $\delta n(I)=\alpha_1\delta n_1(\mathbf{r},I)-\alpha_2\delta n_2(\mathbf{r},I)=\alpha_1\int R_1(\mathbf{r}-\mathbf{r}')|\psi(\mathbf{r}',z)|^2d^2\mathbf{r}'-\alpha_2\int R_2(\mathbf{r}-\mathbf{r}')|\psi(\mathbf{r}',z)|^4d^2\mathbf{r}'$ [29]. Here, $\mathbf{r}$ is the vectorial spatial coordinate, $R_{1,2}(\mathbf{r})$ correspond to the normalized nonlocal response functions, and α_1 and α_2 are positive, which represent the strength of cubic and quintic nonlinearities, respectively. For the reason that the property of competing nonlinearities relies on their relative strengths, we fix $\alpha_1=1$ and only vary α_2 in this Brief Report. For the sake of analytical simplicity and without loss of generality, we consider the case of so-called Gaussian nonlocal response functions [37]: $R_1(\mathbf{r})=(\pi\sigma_1^2)^{-1}\exp(-\mathbf{r}^2/\sigma_1^2)$ and $R_2(\mathbf{r})=(\pi\sigma_2^2)^{-1}\exp(-\mathbf{r}^2/\sigma_2^2)$ with the characteristic width $\sigma_{1,2}$ to represent the degree of nonlocality.

For convenience, we focus on the single vortex soliton ring, which has the form

$$\psi=Ar^m\exp(-r^2/2w^2)\exp(ikz+im\varphi), \tag{2}$$

where $\varphi=\tan^{-1}(y/x)$ and A and w are the amplitude and beam width, respectively. We also set the beam width $w=1$, and only fundamental charged vortex solitons with $m=1$ are studied in the following (high-order charged vortex solitons can be treated in a similar way [18,20]). In this case, the power of the vortex solitons carrying the fundamental topological charge can be obtained by $P=\iint|\psi|^2dx\,dy=\pi A^2w^4$. To investigate the dynamics of the vortex solitons analytically, first, we will employ the Lagrangian (or variational) approach in this Brief Report. It can be shown that the Lagrangian density corresponding to Eq. (1) is of the following form:

$$\mathcal{L} = \frac{i}{2}\left(\psi^*\frac{\partial\psi}{\partial z} - \psi\frac{\partial\psi^*}{\partial z}\right) - \left(\left|\frac{\partial\psi}{\partial x}\right|^2 + \left|\frac{\partial\psi}{\partial y}\right|^2\right) + \frac{\alpha_1}{2}|\psi|^2\int R_1(\mathbf{r}-\mathbf{r}')|\psi(\mathbf{r}',z)|^2 d^2\mathbf{r}' - \frac{\alpha_2}{3}|\psi|^2\int R_2(\mathbf{r}-\mathbf{r}')|\psi(\mathbf{r}',z)|^4 d^2\mathbf{r}'. \quad (3)$$

By the standard variational approach and the Euler-Lagrange equations, one can obtain the corresponding amplitude of vortex beam A,

$$A_{\pm}^2 = \frac{\alpha_1 M \pm \sqrt{3N\left[3\alpha_1^2 w^8 O - 8\alpha_2\left(2w^2+\sigma_1^2\right)^8 P\right]}}{4\alpha_2 w^8\left(2w^2+\sigma_1^2\right)^4 P}, \quad (4)$$

with $M = (729w^18 + 2430w^16\sigma_2^2 + 3240w^14\sigma_2^4 + 2160\, w^12\sigma_2^6 + 720w^10\sigma_2^8 + 96w^8\sigma_2^10)(w^4 + w^2\sigma_1^2 + \sigma_1^4)$, $N = w^8(3w^2 + 2\sigma_2^2)^5$, $O = (w^4 + w^2\sigma_1^2 + \sigma_1^4)^2(3w^2 + 2\sigma_2^2)^5$, and $P = 9w^8 + 21w^6\sigma_2^2 + 24w^4\sigma_2^4 + 15w^2\sigma_2^6 + 2\sigma_2^8$. Here, $A_+(A_-)$ represents the amplitude for the vortex soliton in the upper (lower) branch with a larger (smaller) formation power. The bifurcation of these two solutions comes from the competing effect between cubic and quintic nonlinearities even in local media [38]. However, the vortex solitons in the local cubic-quintic competing nonlinear media are always unstable [39,40].

According to Eq. (4), the formation power of vortex solitons versus degree of nonlocality is shown in Fig. 1 for the special case of $\sigma_1 = \sigma_2 = \sigma$. In addition to the known two bifurcated branches of the solution, there exists a threshold value for the degree of nonlocality, denoted as σ_{th}, above which no solution for any two branches is supported. When the strength of quintic nonlinearity decreases, this threshold value becomes larger as the marked points **A** to **B** shown for $\alpha_2 = 0.01$ and 0.003 in Fig. 1. This again indicates that the upper branch exists only with a nonzero competing quintic nonlinearity. This is why such bifurcated solutions of the vortex solitons in media with competing cubic-cubic nonlocal nonlinearities do not exist [28]. For the lower branch, the supported vortex soliton needs to increase its formation power when the degree of nonlocality increases as the typical scenario happens for solitons in nonlocal media without any competing nonlinearity, i.e., the solid line shown in Fig. 1. However, the formation power for vortex solitons in the upper branch decreases as the degree of nonlocality increases, which is totally contrary to the soliton formation in media with single nonlocal nonlinearity.

With these variational results as the initial conditions, we numerically demonstrate the dynamics of vortex solitons in the lower and upper branches. First, in Fig. 2, we concentrate on the stability for the lower branch. As a comparison, one can see that, when the nonlocality is weak, such as $\sigma = 0.2$ shown in Fig. 2(b), the supported vortex solitons have a smaller stationary propagation distance than that of the case only with local nonlinearities $\sigma = 0$ in Fig. 2(a). However, by increasing the degree of nonlocality, e.g., $\sigma = 0.4$ and $\sigma = 0.7273$ shown in Figs. 2(c) and 2(d), these vortex solitons get benefits from the competing nonlinearities, demonstrating a longer distance for the stable propagation. With a sufficient strong degree of nonlocality, all of these vortex solitons in the lower branch can be stabilized [18,20,28]. For instance, as shown in Fig. 2(e),

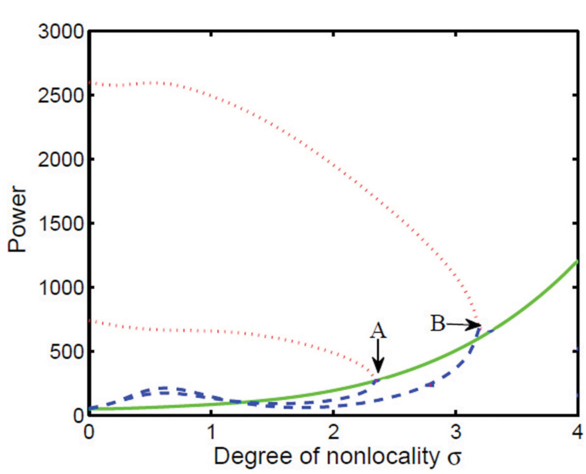

FIG. 1. (Color online) Formation power P versus the degree of nonlocality $\sigma_1 = \sigma_2 = \sigma$ for vortex solitons with different strengths of quintic nonlinearities α_2. Upper (A_+) and lower (A_-) branches are represented with dotted and dashed curves, respectively. These two branches merge at the marked points **A** and **B** at the critical values of $\sigma_{th} = 2.424$ for $\alpha_2 = 0.01$ and $\sigma_{th} = 3.273$ for $\alpha_2 = 0.003$, respectively. The solid line describes the required formation power for the vortex solitons in nonlinear media only with a self-focusing nonlocal cubic nonlinearity ($\alpha_2 = 0$).

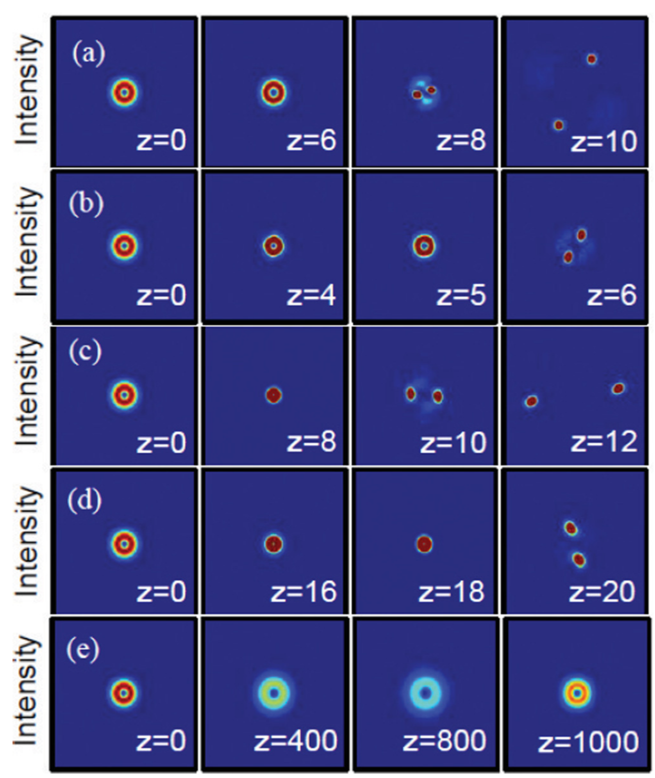

FIG. 2. (Color online) Propagation dynamics for the vortex solitons in the lower branch with different degrees of nonlocalities: (a) $\sigma = 0$, (b) $\sigma = 0.2$, (c) $\sigma = 0.4$, (d) $\sigma = 0.7273$, and (e) $\sigma = 2.2$. The strengths of the nonlinearities are $\alpha_1 = 1$ and $\alpha_2 = 0.01$. Other parameters used are the same as those in Fig. 1.

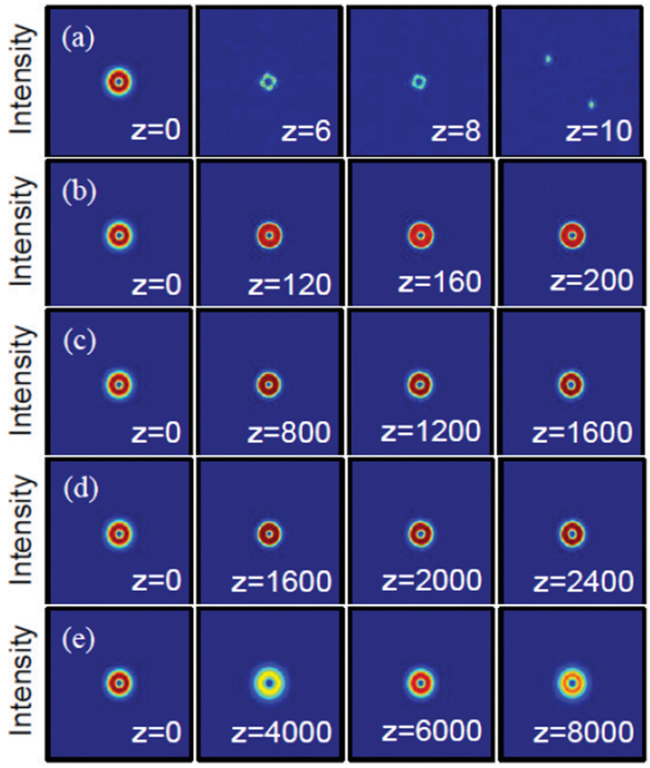

FIG. 3. (Color online) Propagation dynamics for the vortex solitons in the upper branch with different degrees of nonlocalities: (a) $\sigma = 0$, (b) $\sigma = 0.2$, (c) $\sigma = 0.4$, (d) $\sigma = 0.7273$, and (e) $\sigma = 2.424$. The strengths of the nonlinearities are $\alpha_1 = 1$ and $\alpha_2 = 0.01$. Other parameters used are the same as those in Fig. 1.

vortex solitons of the lower branch can propagate with the stationary propagation distance more than $z = 1000$ when $\sigma = 2.2$.

Now, we move to the complicated but interesting results for vortex solitons in the upper branch. In the local case, with $\sigma = 0$ shown in Fig. 3(a), the supported vortex solitons still display an unstable dynamics, i.e., it maintains the vortex ring profile at the beginning but breaks into two-particle clusters quickly at a longer propagation distance. By increasing the degree of nonlocality, e.g., $\sigma = 0.2$, 0.4, and 0.7273 shown in Figs. 3(b)–3(d), these originally unstable upper branch solutions possess a very long stable propagation distance, even longer than that of the solutions in the lower branch shown in Fig. 2. Unexpectedly, as demonstrated in Figs. 2(c) and 2(d), we can have a stable distance as long as $z = 1600$ and 2400 for the vortex solitons in the upper branch when the degrees of nonlocality are much smaller than the threshold value $\sigma_{th} = 2.424$. In this case, vortex solitons are surely stable even in the limit of weak nonlocality [e.g., $\sigma = 0.4$ in Fig. 3(c)]. At the threshold value, the marked point **A** in Fig. 1, where the vortex solitons of the lower and upper branches have the same dynamics, it can propagate stably more than $z = 8000$, although its intensity profile oscillates slightly during the propagation.

In order to illustrate physical mechanisms for the stabilization of vortex solitons in the upper and lower branches, in Fig. 4, we show the nonlinearity-induced refractive index changes $\delta n(I)$ defined in Eq. (1). The first and second rows correspond to $\delta n(I)$ for the upper (a)–(d) and lower (e)–(h) branches. One can see that the nonlinear refractive index change for the upper branch has a double-ring structure in the profile when $\sigma = 0$ as shown in Fig. 4(a); whereas, that for the lower branch only has a single-ring configuration. The difference in the profiles of the refractive index change accounts for the stability and related dynamics for the vortex solitons. Naively, to maintain a stable vortex soliton, a single-ring configuration for the nonlinear refractive index change is needed in order to sustain the soliton profile from collapse. With a large enough nonlocal effect [18–20,24–26], this refractive index change expands accordingly at the price of increasing the formation power, resulting in the stabilization of the vortex solitons in the lower branch. On the contrary, a double-ring configuration in the induced refractive index change provides a separated self-trapping potential, which, in general, breaks down the vortex beam, leading to the formation of particle clusters. But the introduction of nonlocal nonlinearity brings a new balance into this double-ring configuration. Even though the induced refractive index change expands accordingly when the degree of nonlocality increases, the required formation power for the vortex solitons in the upper branch decreases. Due to the balance between nonlocal and quintic nonlinearities as shown in Figs. 4(c)–4(d), now the nonlinear refractive index change has the same profile as a single-ring configuration in the lower branch. This balanced single-ring profile in the refractive index change explains why we have a longer stable propagation distance for the vortex solitons in the upper branch. With a finite, instead of infinite [19] or huge [25,26], degree of nonlocality, we can have stable vortex solitons.

In conclusion, vortex solitons in nonlinear media under competing self-focusing cubic and self-defocusing quintic nonlocal nonlinearities are studied with an arbitrary degree of nonlocality. We demonstrate that the lower branch solutions have similar stability and dynamics as those vortex solitons in the nonlocal nonlinear media without the introduction of a competing quintic term. That is, vortex solitons in the lower branch can always be stabilized with the help of nonlocality but above a critical degree of nonlocality ($\sigma = 2.2$). Instead, even though the vortex solutions in the upper

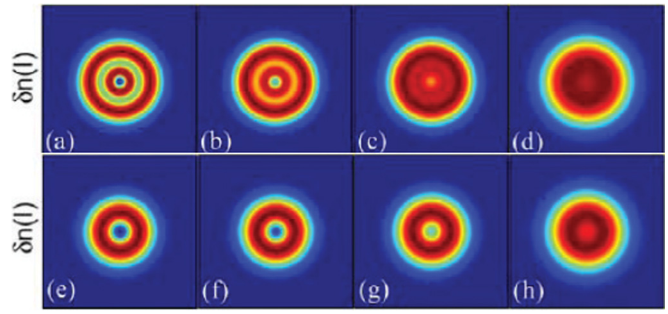

FIG. 4. (Color online) Nonlinearity-induced refractive index changes $\delta n(I)$ for the upper (a)–(d) and lower (e)–(h) branches with different degrees of nonlocalities: (a) and (e) $\sigma = 0$, (b) and (f) $\sigma = 0.2$, (c) and (g) $\sigma = 0.4$, and (d) and (h) $\sigma = 0.7273$. The strengths of the nonlinearities are $\alpha_1 = 1$ and $\alpha_2 = 0.01$. Other parameters used are the same as those in Fig. 1.

branch are conditionally stable, the balance between diffusive nonlocality and defocusing quintic nonlinearity can be met to transfer a double-ring into a single-ring profile in the induced refractive index change. With such a benefit from the bifurcated upper branch, stemmed by the competing cubic-quintic nonlinearities, a stable vortex soliton is demonstrated numerically, only with a finite degree of nonlocality.

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[1] Y. S. Kivshar and G. P. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals* (Academic, San Diego, 2003).

[2] B. A. Malomed, D. Mihalache, F. Wise, and L. Torner, J. Opt. B **7**, R53 (2005).

[3] O. Bang, W. Krolikowski, J. Wyller, and J. J. Rasmussen, Phys. Rev. E **66**, 046619 (2002).

[4] J. Wyller, W. Krolikowski, O. Bang, and J. J. Rasmussen, Phys. Rev. E **66**, 066615 (2002).

[5] Y. Y. Lin, R.-K. Lee, and Y. S. Kivshar, J. Opt. Soc. Am. B **25**, 576 (2008).

[6] M. Shen, Q. Wang, J. Shi, Y. Chen, and X. Wang, Phys. Rev. E **72**, 026604 (2005).

[7] Y. Y. Lin and R.-K. Lee, Opt. Express **15**, 8781 (2007).

[8] Y. Y. Lin, R.-K. Lee, and B. A. Malomed, Phys. Rev. A **80**, 013838 (2009).

[9] M. Shen, H. Ding, Q. Kong, L. Ruan, S. Pang, J. Shi, and Q. Wang, Phys. Rev. A **82**, 043815 (2010).

[10] M. Peccianti, C. Conti, G. Assanto, A. D. Luca, and C. Umeton, Nature (London) **432**, 733 (2004).

[11] C. Rotschild, B. Alfassi, O. Cohen, and M. Segev, Nat. Phys. **2**, 769 (2006).

[12] S. Skupin, M. Saffman, and W. Krolikowski, Phys. Rev. Lett. **98**, 263902 (2007).

[13] P. Pedri and L. Santos, Phys. Rev. Lett. **95**, 200404 (2005).

[14] A. S. Desyatnikov, Y. S. Kivshar, and L. Torner, in *Progress in Optics*, edited by E. Wolf (Elsevier North-Holland, Amsterdam, 2005), Vol. 47, p. 291.

[15] W. J. Firth and D. V. Skryabin, Phys. Rev. Lett. **79**, 2450 (1997).

[16] S. Lopez-Aguayo, A. S. Desyatnikov, and Y. S. Kivshar, Opt. Express **14**, 7903 (2006).

[17] M. Shen, Q. Kong, C.-C. Jeng, L.-J. Ge, R.-K. Lee, and W. Krolikowski, Phys. Rev. A **83**, 023825 (2011).

[18] M. Shen, Y. Y. Lin, C.-C. Jeng, and R.-K. Lee, J. Opt. **14**, 065204 (2012).

[19] C. Rotschild, O. Cohen, O. Manela, M. Segev, and T. Carmon, Phys. Rev. Lett. **95**, 213904 (2005).

[20] D. Briedis, D. E. Petersen, D. Edmundson, W. Krolikowski, and O. Bang, Opt. Express **13**, 435 (2005).

[21] D. Buccoliero, A. S. Desyatnikov, W. Krolikowski, and Y. S. Kivshar, Opt. Lett. **33**, 198 (2008).

[22] F. Ye, Y. V. Kartashov, B. Hu, and L. Torner, Opt. Lett. **35**, 628 (2010).

[23] A. I. Yakimenko, Y. A. Zaliznyak, and Y. S. Kivshar, Phys. Rev. E **71**, 065603(R) (2005).

[24] S. Skupin, O. Bang, D. Edmundson, and W. Krolikowski, Phys. Rev. E **73**, 066603 (2006).

[25] R. Barboza, U. Bortolozzo, G. Assanto, E. Vidal-Henriquez, M. G. Clerc, and S. Residori, Phys. Rev. Lett. **109**, 143901 (2012).

[26] Y. V. Izdebskaya, A. S. Desyatnikov, G. Assanto, and Y. S. Kivshar, Opt. Express **19**, 21457 (2011).

[27] L. Bergé, Phys. Rep. **303**, 259 (1998).

[28] Y. V. Kartashov, V. A. Vysloukh, and L. Torner, Phys. Rev. A **79**, 013803 (2009).

[29] D. Mihalache, D. Mazilu, F. Lederer, L. C. Crasovan, Y. V. Kartashov, L. Torner, and B. A. Malomed, Phys. Rev. E **74**, 066614 (2006).

[30] K.-H. Kuo, Y. Y. Lin, R.-K. Lee, and B. A. Malomed, Phys. Rev. A **83**, 053838 (2011).

[31] I. B. Burgess, M. Peccianti, G. Assanto, and R. Morandotti, Phys. Rev. Lett. **102**, 203903 (2009).

[32] L. Chen, Q. Wang, M. Shen, H. Zhao, Y.-Y. Lin, C.-C. Jeng, R.-K. Lee, and W. Krolikowski, Opt. Lett. **38**, 13 (2013).

[33] Q. Kong, M. Shen, Z. Chen, Q. Wang, R.-K. Lee, and W. Krolikowski, Phys. Rev. A **87**, 063832 (2013).

[34] B. K. Esbensen, A. Wlotzka, M. Bache, O. Bang, and W. Krolikowski, Phys. Rev. A **84**, 053854 (2011).

[35] M. Warenghem, J. F. Blach, and J. F. Henninot, J. Opt. Soc. Am. B **25**, 1882 (2008).

[36] A. Griesmaier, J. Stuhler, T. Koch, M. Fattori, T. Pfau, and S. Giovanazzi, Phys. Rev. Lett. **97**, 250402 (2006).

[37] D. Buccoliero, A. S. Desyatnikov, W. Krolikowski, and Y. S. Kivshar, Phys. Rev. Lett. **98**, 053901 (2007).

[38] N. Dror and B. A. Malomed, Physica D **240**, 526 (2011).

[39] M. Quiroga-Teixeiro and H. Michinel, J. Opt. Soc. Am. B **14**, 2004 (1997).

[40] A. S. Desyatnikov, D. Mihalache, D. Mazilu, B. A. Malomed, C. Denz, and F. Lederer, Phys. Rev. E **71**, 026615 (2005).