Extracting the internal nonlocality from the dilated Hermiticity

Minyi Huang [1,*] Ray-Kuang Lee [2,3,4,†] and Junde Wu [5,‡]

[1]*Department of Mathematical Sciences, Zhejiang Sci-Tech University, Hangzhou 310007, People's Republic of China*
[2]*Institute of Photonics Technologies, National Tsing Hua University, Hsinchu 300, Taiwan, Republic of China*
[3]*Center for Quantum Technology, Hsinchu 30013, Taiwan, Republic of China*
[4]*Physics Division, National Center for Theoretical Sciences, Taipei 10617, Taiwan, Republic of China*
[5]*School of Mathematical Sciences, Zhejiang University, Hangzhou 310027, People's Republic of China*

To effectively realize a $\mathcal{PT}$-symmetric system, one can dilate a $\mathcal{PT}$-symmetric Hamiltonian to some global Hermitian one and simulate its evolution in the dilated Hermitian system. However, with only a global Hermitian Hamiltonian, how do we know whether it is a dilation and is useful for simulation? To answer this question, we consider the problem of how to extract the internal nonlocality in the Hermitian dilation. We unveil that the internal nonlocality brings nontrivial correlations between the subsystems. By evaluating the correlations with local measurements in three different pictures, the resulting different expectations of the Bell operator reveal the distinction of the internal nonlocality. When the simulated $\mathcal{PT}$-symmetric Hamiltonian approaches its exceptional point, such a distinction tends to be most significant. Our results clearly make a distinction between the Hermitian dilation and other global Hamiltonians without internal nonlocality. They also provide the figure of merit to test the reliability of the simulation, as well as to verify a $\mathcal{PT}$-symmetric (sub)system.

I. INTRODUCTION

Nowadays, both in classical and quantum physics, we are witnessing a growing interest in discussing $\mathcal{PT}$-symmetric systems. Historically, parity-time ($\mathcal{PT}$)-symmetric systems were first introduced to permit entirely real spectra even in the non-Hermitian setting [1]. With the experimental controls in gain and loss, photonic systems have been used to simulate $\mathcal{PT}$-symmetric wave phenomena with an equivalence between single-particle quantum mechanics and classical wave equation [2–6]. Interestingly, $\mathcal{PT}$-symmetry has also found powerful applications in circuit design [7]. The concept of $\mathcal{PT}$-symmetry was generalized to pseudo-Hermiticity [8–11] and anti-$\mathcal{PT}$-symmetry [12,13]. Moreover, $\mathcal{PT}$-symmetric theory extended profoundly to the research of non-Hermiticity, with fruitful results and novel phenomena [14]. For example, in the field of dynamics and band topology, the skin effect was introduced in the complex spectra of non-Hermitian systems [15–18].

Despite the initial motivation to establish an alternative framework of quantum theory, we can also take $\mathcal{PT}$-symmetric systems as effective descriptions of large Hermitian systems in some subspaces, similar to the Feshbach formalism dealing with an effective description of open quantum systems [19]. By using the Naimark dilation theorem, one can always find some four-dimensional Hermitian Hamiltonians to effectively realize two-dimensional unbroken $\mathcal{PT}$-symmetric systems [20]. Then, such a methodology can be generalized to simulate any finite-dimensional $\mathcal{PT}$-symmetric systems [21,22]. By evolving states under the Hermitian dilation Hamiltonians, and projecting out the ancillary systems, this paradigm successfully simulates the evolution of unbroken $\mathcal{PT}$-symmetric Hamiltonians in subspaces. It endows a direct physical meaning of $\mathcal{PT}$-symmetric quantum systems in the sense of open systems. As for the broken $\mathcal{PT}$-symmetry, there are at least two different approaches to the simulation. One way is utilizing weak measurement as an approximation paradigm for the broken $\mathcal{PT}$-symmetric systems [23], while the other way is simulating the evolution of broken $\mathcal{PT}$-symmetric systems with time-dependent Hamiltonians, connecting the topology and dynamics [24].

In the simulation of $\mathcal{PT}$-symmetric systems, Hermitian dilation Hamiltonians play a key role. Generally, these Hamiltonians are inseparable and act on certain global systems composed of two subsystems [21,22]. This implies that there exist nonlocal correlations between the subsystems. To investigate such correlations, we propose an approach to extract the internal nonlocality when the global Hermitian Hamiltonian is shared with Alice and Bob. With only local measurements performed by Alice and Bob, we show that the expectations of the Bell operator differ in different correlation pictures. It gives a higher value of the upper bound when both the classical and local Hermitian pictures are considered, but a lower value of the upper bound for the simulation picture. Moreover, when the simulated $\mathcal{PT}$-symmetric Hamiltonian approaches its exceptional point, the value of this upper bound gives the largest departure from the local Hermitian systems. Along the lines of quantum information approaches [25–29],

*hmy@zstu.edu.cn
†rklee@ee.nthu.edu.tw
‡wjd@zju.edu.cn

our work provides a way to distinguish isospectral Hermitian Hamiltonians with and without internal nonlocality. With the ability to know whether a $\mathcal{PT}$-symmetric subsystem is embedded inside a global Hermitian one, our results provide the figure of merit to test the reliability of the simulation, as well as the verification for a $\mathcal{PT}$-symmetric (sub)system.

The remainder of this paper is organized as follows. In Sec. II, we introduce the preliminaries on the related notions of $\mathcal{PT}$-symmetric systems and the CHSH (Clauser, Horne, Shimony, and Holt) scenario. In Sec. III, the internal nonlocality is illustrated by investigating the correlation measurements between Alice and Bob for the classical, local Hermitian, and simulation picture, respectively. Discussions on the physical implications of these three pictures and their essential difference from the CHSH scenario are given in Sec. IV. Finally, in Sec. V we conclude with our results.

II. PRELIMINARIES

A. Basic notions and the simulation by dilation

The concepts of parity, time-reversal, and $\mathcal{PT}$-symmetric operators have been studied since the early age of quantum mechanics. A linear operator H is said to be $\mathcal{PT}$-symmetric if $H\mathcal{PT} = \mathcal{PT}H$, where $\mathcal{P}$ and $\mathcal{T}$ are parity and time-reversal operators. Here, we focus on finite-dimensional spaces, in which a $\mathcal{PT}$-symmetric operator H is said to be unbroken if it is similar to a real diagonal operator; H is said to be broken $\mathcal{PT}$-symmetric if it cannot be diagonalized or has complex eigenvalues [21]. $\mathcal{PT}$-symmetric operators are usually non-Hermitian but satisfy the condition of pseudo-Hermiticity [11].

The simulation of $\mathcal{PT}$-symmetric systems is closely related to the mathematical concept of Hermitian dilation [20]. Let H be a $\mathcal{PT}$-symmetric operator on $\mathbb{C}^n$, and let $\hat{H}$ be a Hermitian operator on $\mathbb{C}^m$, with $m > n$. P_1 is an operator defined by $P_1 : \mathbb{C}^m \to \mathbb{C}^n$, $P_1[\begin{smallmatrix}\phi_1\\\phi_2\end{smallmatrix}] = \phi_1$, where $\phi_1 \in \mathbb{C}^n$ and $\phi_2 \in \mathbb{C}^{m-n}$. Let $X_{\hat{H}} = \{x : x \in \mathbb{C}^m, P_1\hat{H}x = HP_1x, P_1e^{-it\hat{H}}x = e^{-itH}P_1x\}$. If $P_1X_{\hat{H}} = \mathbb{C}^n$, then we say that $\hat{H}$ is a Hermitian dilation of H.

By evolving the Hermitian dilation Hamiltonian $\hat{H}$ on a large space, the evolution of a $\mathcal{PT}$-symmetric non-Hermitian system can be realized in the subspace. In addition, only unbroken $\mathcal{PT}$-symmetric operators preserve such a property, hence our discussions mainly focus on the case of unbroken $\mathcal{PT}$-symmetry. Actually, for any state $|\psi\rangle$, the definition of $\hat{H}$ can ensure the following equation (unnormalized for convenience):

$$e^{-it\hat{H}}(|0\rangle|\psi\rangle + |1\rangle|\tau\psi\rangle) = |0\rangle|e^{-itH}\psi\rangle + |1\rangle|\tau e^{-itH}\psi\rangle,$$

where τ is an operator linked to the metric operator [21]. The equation above clearly shows that there are two subsystems. Moreover, by projecting out $|1\rangle|\tau e^{-itH}\psi\rangle$, the effect of a non-Hermitian Hamiltonian can be realized by $|0\rangle|e^{-itH}\psi\rangle$ in the latter subsystem. An experimental realization of such an evolution effect in the subsystem is called a simulation, in which the preparation of Hermitian dilation Hamiltonian plays an important role. As for more technical and experimental details, see Refs. [21,26].

In general, there exist various $\hat{H}$ to satisfy the dilation scenario. Without loss of generality, we adopt the following one by requiring

$$e^{-it\hat{H}}(|1\rangle|\psi\rangle - |0\rangle|\tau\psi\rangle) = |1\rangle|e^{-itH}\psi\rangle - |0\rangle|\tau e^{-itH}\psi\rangle.$$

Now, the resulting $\hat{H}$ can achieve a simple form,

$$\hat{H} = I_2 \otimes \Lambda + i\sigma_y \otimes \Omega, \tag{1}$$

$$\Lambda = (H\tau^{-\frac{1}{2}} + \tau^{\frac{1}{2}}H)(\tau^{-\frac{1}{2}} + \tau^{\frac{1}{2}})^{-1}, \tag{2}$$

$$\Omega = (H - \tau^{\frac{1}{2}}H\tau^{-\frac{1}{2}})(\tau^{-\frac{1}{2}} + \tau^{\frac{1}{2}})^{-1}, \tag{3}$$

in which the details about τ can be referred to Refs. [20–22].

According to Eq. (1), the Hermitian dilation Hamiltonian $\hat{H}$ is inseparable. That is, $\hat{H}$ cannot be written as a tensor product of two local operators. As a consequence, it implies the possibility of investigating the correlations that $\hat{H}$ brings to the subsystems. Moreover, $\hat{H}$ is isospectral to H. It means that the Hermitian dilation Hamiltonian has the same eigenvalues as the simulated $\mathcal{PT}$-symmetric Hamiltonian with twofold spectra, i.e., it has two multiplicities of eigenvalues. Such a property implicitly allows us to use the measurements on the large space to simulate the measurements of the $\mathcal{PT}$-symmetric system [23]. Briefly speaking, by measuring the Hermitian dilation Hamiltonian $\hat{H}$, one can read out the eigenvalues of the $\mathcal{PT}$-symmetric system.

B. Two-dimensional model

To illustrate our proposed concept in a clear way, we consider a two-dimensional $\mathcal{PT}$-symmetric Hamiltonian as an example by following [20,30], i.e.,

$$H = E_0 I_2 + s\begin{bmatrix} i\sin\alpha & 1 \\ 1 & -i\sin\alpha \end{bmatrix}. \tag{4}$$

The corresponding eigenvalues for this two-dimensional non-Hermitian system are $\lambda_{\pm} = E_0 \pm s\cos\alpha$. Moreover, there exists an exceptional point when $\alpha = \frac{\pi}{2}$, in which case the Hamiltonian is no longer diagonalized. When $\alpha \neq \frac{\pi}{2}$, the Hamiltonian H has real eigenvalues and can be diagonalized. Hence, $\mathcal{PT}$-symmetry is unbroken. Specifically, when $\alpha = 0$, the Hamiltonian returns to the Hermitian one.

By applying the dilation process given in Eqs. (1)–(3), the corresponding Hermitian dilation Hamiltonian $\hat{H}$ has the form

$$\hat{H} = I_2 \otimes \Lambda + i\sigma_y \otimes \Omega, \tag{5}$$

$$\Lambda = E_0 I_2 + \frac{\omega_0}{2}\cos\alpha\,\sigma_x, \tag{6}$$

$$\Omega = i\frac{\omega_0}{2}\sin\alpha\,\sigma_z, \tag{7}$$

where $\omega_0 = 2s\cos\alpha$ [20,21].

C. CHSH scenario

As the joint correlation measurements will be performed locally by Alice and Bob, it is instructive to briefly recall Bell's nonlocality and the related CHSH scenario [31,32]. In a standard Bell's test on nonlocality, two (sub)systems shared by Alice and Bob are spatially separated. By performing local measurements, Alice obtains several possible outcomes from her subsystem, denoted as a, with the outcomes denoted as

b from Bob's measurements on his subsystem. Due to the randomness in the local measurements, the outcomes a and b may have different values. Nevertheless, these outcomes are in general governed by a probability distribution $p(ab|ij)$, where the local measurements are labeled with the index i, j. Usually the joint probability distribution reveals that

$$p(ab|ij) \neq p(a|i)p(b|j). \tag{8}$$

It implies that the results a and b are not independent, even when Alice and Bob are spacelike separated. However, a classical correlation theory does not admit nonlocality. Hence, a possible explanation is that some dependence between the subsystems was established when they interacted in the past, eventually leading to the inequality shown in Eq. (8). Such an explanation also suggests that if we take into account all the past factors, described by some random variable ν, then the joint probability distribution for a and b can be factorized as

$$p(ab|ij, \nu) = p(a|i, \nu)p(b|j, \nu). \tag{9}$$

Apparently, Eq. (9) shows that a and b are independent, which is consistent with the classical (local) correlation theory. On the other hand, by denoting $q(\nu)$ as the probability distribution of ν, one can have

$$p(ab|ij) = \int_N d\nu \, q(\nu)p(a|i, \nu)p(b|j, \nu), \tag{10}$$

which is the condition for locality in the context of Bell's test.

Equation (10) is the key to deriving the well-known CHSH inequality. Suppose Alice can perform two local measurements denoted as A_i, $i \in \{0, 1\}$, while Bob can also perform two local measurements B_j, $j \in \{0, 1\}$. The possible outcomes of A_i and B_j have two values labeled $a, b \in \{+1, -1\}$. Now, let $\langle A_i B_j \rangle = \sum_{a,b} ab \, p(ab|ij)$ be the expectation value of the product ab for given measurements $A_i B_j$. Here, $A_i B_j$ is often called the correlation function. With these notions, one can further define the following Bell operator:

$$S = A_0 B_0 + A_1 B_0 + A_0 B_1 - A_1 B_1. \tag{11}$$

According to Eq. (10) and some further calculations, one can find that

$$\langle A_0 B_0 \rangle + \langle A_1 B_0 \rangle + \langle A_0 B_1 \rangle - \langle A_1 B_1 \rangle \leqslant 2, \tag{12}$$

which is known as the (classical) CHSH inequality. It is noted that the discussion above is abstract and has nothing to do with how to realize the measurements or whether the systems are classical or quantum. The only ingredient is the classical (local) correlations.

However, let us consider the two subsystems measured by Alice and Bob, which are two qubits in the singlet state $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, where $|0\rangle$ and $|1\rangle$ are the eigenstates of σ_z for the eigenvalues of $+1$ and -1. Suppose that the A_0 and A_1 correspond to the measurements of spin in the orthogonal directions e_0 and e_1, respectively. Similarly, B_0 and B_1 correspond to the measurements in the directions $-\frac{1}{\sqrt{2}}(e_0 + e_1)$ and $\frac{1}{\sqrt{2}}(-e_0 + e_1)$. Then, we have

$$\langle A_0 B_0 \rangle + \langle A_1 B_0 \rangle + \langle A_0 B_1 \rangle - \langle A_1 B_1 \rangle = 2\sqrt{2}, \tag{13}$$

as $\langle A_0 B_0 \rangle = \langle A_1 B_0 \rangle = \langle A_0 B_1 \rangle = -\langle A_1 B_1 \rangle = \frac{1}{\sqrt{2}}$. As one can see, the violation of the classical bound given in Eq. (12) reveals the nonlocal character of quantum theory.

III. THE INTERNAL NONLOCALITY IN SIMULATING PT-SYMMETRIC SYSTEMS

We know that in the simulation paradigm, a $\mathcal{PT}$-symmetric (pseudo-Hermitian) system can be embedded into a global Hermitian one. On the contrary, with only a global Hermitian Hamiltonian at hand, how can we know whether a pseudo-Hermitian Hamiltonian is embedded inside, or if such a global Hermitian Hamiltonian is composed by local Hermitian ones? Moreover, can we have the figure of merit to test the reliability of the simulation paradigm?

A natural way to answer these questions is to investigate the internal nonlocality of the global Hermiticity. As Eq. (1) suggests, the Hermitian dilation Hamiltonian may bring nonlocal correlations to the subsystems. Inspired by the CHSH scenario, we propose here a CHSH-like discussion of the nonlocality for the global Hamiltonian. Due to the fact that the nonlocality does not directly come from the entangled states but rather from the Hermitian dilation, our formulation differs considerably from earlier studies in essence. To distinguish between our setting and the CHSH scenario on quantum states, we coined the term *internal nonlocality*.

A. The simulation picture

Without loss of generality, let us take the two-dimensional model given in Eq. (5) as an example. In the simulation picture, the Hermitian dilation Hamiltonian $\hat{H}$ is assumed to be shared by Alice and Bob. Suppose Alice makes two local measurements denoted as A_0 and A_1, while Bob also makes two local measurements, denoted B_0 and B_1. By adopting similar ideas to those used in the CHSH scenario, one can consider the correlation functions $B_i A_j$. The reason why B_i comes before A_j is that Alice is assumed to be in charge of the second subsystem, which simulates the $\mathcal{PT}$-symmetric Hamiltonian. Due to the fact that they are actually "measuring" the global Hamiltonian, Alice and Bob only need to use local states to obtain the measurement results. Let Alice have the local state $\{|u_+\rangle = u|0\rangle + v|1\rangle\}$ for A_0 and $\{|u_-\rangle = \bar{v}|0\rangle - \bar{u}|1\rangle\}$ for A_1, while Bob has two local states $\{|0\rangle$ and $|1\rangle\}$ for B_0 and B_1, respectively. Hence the expectations of $B_i A_j$ can be calculated as follows:

$$\langle B_0 A_0 \rangle = \mathrm{Tr}(|0\rangle\langle 0| \otimes |u_+\rangle\langle u_+|)\hat{H}, \tag{14}$$

$$\langle B_1 A_0 \rangle = \mathrm{Tr}(|1\rangle\langle 1| \otimes |u_+\rangle\langle u_+|)\hat{H}, \tag{15}$$

$$\langle B_0 A_1 \rangle = \mathrm{Tr}(|0\rangle\langle 0| \otimes |u_-\rangle\langle u_-|)\hat{H}, \tag{16}$$

$$\langle B_1 A_1 \rangle = \mathrm{Tr}(|1\rangle\langle 1| \otimes |u_-\rangle\langle u_-|)\hat{H}. \tag{17}$$

Now, one can further consider the expectation value of the Bell operator:

$$\begin{aligned}
&\langle B_0 A_0 \rangle + \langle B_0 A_1 \rangle + \langle B_1 A_0 \rangle - \langle B_1 A_1 \rangle \\
&= \mathrm{Tr}[|0\rangle\langle 0| \otimes I_2 + |1\rangle\langle 1| \otimes (|u|^2 - |v|^2)\sigma_z \\
&\quad + |1\rangle\langle 1| \otimes 2(u\bar{v}|0\rangle\langle 1| + \bar{u}v|1\rangle\langle 0|)]\hat{H} \\
&= \mathrm{Tr}[|0\rangle\langle 0| \otimes \Lambda + |1\rangle\langle 1| \otimes 2(u\bar{v}|0\rangle\langle 1| + \bar{u}v|1\rangle\langle 0|)\Lambda] \\
&= 2E_0 + (\bar{u}v + u\bar{v})\omega_0 \cos\alpha.
\end{aligned} \tag{18}$$

Here, for the last term shown in Eq. (18), one also knows

$$|(\bar{u}v + u\bar{v})\,\omega_0 \cos\alpha| \leqslant |2s\cos^2\alpha|, \qquad (19)$$

where the identity holds if and only if $u = \pm v$. When the $\mathcal{PT}$-symmetric Hamiltonian approaches the exceptional point, that is, $\alpha \to \frac{\pi}{2}$, Eq. (18) gives only the value $2E_0$. However, Eq. (18) can reach $2E_0 \pm 2s$ with $\alpha = 0$ when we have a Hermitian Hamiltonian. This means that the $\mathcal{PT}$-symmetric system gives the largest departure from a Hermitian system when it tends to be broken. The unbroken $\mathcal{PT}$-symmetry can thus be viewed as an intermediate case.

B. The classical and local Hermitian pictures

In addition to the simulation picture, we study the same setting but with another two pictures, namely the classical and local Hermitian pictures, in order to give a clear illustration of the internal nonlocality. First, the classical picture here means that one just skips all the details of quantum mechanics but only considers a classical description of what Alice and Bob do. The only thing we ask for is to have the picture be consistent with the simulation picture. That is, we assume that one of the two observers, e.g., Alice, has a "$\mathcal{PT}$-symmetric like" subsystem, and the joint measurements of Alice and Bob depict the characteristics of measuring the global Hamiltonian $\hat{H}$. Indeed, such a consistency rule plays a key role in giving a classical picture. A natural consequence of this rule is to assume the measurement results of A_j are just $\lambda_\pm$, namely the eigenvalues of the $\mathcal{PT}$-symmetric Hamiltonian H. At the same time, the results of B_i should be 1, such that the correlation functions B_iA_j trivially give the eigenvalues of $\hat{H}$. Now A_i and B_i are determined, equivalently completing the classical picture. Moreover, since Bob's results always give 1, apparently the two observers' results and the corresponding probability distributions are independent. Thus, we do have a classical local picture.

Let us come back to calculate the expectation of the Bell operator. As the results of A_i are the eigenvalues $\lambda_\pm$ and the result of B_i is 1, we have

$$\begin{aligned}
&\langle B_0A_0\rangle + \langle B_0A_1\rangle + \langle B_1A_0\rangle - \langle B_1A_1\rangle \\
&= \int [B_0(\nu)(A_0 + A_1)(\nu) + B_1(\nu)(A_0 - A_1)(\nu)]d\nu \\
&= \int [(A_0 + A_1)(\nu) + (A_0 - A_1)(\nu)]d\nu \\
&= 2E_0 + \omega_0\,(p_+ - p_-),
\end{aligned} \qquad (20)$$

where $p_\pm$ are the probabilities corresponding to the situations when the results of A_0 are $\lambda_\pm$.

Secondly, let us consider the local Hermitian picture. Now, the randomness comes from the global Hamiltonian $\hat{H}'$, which is in a tensor product form of two local Hermitian Hamiltonians. To have this local Hermitian picture be consistent with the simulation, one can assume that $\hat{H}'$ has the same eigenvalues as $\hat{H}$ and one of the local Hamiltonians has the same eigenvalues as H. Hence, we have $\hat{H}' = I \otimes H_h$, where $H_h = \lambda_+|s_+\rangle\langle s_+| + \lambda_-|s_-\rangle\langle s_-|$ and $|s_\pm\rangle$ are two orthogonal states. In contrast to $\hat{H}$, the form of $\hat{H}'$ implies that it does not have internal nonlocality. It also implies that by distinguishing

the isospectral global Hamiltonians $\hat{H}$ and $\hat{H}'$, one can distinguish a $\mathcal{PT}$-symmetric Hamiltonian H from an isospectral Hermitian Hamiltonian H_h.

Again, by substituting the $\hat{H}'$ in the local Hermitian picture to Eqs. (14)–(17), the expectation of the Bell operator is

$$\begin{aligned}
&\langle B_0A_0\rangle + \langle B_1A_0\rangle + \langle B_0A_1\rangle - \langle B_1A_1\rangle \\
&= \mathrm{Tr}(I \otimes |u_+\rangle\langle u_+|)(I \otimes H_h) \\
&\quad + \mathrm{Tr}[(|0\rangle\langle 0| - |1\rangle\langle 1|) \otimes |u_-\rangle\langle u_-|](I \otimes H_h),
\end{aligned} \qquad (21)$$

which can be further reduced to

$$2\langle u_+|H_h|u_+\rangle = 2\lambda_+|\langle u_+|s_+\rangle|^2 + 2\lambda_-|\langle u_+|s_-\rangle|^2. \qquad (22)$$

As $\lambda_\pm = E_0 \pm \frac{\omega_0}{2}$, we can denote $p_\pm = |\langle u_+|s_\pm\rangle|^2$ and reach

$$2E_0 + \omega_0\,(p_+ - p_-). \qquad (23)$$

By comparing Eq. (23) with Eqs. (18) and (20), all the expectations in the three pictures contain two terms. The common term $2E_0$ is the sum of the two eigenvalues λ_+ and λ_-, while the other one represents a deviation term. This deviation term is the same for the classical and local Hermitian pictures, as both of them do not support the internal nonlocality. Moreover, we also have

$$|\omega_0(p_+ - p_-)| = |2s(p_+ - p_-)\cos\alpha| \leqslant |2s\cos\alpha|, \qquad (24)$$

which means that these two pictures give a larger value of the upper bound than that obtained in the simulation picture.

IV. DISCUSSIONS

Here, we discuss the physical implications behind our results by contrasting them with those of the CHSH scenario [32]. Even though the generalization of the CHSH scenario to $\mathcal{PT}$-symmetric settings can be found in the literature [33], these approaches essentially differ from our discussions. In the CHSH scenario, the two observers share some entangled states and perform local measurements to explore the correlations. On the contrary, in our setting, the resource of correlations comes from the Hermitian dilation Hamiltonian rather than states.

Moreover, in the CHSH scenario, the observers do perform several local measurements. For example, Alice can measure the spin in the e_0 and e_1 directions. However, in our scenario, Alice performs two "local measurements" with two orthogonal local states $|u_+\rangle$ and $|u_-\rangle$. According to von Neumann's measurement theory, these two states can only represent one measurement rather than two. Furthermore, our randomness and correlations come from the global Hamiltonian. Hence, Alice and Bob can obtain "measurement results" simply by inputting different states, reaching a similar effect to the measurements in the CHSH scenario.

The most significant distinction between our discussions and CHSH's is that our scenario is concretely constructed and logically derived by a consistency rule, which reflects the natural ideas and requirements in simulations of $\mathcal{PT}$-symmetric systems. This explains why the measurement results are *a posteriori*, determined by the consistency rule in the classical picture, while in the CHSH case they are *a priori* known. It also explains why the classical and local Hermitian pictures have the same bounds. Both pictures are constructed

to be consistent with the simulation. The classical picture gives a general and abstract description of Alice's and Bob's measurements as well as the correlations in simulation, from the perspective of locality. The local Hermitian picture can be viewed as a quantum realization of this classical (local) description. Hence the same upper bounds of the two pictures are reasonable.

It is also worth noting that the expectation of the Bell operator exists in a larger range for the classical and local Hermitian cases, rather than in the simulation case. At first glance, the results are counterintuitive as the latter case possesses internal nonlocality. Indeed, nonlocal correlations yield a larger range for the upper bound in the CHSH scenario. However, our scenario is based on the consistency rule utilizing $\hat{H}$ to simulate H and the measurements of $\hat{H}$ to simulate measurements of H. The results of Eqs. (18), (20), and (23) are all essentially characterizing the average deviation from the mean value $2E_0$ in the measuring process. Note that $\hat{H}$ correlates the subsystems, and the internal nonlocality can therefore be viewed to impose some internal constraints on the system. As a result, it is reasonable to have a smaller deviation term in the simulation picture. Moreover, when approaching the exceptional point, an unbroken $\mathcal{PT}$-symmetric system shows the largest departure from Hermitian systems. The minimal deviation at the exceptional point is consistent with such an intuition.

It should also be noted that the results of this paper mainly focus on the two-dimensional case in Eq. (5). However, Eq. (5) is a special case of Eq. (1). Hence, the analogy of the Hermitian dilation Hamiltonians, as well as the isospectral property, implies that the classical picture can be generalized in general. That is, for a higher-dimensional Hamiltonian in Eq. (1), we can similarly assume Alice's results to be the eigenvalues and Bob's result to always be 1, establishing a classical picture. By choosing meaningful Bell operators, a discussion of internal nonlocality is natural in higher-dimensional spaces.

Before the conclusion, we propose two potential applications. First, our results provide a figure of merit to know and test the reliability of the simulation. Suppose we have a set of devices that can produce the Hermitian dilation Hamiltonian and simulate a $\mathcal{PT}$-symmetric system. One may wonder whether the device is reliable, or if it faithfully realizes the simulation design. Apparently, this question is closely related to whether the Hermitian dilation Hamiltonian is well-prepared. To see this, one may have Alice and Bob perform the joint correlation measurements, comparing the results to Eq. (19). If the obtained value of the upper bound is larger than that given in Eq. (19), then the device cannot produce the

needed global Hamiltonian and cannot be used for simulation. Otherwise, it is likely to be reliable.

Moreover, our results can also help in the verification problem of a $\mathcal{PT}$-symmetric system. Consider the following scenario. Let Alice have a system, which is either a simulated $\mathcal{PT}$-symmetric or an isospectral Hermitian one. Let Bob be in charge of another system, which either serves as an ancillary subsystem in simulation or a completely independent Hermitian system. Can they verify whether Alice's system is $\mathcal{PT}$-symmetric or Hermitian just by making measurement? Note that the isospectral property prevents one from seeing the difference by simply reading out the eigenvalues. Moreover, there exist infinitely many isospectral Hermitian Hamiltonians. To this end, Alice and Bob can locally measure the global Hermitian Hamiltonian and evaluate the joint correlation measurements. If the randomness comes from the classical or local Hermitian pictures, they can obtain a large deviation from the mean value. Thus, they know that the system is not simulated to be $\mathcal{PT}$-symmetric.

V. CONCLUSION

In summary, we propose an operational way to explore the internal properties of $\mathcal{PT}$-symmetric systems, as well as their Hermitian dilations, by constructing a nonlocal scenario between Alice and Bob. It is illustrated how to construct correlation pictures based on some concrete procedures such as simulation, proposing a different aspect of investigating nonlocality. By performing local measurements, the resulting expectation values make it possible to extract the internal nonlocality in the global Hermiticity. The ranges in different pictures clearly show the departure of $\mathcal{PT}$-symmetric systems from classical and Hermitian quantum systems, for which the latter two share the same bound. The extremal property of the exceptional point is obtained in the simulation picture. These results not only show the characteristics of the internal nonlocality, but they also can have potential applications. In addition, despite focusing on the discussion of $\mathcal{PT}$-symmetric systems, it is possible to generalize our discussions to the simulation of other non-Hermitian systems.

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