# Solvable dilation model of time-dependent $\mathcal{PT}$-symmetric systems

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The dilation method is a practical way to experimentally simulate non-Hermitian, especially $\mathcal{PT}$-symmetric, quantum systems. However, the time-dependent dilation problem cannot be explicitly solved in general. In this paper, we present a simple yet nontrivial exactly solvable dilation problem for two-dimensional time-dependent $\mathcal{PT}$-symmetric Hamiltonians. Our system is initially set in the unbroken $\mathcal{PT}$-symmetric phase, then goes across the so-called exceptional point, and ends in the broken $\mathcal{PT}$-symmetric phase. For this system, the dilated Hamiltonian and the evolution of $\mathcal{PT}$-symmetric system are analytically worked out. By investigating the large-time behaviors, we give an effective method to choose and adjust the dilation parameters. Our result also shows that the exceptional points do not have much physical relevance in a *time-dependent* system.

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## I. INTRODUCTION

In recent years, researchers have witnessed growing interest in discussing non-Hermitian systems, especially in the field of dynamics and topology [1]. Lots of work has been done, and many intriguing properties of non-Hermitian systems have been revealed and discussed. Related topics, such as the skin effect, have been attracting increasing attention [2–8].

As one of the most important classes of non-Hermitian systems, $\mathcal{PT}$-symmetric systems are of great interest both theoretically and experimentally. Systematic studies of such systems began in 1998, with Bender and his colleagues' discussion of the reality of the eigenvalues of $\mathcal{PT}$-symmetric Hamiltonians [9]. Since then, much work has been done to investigate $\mathcal{PT}$-symmetric quantum systems, among which Mostafazadeh generalized $\mathcal{PT}$-symmetric theory to pseudo-Hermitian theory [10–13]. Recently, there have also been discussions on anti-$\mathcal{PT}$-symmetric systems [14,15].

In general, $\mathcal{PT}$-symmetric systems are non-Hermitian, and it is possible to use large Hermitian systems to simulate such non-Hermitian systems. The simulation of $\mathcal{PT}$-symmetric systems is tightly related to the mathematical concept of operator dilation. In 2008, Günther and Samsonov showed that a special two-dimensional unbroken $\mathcal{PT}$-symmetric Hamiltonian can be dilated, and their results were experimentally realized [16,17]. Later, the result was generalized to any finite-dimensional case [18,19]. As for the broken $\mathcal{PT}$ symmetry, there are also different approaches. One way is to utilize weak measurement, which can be viewed as an approximation paradigm [20]; another way is to simulate the time-dependent broken $\mathcal{PT}$-symmetric systems with the time-dependent Hermitian systems [21,22]. In fact, time-dependent $\mathcal{PT}$-symmetric systems are important research issues in their own right, e.g., the Floquet theory such as in [23,24] and many other features of these systems [25–37]. In particular, the work using Dyson maps by Fring and collaborators implies that exceptional points (EPs) do not play an essential role in such time-dependent systems [33–37]. The discussion of time-dependent dilation gives an important approach for investigating the topology and dynamics of non-Hermitian systems. However, the problem is that usually, the time-evolution operator and the dilated Hamiltonian cannot be analytically worked out, owing to the fact that the Hamiltonian at different times cannot be diagonalized in the same eigenstates [21,38].

In this paper, we discuss a solvable example for the time-dependent dilation problem. All the relevant matrix operators are worked out explicitly. Our model shows that the exceptional points have no physical significance in a *time-dependent* system as the dynamics throughout smoothly evolves.

This paper is organized as follows. In Sec. II, we briefly review the elements of dilation. In Sec. III, we discuss the time-dependent dilation problem and give a solvable model.

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In Sec. IV, we present the detailed results of a special case. We offer some discussion in Sec. V and conclude our results in In Sec. VI.

## II. THE CONCEPT OF DILATION

In this section, we briefly recap the dilation method described in [16,21,22]. Consider an $n$-dimensional, time-dependent non-Hermitian Hamiltonian $H(t)$; it governs an evolution by the Schrödinger equation,

$$i\dot{\psi}(t) = H(t)\psi(t), \tag{1}$$

where the overdot denotes the time derivative. Units with $\hbar = 1$ are adopted. For simplicity, we may suppress the time variable. To simulate such a system in experiments, we dilate the state into $2n$ dimensions, i.e., $\Psi := [\begin{smallmatrix}\psi \\ \tau\psi\end{smallmatrix}]$, where $\tau$ is an ancillary matrix to be specified later. The dilated vector $\Psi$ evolves under a *Hermitian* Hamiltonian

$$i\dot{\Psi}(t) = \mathbb{H}(t)\Psi(t). \tag{2}$$

Here $\mathbb{H} := [\begin{smallmatrix}h_1 & h_2 \\ h_2^\dagger & h_4\end{smallmatrix}]$, with $h_1 = h_1^\dagger$ and $h_4 = h_4^\dagger$. Equations (1) and (2) yield the following conditions:

$$h_1 + h_2\tau = H, \tag{3}$$

$$h_2^\dagger + h_4\tau = i\dot{\tau} + \tau H. \tag{4}$$

It follows from Eq. (4) that

$$h_2 = -i\dot{\tau}^\dagger + H^\dagger\tau^\dagger - \tau^\dagger h_4. \tag{5}$$

By substituting Eq. (5) into Eq. (3), we have

$$h_1 = H + i\dot{\tau}^\dagger\tau - H^\dagger\tau^\dagger\tau + \tau^\dagger h_4\tau. \tag{6}$$

Thus, the dilated Hamiltonian $\mathbb{H}$ is determined by $h_4$ and $\tau$. Apparently, $h_4$ can be an arbitrary $n\times n$ Hermitian matrix, and one needs to find only $\tau$. By the Hermiticity of $h_1$, we have

$$i\frac{d}{dt}(\tau^\dagger\tau) = H^\dagger(\mathbb{1} + \tau^\dagger\tau) - (\mathbb{1} + \tau^\dagger\tau)H. \tag{7}$$

If we denote

$$\eta(t) := (\mathbb{1} + \tau^\dagger\tau), \tag{8}$$

then

$$i\dot{\eta} = H^\dagger\eta - \eta H. \tag{9}$$

By construction, $\eta$ is positive definite. If $\eta$ happens to be time independent, Eq. (9) indicates that the non-Hermitian Hamiltonian is $\mathcal{PT}$ symmetric, $H = H_{\mathcal{PT}}$ with unbroken $\mathcal{PT}$ symmetry,

$$H_{\mathcal{PT}}^\dagger\eta = \eta H_{\mathcal{PT}}. \tag{10}$$

Equation (10) is often called pseudo-Hermiticity in the literature. On the other hand, when $\eta$ is positive definite, the condition of $\eta$-pseudo-Hermiticity is equivalent to unbroken $\mathcal{PT}$ symmetry in finite-dimensional spaces [19]. We will use the term "$\mathcal{PT}$ symmetry" throughout this paper. In general, an arbitrary non-Hermitian $H(t)$ is the combination of an (unbroken) $\mathcal{PT}$-symmetric Hamiltonian $H_{\mathcal{PT}}$ and a gauge term [22,39],

$$H = H_{\mathcal{PT}} - \frac{i}{2}\eta^{-1}\dot{\eta}. \tag{11}$$

Usually, $\eta$ is called the metric operator. The key to dilate a non-Hermitian system is to find a metric operator such that Eq. (9) holds. An important observation is that the matrix $\tau$ exists in Eq. (8) if and only if $(\eta - \mathbb{1})$ is semipositive definite. Or equivalently, all the eigenvalues of the Hermitian matrix $(\eta - \mathbb{1})$ are non-negative. In this case, we can always write the solution of Eq. (8) in the polar decomposition as $\tau = U\sqrt{\eta - \mathbb{1}}$, where $U$ is an arbitrary unitary matrix. A different choice of $U$ will lead to a different but equivalent dilation. For simplicity, we choose a Hermitian $\tau$ with $U = \mathbb{1}$. Obtaining $\tau$, we can further construct the large Hermitian Hamiltonian $\mathbb{H}$. Note that the Hermitian Hamiltonian $\mathbb{H}$ is not determined because $h_4$ is an arbitrary $n\times n$ Hermitian matrix. One simple way to specify $h_4$ is to take it as the Hermitian part of $H$, $h_4 = \frac{1}{2}(H + H^\dagger)$. Another way is to follow Ref. [21] and demand $h_4 = [H + (i\dot{\tau} + \tau H)\tau]\eta^{-1}$. The above formalism allows us to simulate, that is, effectively realize, non-Hermitian systems using larger Hermitian systems.

To find $\eta$, let us take

$$\eta(t) = \zeta^\dagger(t)\zeta(t),$$

where $\zeta(t)$ is a matrix satisfying the following differential equation [22]:

$$i\dot{\zeta}^\dagger(t) = H^\dagger(t)\zeta^\dagger(t). \tag{12}$$

The solutions to Eq. (12) can be easily constructed from the solutions to the dual Schrödinger equation whose Hamiltonian is $H^\dagger(t)$. Note that the initial value of $\zeta(0)$ is arbitrary as long as all the moduli of its eigenvalues are not smaller than 1. Different choices of $\zeta(0)$ lead to different but equivalent $\eta(t)$ and the dilation Hamiltonian $\mathbb{H}(t)$ [22].

In general, a closed form of the solution to Eq. (12) is hard to find. In the next section, we discuss a simple but nontrivial two-dimensional model whose dilation problem can be solved exactly.

## III. A SOLVABLE MODEL

In this section, we illustrate the general ideas using a concrete example. We start with a $2\times 2$ time-dependent non-Hermitian Hamiltonian $H_\omega(t)$ and solve the Schrödinger equation governed by it. Our goal is to obtain the dilated Hermitian Hamiltonian $\mathbb{H}$ as explicitly as possible. Equations (5) and (6) show that $\mathbb{H}$ is determined by $\tau$. The key step to obtain $\tau$ is to find the metric operator $\eta$ determined by Eq. (9). The discussion near the end of Sec. II shows that $\eta$ can be constructed by the solutions to the dual Schrödinger equation. Finally, by taking a square root of $(\eta - \mathbb{1})$, we get $\tau$.

The $2\times 2$ time-dependent Hamiltonian is as follows:

$$H_\omega(t) = \begin{bmatrix} E + i\omega t & 1 \\ 1 & E - i\omega t \end{bmatrix}, \tag{13}$$

where $E$ and $\omega$ are real parameters. The parity operator is chosen to be the first Pauli matrix, and the time-reversal operator is chosen to be the complex conjugation (or Hermitian conjugation since the matrix is symmetric),

$$\mathcal{P} = \sigma_x, \quad \mathcal{T} = * \text{ or } \dagger. \tag{14}$$

One can verify that $H_\omega(t)$ is $\mathcal{PT}$ symmetric, that is,

$$H_\omega(t)\mathcal{PT} = \mathcal{PT}H_\omega(t). \tag{15}$$

Such an example can be used to discuss the Jarzynski equality [40] and the time-dependent $\mathcal{PT}$-symmetric quantum mechanics [22,39]. The instantaneous eigenvalues are

$$\lambda(t) = E \pm \sqrt{1-(\omega t)^2}. \tag{16}$$

When $\omega t < 1$, the $\mathcal{PT}$ symmetry is unbroken, and both eigenvalues are real. At the EP, $\omega t = 1$, the Hamiltonian is not diagonalizable. When $\omega t > 1$, the eigenvalues are complex. Thus, the $\mathcal{PT}$ symmetry is broken for $\omega t \geqslant 1$. One may expect that some critical phenomena happen at the EP. On the contrary, as we will see later, that is not the case. The dynamics evolves smoothly even when $\omega t$ crosses the EP.

We divide the process of solving dilation into several sections. In Sec. III A, we solve the Schrödinger equation governed by $H_\omega(t)$,

$$i\dot{\psi}(t) = H_\omega(t)\psi(t). \tag{17}$$

In Sec. III B, we solve the dual Schrödinger equation to construct $\eta$. In Sec. III C, we give the form of $\tau$ by taking the square root of $(\eta - \mathbb{1})$. Thus, by arbitrarily choosing a Hermitian matrix $h_4$, one can obtain $\mathbb{H}$, and the dilation problem is solved. In Sec. III D, we discuss some large-time behavior which dictates when a given dilation may fail.

### A. Solutions to the Schödinger equation (17)

We write the solution to Eq. (17) in the following component form:

$$\psi(t) = \begin{bmatrix} x_\uparrow(t) \\ x_\downarrow(t) \end{bmatrix}.$$

Now Eq. (17) gives two combined equations,

$$i\dot{x}_\uparrow(t) = (E + i\omega t)x_\uparrow(t) + x_\downarrow(t), \tag{18}$$

$$i\dot{x}_\downarrow(t) = x_\uparrow(t) + (E - i\omega t)x_\downarrow(t). \tag{19}$$

By substituting Eq. (18) into Eq. (19) and eliminating $x_\downarrow(t)$, we get a second-order differential equation,

$$\ddot{x}_\uparrow(t) + 2iE\dot{x}_\uparrow(t) + [1 - E^2 - \omega(1 + wt^2)]x_\uparrow(t) = 0. \tag{20}$$

By changing variables,

$$z := \omega t^2, \quad w(z) := \sqrt{t}\, e^{iEt} x_\uparrow(t), \tag{21}$$

we obtain a Whittaker equation,

$$w''(z) + \left(-\frac{1}{4} + \frac{1-\omega}{4\omega z} + \frac{3}{16z^2}\right)w(z) = 0. \tag{22}$$

The general solution can be represented by the Whittaker functions,

$$w(z) = C_0 W_{\kappa,\mu}(z) + C_1 W_{-\kappa,\mu}(e^{i\pi}z), \tag{23}$$

with

$$\kappa = -\tfrac{1}{4} + \tfrac{1}{4\omega}, \quad \mu = \tfrac{1}{4}.$$

Here we follow the notations in Ref. [41]. In terms of the original variables, we have

$$x_\uparrow(t) = C_0 \frac{e^{-iEt}}{\sqrt{t}} W_{\kappa,\mu}(\omega t^2) + C_1 \frac{e^{-iEt}}{\sqrt{t}} W_{-\kappa,\mu}(-\omega t^2). \tag{24}$$

For simplicity, let us define two linearly independent solutions as

$$x_\uparrow^{(0)}(t) := \frac{e^{-iEt}}{\sqrt{t}} W_{\kappa,\mu}(\omega t^2), \quad x_\uparrow^{(1)}(t) := \frac{e^{-iEt}}{\sqrt{t}} W_{-\kappa,\mu}(-\omega t^2). \tag{25}$$

Note that there is no singularity as $t \to 0$ because $W_{\pm\kappa,\mu}(\pm\omega t^2) \propto \sqrt{t}$ for small $t$ ([41], (13.14.18)). In principle, the corresponding lower components $x_\downarrow^{(i)}$ can be solved similarly by eliminating $x_\uparrow^{(i)}$ from Eq. (18). Moreover, the coefficients in $x_\downarrow^{(i)}$ are determined by the corresponding $x_\uparrow^{(i)}$. After a lengthy calculation (see Appendix A), compact results are found,

$$x_\downarrow^{(0)}(t) = -2ie^{-iEt}\sqrt{\frac{\omega}{t}} W_{\kappa',\mu}(\omega t^2),$$
$$x_\downarrow^{(1)}(t) = \frac{e^{-iEt}}{2\sqrt{\omega t}} W_{-\kappa',\mu}(-\omega t^2), \tag{26}$$

where $\kappa' = \frac{1}{4} + \frac{1}{4\omega}$. Note that all four Whittaker functions are smooth near the EP, implying that nothing special happens. Furthermore, since the Wronskian $\mathcal{W}$ of the Whittaker functions is a constant [41], i.e.,

$$\mathcal{W}\{W_{\kappa,\mu}(z), W_{-\kappa,\mu}(e^{\pi i}z)\} = e^{-\kappa\pi i},$$

these two solutions will never coalesce.

It is also known that for some special values of parameters, the Whittaker function can be truncated to Hermite polynomials [41],

$$W_{\frac{1}{4}+\frac{n}{2},\frac{1}{4}}(z) = \frac{e^{-z/2}}{2^n} z^{1/4} H_n(\sqrt{z}). \tag{27}$$

Therefore, when $\frac{1}{2\omega}$ is an integer, one of the solutions reduces to elementary functions.

### B. The metric operator

To determine the metric operator $\eta(t)$, we need to solve Eq. (12), where $H^\dagger(t)$ is now given by $H_\omega^\dagger(t)$. The column vectors of $\zeta^\dagger$ are just the solutions to the "dual Schrödinger equation,"

$$i\dot{\phi}(t) = H_\omega^\dagger(t)\phi(t). \tag{28}$$

On the other hand, one can derive from the $\mathcal{PT}$-symmetry condition in Eq. (15) that

$$\sigma_x H_\omega = H_\omega^\dagger \sigma_x.$$

Let $\psi$ be a solution to the Schrödinger equation (17); then

$$H_\omega^\dagger \sigma_x \psi = \sigma_x H_\omega \psi = i\frac{d}{dt}(\sigma_x\psi). \tag{29}$$

That is, $\phi = \sigma_x\psi$ is the solution to the dual Schrödinger equation (28). If we define

$$y^{(0)} := \sigma_x x^{(1)}, \quad y^{(1)} := \sigma_x x^{(0)},$$

then the explicit forms of $y^{(i)}$ are

$$y^{(0)}(t) = \frac{e^{-iEt}}{\sqrt{t}}\begin{bmatrix}\frac{1}{2\sqrt{\omega}}W_{-\kappa',\mu}(-\omega t^2)\\ W_{-\kappa,\mu}(-\omega t^2)\end{bmatrix}, \tag{30}$$

$$y^{(1)}(t) = \frac{e^{-iEt}}{\sqrt{t}}\begin{bmatrix}-2i\sqrt{\omega}W_{\kappa',\mu}(\omega t^2)\\ W_{\kappa,\mu}(\omega t^2)\end{bmatrix}. \tag{31}$$

The two column vectors of $\zeta^\dagger$ are both linear combinations of $y^{(0)}$ and $y^{(1)}$. To determine $\zeta^\dagger$, one should determine four linear combination coefficients. For simplicity, we consider a solution to Eq. (12) with only two parameters, $D_0$ and $D_1$,

$$\zeta^\dagger(t) = \begin{bmatrix}D_0 y_\uparrow^{(0)}(t) & D_1 y_\uparrow^{(1)}(t)\\ D_0 y_\downarrow^{(0)}(t) & D_1 y_\downarrow^{(1)}(t)\end{bmatrix}. \tag{32}$$

With this choice, we have

$$\eta = \zeta^\dagger\zeta = |D_0|^2|y^{(0)}\rangle\langle y^{(0)}| + |D_1|^2|y^{(1)}\rangle\langle y^{(1)}|, \tag{33}$$

where the bras and kets are conventional Dirac notation with $|\cdot\rangle = \langle\cdot|^\dagger$. From the definition of $\eta$ in Eq. (8), $(\eta - \mathbb{1})$ must be semipositive definite. Thus, it requires that all the eigenvalues of $\eta$ are not smaller than 1. Otherwise, one cannot find an appropriate matrix $\tau$ such that Eq. (7) holds. In this case, dilation fails. This imposes constraints on $D_i$. In this model, for any finite time interval, one can always find a set of appropriate $D_i$ such that the eigenvalues of $\eta$ are not smaller than 1; that is, the dilation can be valid over any given time interval. For this purpose, note that $\eta = \zeta^\dagger\zeta$ has the eigenvalues

$$\lambda_\pm = \frac{l}{2} \pm \sqrt{\frac{l^2}{4} - |D_0 D_1|^2\Delta}, \tag{34}$$

where we define

$$l := |D_0|^2\|y^{(0)}\|^2 + |D_1|^2\|y^{(1)}\|^2,$$
$$\Delta := \|y^{(0)}\|^2\|y^{(1)}\|^2 - \left|\langle y^{(0)}|y^{(1)}\rangle\right|^2,$$

with the conventional notation $\|\cdot\| := \sqrt{\langle\cdot|\cdot\rangle}$.

There are various ways to choose appropriate $D_i$ so that the eigenvalues are not smaller than 1. Moreover, since $\lambda_+ \geqslant \lambda_-$, one needs to guarantee only that $\lambda_- \geqslant 1$. For example, one may choose $D_0 = D_1 = D$. According to the Schwartz inequality, $\Delta > 0$. We denote $\tilde{l} := \|y^{(0)}\|^2 + \|y^{(1)}\|^2$; then $\lambda_- \geqslant 1$ is equivalent to

$$|D|^2 \geqslant \frac{\tilde{l} + \sqrt{\tilde{l}^2 - 4\Delta}}{2\Delta}.$$

Note that right-hand side is a continuous function of $\tilde{l}$ and $\Delta$. Thus, in any finite time interval, it has a maximal value. We may always choose $|D|^2$ so that it is larger than this maximum to ensure the dilation is valid.

### C. The dilation matrix $\tau$

After solving the metric operator $\eta(t)$, we are ready to find the dilation operator $\tau(t)$ in the dilated state vectors $\Psi$ and Hamiltonian $\mathbb{H}$. As discussed before, we choose a Hermitian $\tau$ for simplicity,

$$\tau = \sqrt{\eta - \mathbb{1}} = \begin{bmatrix}d + c & a - ib\\ a + ib & d - c\end{bmatrix}. \tag{35}$$

It can be shown that (see Appendix C for more details)

$$a = \frac{X}{\sqrt{2}\sqrt{W + \sqrt{W^2 - X^2 - Y^2 - Z^2}}},$$
$$b = \frac{Y}{\sqrt{2}\sqrt{W + \sqrt{W^2 - X^2 - Y^2 - Z^2}}},$$
$$c = \frac{Z}{\sqrt{2}\sqrt{W + \sqrt{W^2 - X^2 - Y^2 - Z^2}}},$$
$$d = \frac{\sqrt{W + \sqrt{W^2 - X^2 - Y^2 - Z^2}}}{\sqrt{2}}, \tag{36}$$

where $X$, $Y$, $Z$, and $W$ are matrix elements of $\eta - \mathbb{1}$,

$$\tau^2 = \eta - \mathbb{1} = \begin{bmatrix}W + Z & X - iY\\ X + iY & W - Z\end{bmatrix}. \tag{37}$$

According to Eq. (33), the explicit forms of the matrix elements of $\tau^2$ are

$$\begin{aligned}X &= \tfrac{1}{2}[|D_0|^2(y_\uparrow^{(0)}y_\downarrow^{(0)*} + y_\uparrow^{(0)*}y_\downarrow^{(0)})\\ &\quad + |D_1|^2(y_\uparrow^{(1)}y_\downarrow^{(1)*} + y_\uparrow^{(1)*}y_\downarrow^{(1)})],\\ Y &= \tfrac{i}{2}[|D_0|^2(y_\uparrow^{(0)}y_\downarrow^{(0)*} - y_\uparrow^{(0)*}y_\downarrow^{(0)})\\ &\quad + |D_1|^2(y_\uparrow^{(1)}y_\downarrow^{(1)*} - y_\uparrow^{(1)*}y_\downarrow^{(1)})],\\ Z &= \tfrac{1}{2}[|D_0|^2(|y_\uparrow^{(0)}|^2 - |y_\downarrow^{(0)}|^2)\\ &\quad + |D_1|^2(|y_\uparrow^{(1)}|^2 - |y_\downarrow^{(1)}|^2)],\\ W &= \tfrac{1}{2}(|D_0|^2\|y^{(0)}\|^2 + |D_1|^2\|y^{(1)}\|^2) - 1,\end{aligned} \tag{38}$$

and

$$\begin{aligned}&W^2 - X^2 - Y^2 - Z^2\\ &\quad = 1 - (|D_0|^2\|y^{(0)}\|^2 + |D_1|^2\|y^{(1)}\|^2)\\ &\qquad + |D_0 D_1|^2(\|y^{(0)}\|^2\|y^{(1)}\|^2 - |\langle y^{(0)}|y^{(1)}\rangle|^2).\end{aligned} \tag{39}$$

With a proper choice of $h_4$, one can get the explicit dilated Hamiltonian $\mathbb{H}$. Regardless of the choice of $h_4$, Eqs. (5) and (6), as well as the more or less complicated form of $\eta$, make the final form of $\mathbb{H}$ too long to present here.

### D. Large-time behavior

The above discussion applies to only the situation of a finite time interval. When time $t$ tends to be infinitely large, the simulation will always fail in this model. We may see this by studying the large-time behaviors.

Applying formulas in Ref. [41], we can derive the large-time behaviors for the eigenvalues of $\eta$ as $t \to \infty$ (see Appendix B for details),

$$\lambda_+ \sim \frac{\sqrt{\omega}}{(\omega t^2)^{\frac{1}{2\omega}}}|D_0|^2 e^{\omega t^2}, \tag{40}$$

$$\lambda_- \sim |D_1|^2 4\omega^{\frac{3}{2}}(\omega t^2)^{\frac{1}{2\omega}}e^{-\omega t^2}. \tag{41}$$

The decaying factor $e^{-\omega t^2}$ in $\lambda_-$ indicates that the eigenvalue will be smaller than 1 for sufficiently large time, regardless of the choices of $D_i$. That is, all dilation schemes will break down eventually. To quantitatively see when the dilation will fail, we will discuss a special case in detail.

## IV. THE SPECIAL CASE WITH $\omega = \frac{1}{2}$

In this section, we present the results for a special case with $\omega = \frac{1}{2}$ in detail. The Hamiltonian becomes

$$H_{\frac{1}{2}}(t) = \begin{bmatrix} E + \frac{1}{2}ti & 1 \\ 1 & E - \frac{1}{2}ti \end{bmatrix}. \tag{42}$$

The eigenvalues of $H_{\frac{1}{2}}(t)$ are $\lambda_\pm = E \pm \sqrt{1 - \frac{t^2}{4}}$. We consider a finite time interval $t \in [0, 4]$. When $t \in [0, 2)$, the eigenvalues $\lambda_\pm$ are real, and $\mathcal{PT}$ symmetry is unbroken. When $t \in (2, 4]$, $\lambda_\pm$ are complex, and thus, $\mathcal{PT}$ symmetry is broken. The EP occurs at $t = 2$.

For this case, the solution in Eq. (24) is

$$x_\uparrow(t) = C_0 \frac{e^{-iEt}}{\sqrt{t}} W_{\frac{1}{4},\frac{1}{4}}\left(\frac{1}{2}t^2\right) + C_1 \frac{e^{-iEt}}{\sqrt{t}} W_{-\frac{1}{4},\frac{1}{4}}\left(-\frac{1}{2}t^2\right). \tag{43}$$

Note that $W_{\frac{1}{4},\frac{1}{4}}(\frac{1}{2}t^2) \propto \sqrt{t}\, e^{-\frac{t^2}{4}}$ and $W_{-\frac{1}{4},\frac{1}{4}}(-\frac{1}{2}t^2)$ is a linear combination of $\sqrt{t}\, e^{-\frac{t^2}{4}}$ and $\sqrt{t}\, e^{-\frac{t^2}{4}} \mathrm{Erfi}(\frac{t}{\sqrt{2}})$ ([41], (13.18.7) and (13.18.16)), where Erfi($z$) is the imaginary error function (we adopt the definition in Ref. [41], i.e., $\mathrm{Erfi}(z) := \int_0^z e^{x^2} dx$). Therefore, for convenience, we choose one solution vector as

$$|x^{(0)}(t)\rangle = e^{-iEt - \frac{t^2}{4}} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \tag{44}$$

with

$$\alpha = 1, \quad \beta = -it,$$

and the other solution vector as

$$|x^{(1)}(t)\rangle = e^{-iEt - \frac{t^2}{4}} \begin{bmatrix} \gamma \\ \delta \end{bmatrix}, \tag{45}$$

where

$$\gamma = -i\sqrt{2}\,\mathrm{Erfi}\left[\frac{t}{\sqrt{2}}\right], \quad \delta = e^{\frac{t^2}{2}} - \sqrt{2}\,t\,\mathrm{Erfi}\left[\frac{t}{\sqrt{2}}\right].$$

To determine the metric operator $\eta(t)$, we take two independent solutions to the dual Schrödinger equation as

$$|y^{(0)}(t)\rangle = e^{-iEt - \frac{t^2}{4}} \begin{bmatrix} \delta \\ \gamma \end{bmatrix}, \tag{46}$$

$$|y^{(1)}(t)\rangle = e^{-iEt - \frac{t^2}{4}} \begin{bmatrix} \beta \\ \alpha \end{bmatrix}, \tag{47}$$

where $\alpha, \beta, \gamma$, and $\delta$ are defined in Eqs. (44) and (45).

It can be verified that

$$|x^{(0)}(0)\rangle = |y^{(0)}(0)\rangle = |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$|x^{(1)}(0)\rangle = |y^{(1)}(0)\rangle = |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

This justifies our labeling on $|x\rangle$ and $|y\rangle$. In addition, as time increases, $|y^{(0)}\rangle$ and $|x^{(1)}\rangle$ tend to be infinitely large, while $|y^{(1)}\rangle$ and $|x^{(0)}\rangle$ tend to vanish.

Now the problem is to find appropriate parameters $D_i$ which give a successful dilation in a finite time interval. As shown in Sec. III B, this reduces to finding $D_i$ such that $\lambda_- \geqslant 1$ over the target time domain. The large-time behaviors often help us to choose the parameters $D_i$. To see this, we deduce from Eq. (34) that $\lambda_- \geqslant 1$ is equivalent to

$$2 \leqslant l \leqslant 1 + |D_0|^2 |D_1|^2, \tag{48}$$

where we substitute the result of $\Delta = 1$ in this model. Now suppose that we want to obtain the $D_i$ which gives a successful dilation over the time interval [0,4]. Note that for large $t$, e.g., $t = 4$ in the present case, $\|y^{(1)}\|$ is small. Therefore, Eq. (48) may be approximated as

$$|D_0|^2 \geqslant \max \frac{2}{\|y^{(0)}(t)\|^2} \approx 3.43, \tag{49}$$

$$|D_1|^2 \geqslant \max \|y^{(0)}(t)\|^2 \approx 237.80. \tag{50}$$

Thus, a possible way to choose parameters is to take $D_i$ satisfying Eqs. (49) and (50) and verify whether these parameters really ensure the dilation. For example, we can take $D_0^2 = 3.5$ and $D_1^2 = 238$. The smaller eigenvalue of the metric operator $\lambda_-(t)$ is plotted in the top panel of Fig. 1. The dilation fails around 4.0001.

For this model, $y^{(0)}$ dominates for large time. When $|D_0|$ is fixed, in order to extend the valid time for the dilation to $t_0$, we simply need to choose

$$|D_1|^2 \geqslant \|y^{(0)}(t_0)\|^2. \tag{51}$$

In the bottom panel of Fig. 1, we plot a case which is valid up to $t = 4.5$.

Note that Eq. (51) gives a very good estimation of the breakdown time for $t_0 = 4$ or lager. This is because $y^{(0)}$ is exponentially large for large $t$. However, the asymptotic analysis may need to be fine-tuned for not so large a dilation interval. For example, if we consider the time interval [0, 2.1] and $D_0^2 = 3.5$, we may guess that $|D_1|^2 \geqslant |y^{(0)}(2.1)|^2 \approx 4.129$ according to Eq. (51). However, the dilation actually breaks down earlier at $t \approx 2.003$ for $D_1^2 = 4.13$. In such a situation, we must use the full expression in Eq. (48) to make a better estimation, which is equivalent to

$$|D_1|^2 \geqslant \frac{|D_0|^2 \|y^{(0)}\|^2 - 1}{|D_0|^2 - \|y^{(1)}\|^2}. \tag{52}$$

For the dilation to be valid up to $t = 2.1$, we must take $|D_1|^2 \geqslant \frac{3.5\|y^{(0)}(2.1)\|^2 - 1}{3.5 - \|y^{(1)}(2.1)\|^2} \approx 4.633$. The actual breakdown time is $t \approx 2.1003$ with $D_1^2 = 4.634$. The smaller eigenvalues of the metric operator of both choices are plotted in Fig. 2. Once again, both functions are smooth around the exceptional point at $t = 2$.

## V. DISCUSSION

The key to specify a metric operator is to determine the parameters $D_0$ and $D_1$. They can take either the same or different values. An advantage of taking different values is improving the efficiency of dilation. According to Eq. (2), for the state $\Psi = [\begin{smallmatrix} \psi \\ \tau\psi \end{smallmatrix}]$, the $\mathcal{PT}$-symmetric system is simulated for the upper components. Hence, the dilation efficiency can

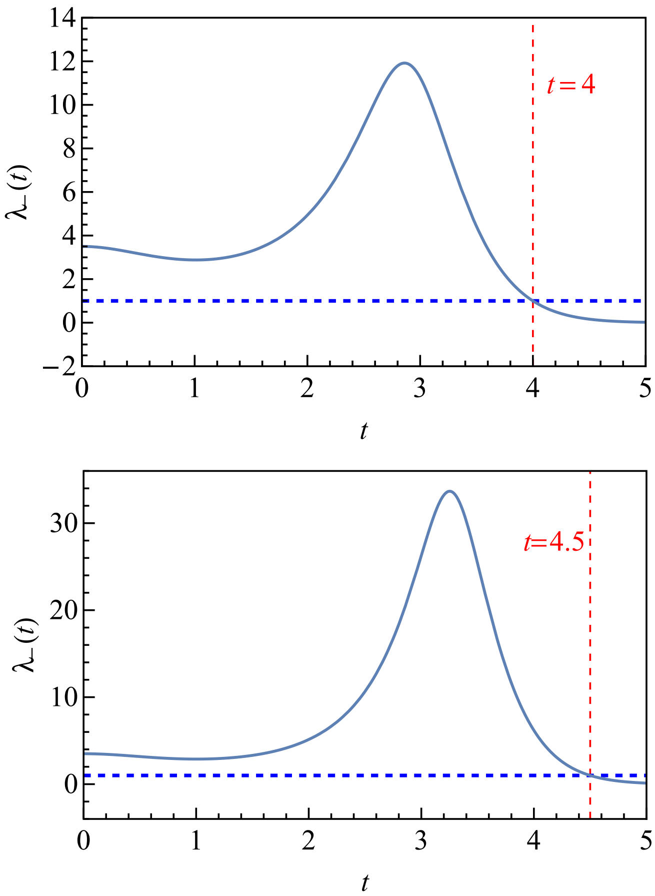


FIG. 1. The smaller eigenvalue $\lambda_-$ of the metric $\eta$ as a function of time for $t \in [0, 5]$. Here we choose $D_0^2 = 3.5$ for both panels. In the top panel, $D_1^2 = 238$. The dilation is valid up to $t \approx 4.0001$. In the bottom panel, $D_1^2 = 1474$, and the dilation is valid up to $t \approx 4.5$. The solid lines are the smaller eigenvalue $\lambda_-$ of the metric operator $\eta$. The blue dashed lines are the constant 1. The dilations are valid when they are above the horizontal blue dashed lines. The vertical red dashed lines mark the time limits when the dilations fail. Note that both eigenvalues are smooth at the exceptional point $t = 2$.

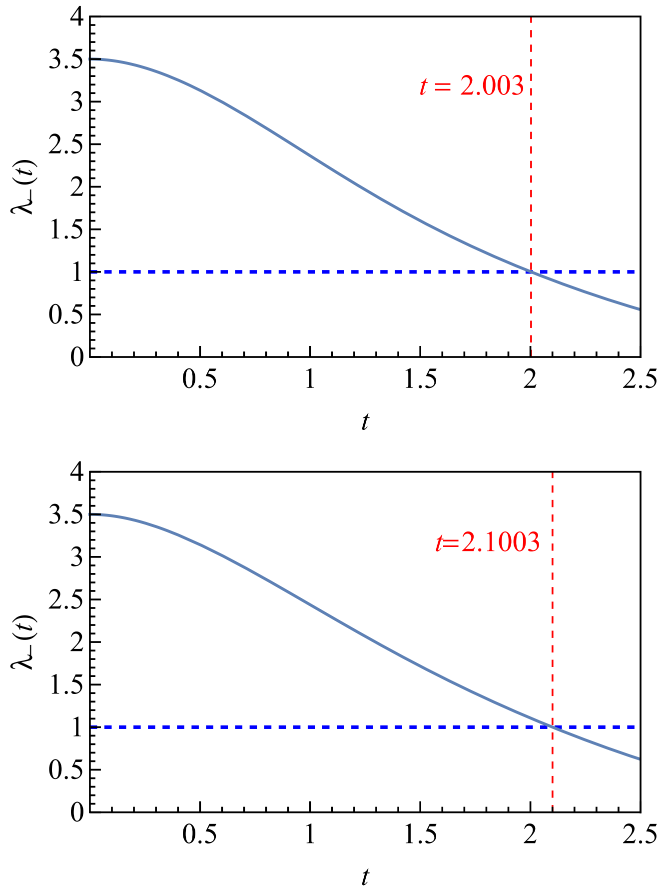


FIG. 2. The smaller eigenvalue $\lambda_-$ of $\eta$ as a function of time for $t \in [0, 2.5]$. Here we choose $D_0^2 = 3.5$ for both panels. For the top panel, $D_1^2 = 4.13$, and the dilation is valid up to $t \approx 2.003$. For the bottom panel, $D_1^2 = 4.634$, and the dilation is valid up to $t \approx 2.1003$. The solid lines are the smaller eigenvalue of $\eta$. The blue dashed lines are the constant 1. The dilations are valid when the solid lines are above the horizontal blue dashed lines. The vertical red dashed lines mark the time when the dilations fail. Note again that nothing dramatic happens at the exceptional point $t = 2$.

be characterized by $\frac{\langle\psi|\psi\rangle}{\langle\psi|\mathbb{1}+\tau^\dagger\tau|\psi\rangle} = \frac{\langle\psi|\psi\rangle}{\langle\psi|\eta|\psi\rangle}$. In our discussion of the time interval [0,4], $|D_1|^2 \geqslant 238$. If we choose $|D_0| = |D_1|$, the dilation efficiency for any state can be estimated by $\frac{1}{238^2}$. However, if we take $|D_0|^2 = 3.5$, then for state $[\frac{|0\rangle}{\tau(0)|0\rangle}]$, its dilation efficiency can be characterized by $\frac{1}{3.5^2} > \frac{1}{238^2}$. Thus, the dilation efficiency is improved.

The EP plays a central role in many studies on non-Hermitian Hamiltonians. A critical phenomenon is expected at the exceptional point in a *time-independent* system because the energy eigenstates coalesce and the Hamiltonian becomes nondiagonalizable. However, in a *time-dependent* system like ours, the instantaneous eigenstates are not solutions to the Schrödinger equation. The critical behavior of the eigenstates does not directly translate to the dynamical states of the system. Therefore, one should not expect anything special to happen at the exceptional point. As shown in our model, the exceptional point is as normal as any other point within the dilation interval.

As shown in Sec. III, for the given Hamiltonian in Eq. (13) and any finite time interval, one can always find appropriate $\eta$ (and $\tau$) such that the dilation is valid. However, as $t$ tends to infinity, the dilation will eventually break down. Note that this breakdown time can be arbitrarily postponed by different dilation parameters $D_i$. Thus, it cannot be an *intrinsic* critical point of the original non-Hermitian system. Rather, such a breakdown is only a limitation of our dilation technique.

It might be interesting from a mathematical point of view to see what happens to the dilated Hamiltonian $\mathbb{H}$ after the breakdown time. If we keep the ancillary matrix as $\tau = U\sqrt{\eta - \mathbb{1}}$ with an arbitrary unitary matrix $U$, then the dilated Hamiltonian $\mathbb{H}$ defined by Eqs. (5) and (6) is no longer Hermitian. To see this, note that $\sqrt{\eta - \mathbb{1}}$ is not Hermitian for $\lambda_- < 1$. Therefore,

$$\tau^\dagger\tau + \mathbb{1} = (\sqrt{\eta - \mathbb{1}})^\dagger(\sqrt{\eta - \mathbb{1}}) + \mathbb{1} \neq \eta.$$

Plugging the above inequality into Eq. (6), in general, we have that

$$h_1 - h_1^\dagger \neq i\dot{\eta} - H^\dagger\eta + \eta H = 0. \tag{53}$$

Further note that the non-Hermiticity of $\mathbb{H}$ cannot be saved by any choice of $h_4$.

## VI. CONCLUSION

In summary, a two-dimensional solvable model for *time-dependent* $\mathcal{PT}$-symmetric systems was shown to have an explicit scheme to dilate into a four-dimensional Hermitian system. Furthermore, by investigating the large-time behaviors, we gave an effective method to choose and adjust the dilation parameters. A good estimation of the breakdown time for the dilation was also derived. As the dilated Hermitian Hamiltonians play an important role in the simulation of $\mathcal{PT}$-symmetric systems, our results may shed new light on the study of *time-dependent* $\mathcal{PT}$-symmetric systems.

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## APPENDIX A: THE LOWER COMPONENT OF THE SOLUTION TO THE SCHRÖDINGER EQUATION

In this Appendix, we show how to derive the lower component of the solution to the Schrödinger equation in Eq. (26). Using Eq. (19), we easily express Eq. (18) as a second-order differential equation for $x_\downarrow$, which is very similar to that for $x_\uparrow$ in (20),

$$\ddot{x}_\downarrow + 2iE\dot{x}_\downarrow + [1 - E^2 + \omega(1 - \omega t^2)]x_\downarrow = 0. \tag{A1}$$

The only difference between Eqs. (20) and (A1) is the sign of $\omega$. Thus, we can immediately read off that the general solution for $x_\downarrow$ is the linear combination of two Whittaker functions,

$$x_\downarrow(t) = C_3\frac{e^{-iEt}}{\sqrt{t}}W_{\kappa',\mu}(\omega t^2) + C_4\frac{e^{-iEt}}{\sqrt{t}}W_{-\kappa',\mu}(-\omega t^2), \tag{A2}$$

where $\kappa' := \frac{1}{4} + \frac{1}{4\omega}$. We may guess that

$$x_\downarrow^{(0)}(\omega, t) = C_3\frac{e^{-iEt}}{\sqrt{t}}W_{\kappa',\frac{1}{4}}(\omega t^2), \tag{A3}$$

$$x_\downarrow^{(1)}(\omega, t) = C_4\frac{e^{-iEt}}{\sqrt{t}}W_{-\kappa',\frac{1}{4}}(-\omega t^2). \tag{A4}$$

To determine the correct constants $C_3$ and $C_4$, let us plug Eq. (25) into Eq. (18),

$$x_\downarrow = i\dot{x}_\uparrow - (E + i\omega t)x_\uparrow. \tag{A5}$$

Applying the identity ([41], (13.15.25)) with $n = 1$,

$$\frac{d}{dz}\left[e^{-\frac{1}{2}z}z^{\mu-\frac{1}{2}}W_{\kappa,\mu}(z)\right] = -e^{-\frac{1}{2}z}z^{\mu-1}W_{\kappa+\frac{1}{2},\mu-\frac{1}{2}}(z),$$

and recognizing that ([41], (13.14.31))

$$W_{\kappa,\mu}(z) = W_{\kappa,-\mu}(z),$$

we get from Eq. (A5) that

$$x_\downarrow^{(0)} = -2i\omega^{\frac{1}{2}}\frac{e^{-iEt}}{\sqrt{t}}W_{\kappa',\frac{1}{4}}(\omega t^2). \tag{A6}$$

That is,

$$C_3 = -2i\sqrt{\omega}.$$

Similarly, by applying ([41], (13.15.22)), with $n = 1$,

$$\frac{d}{dz}\left[e^{\frac{1}{2}z}z^{\mu-\frac{1}{2}}W_{\kappa,\mu}(z)\right] = -\left(\frac{1}{2} - \kappa - \mu\right)e^{\frac{1}{2}z}z^{\mu-1}W_{\kappa-\frac{1}{2},\mu-\frac{1}{2}}(z),$$

we obtain

$$x_\downarrow^{(1)} = \frac{1}{2\sqrt{\omega}}\frac{e^{-iEt}}{\sqrt{t}}W_{-\kappa',\frac{1}{4}}(-\omega t^2).$$

Namely, $C_4 = \frac{1}{2\sqrt{\omega}}$. Putting them together, the explicit forms of the two independent solutions to the Schrödinger equation are

$$x^{(0)}(t) = \frac{e^{-iEt}}{\sqrt{t}}\begin{bmatrix} W_{-\frac{1}{4}+\frac{1}{4\omega},\frac{1}{4}}(\omega t^2) \\ -2i\sqrt{\omega}W_{\frac{1}{4}+\frac{1}{4\omega},\frac{1}{4}}(\omega t^2)\end{bmatrix}, \tag{A7}$$

$$x^{(1)}(t) = \frac{e^{-iEt}}{\sqrt{t}}\begin{bmatrix} W_{\frac{1}{4}-\frac{1}{4\omega},\frac{1}{4}}(-\omega t^2) \\ \frac{1}{2\sqrt{\omega}}W_{-\frac{1}{4}-\frac{1}{4\omega},\frac{1}{4}}(-\omega t^2)\end{bmatrix}, \tag{A8}$$

where $-\omega t^2 = e^{i\pi}\omega t^2$ for the branch cut of the Whittaker function.

## APPENDIX B: LARGE-TIME BEHAVIORS

In this Appendix, we derive the large-time behaviors of the dilated system. Applying the identity ([41], (13.14.21))

$$W_{\kappa,\mu} \sim e^{-\frac{1}{2}z}z^\kappa, \quad \text{as} \quad z \to \infty, \quad |\arg z| < \frac{3}{2}\pi - \delta, \tag{B1}$$

to the solution of the Schrödinger equation in Eqs. (A7) and (A8), we get the large-time behaviors,

$$x^{(0)} \sim e^{-iEt-\frac{1}{2}\omega t^2}\omega^{-\frac{1}{4}+\frac{1}{4\omega}}t^{\frac{1}{2\omega}-1}\begin{bmatrix} 1 \\ -2i\omega t\end{bmatrix}, \tag{B2}$$

$$x^{(1)} \sim e^{-iEt+\frac{1}{2}\omega t^2}(e^{i\pi}\omega)^{\frac{1}{4}-\frac{1}{4\omega}}t^{-\frac{1}{2\omega}}\begin{bmatrix} 1 \\ \frac{1}{2i\omega t}\end{bmatrix}. \tag{B3}$$

Accordingly, the large-time behaviors of the solutions to the "dual equation" are

$$y^{(0)} \sim e^{-iEt+\frac{1}{2}\omega t^2}(-\omega)^{\frac{1}{4}-\frac{1}{4\omega}}t^{-\frac{1}{2\omega}}\begin{bmatrix} \frac{1}{2i\omega t} \\ 1\end{bmatrix}, \tag{B4}$$

$$y^{(1)} \sim e^{-iEt-\frac{1}{2}\omega t^2}\omega^{-\frac{1}{4}+\frac{1}{4\omega}}t^{\frac{1}{2\omega}-1}\begin{bmatrix} -2i\omega t \\ 1\end{bmatrix}. \tag{B5}$$

The large-time behaviors of $y^{(i)}$ imply that one of the eigenvalues of $\eta$ will tend to vanish. This implies that the dilation will fail when $t$ is sufficiently large. In fact, using Eqs. (B4) and (B5), we see that

$$y_{\uparrow\downarrow}^{(0)} \to \infty, \quad y_{\uparrow\downarrow}^{(1)} \to 0, \quad y_\downarrow^{(0)} \sim 2i\omega t y_\uparrow^{(0)}, \quad y_\downarrow^{(1)} \sim -\frac{1}{2i\omega t}y_\uparrow^{(1)}.$$

Thus, we can further obtain

$$l = D_0^2\|y^{(0)}\|^2 + D_1^2\|y^{(1)}\|^2 \sim 4\omega^2 t^2 |D_0 y_\uparrow^{(0)}|^2 \tag{B6}$$

and

$$\Delta = y_\uparrow^{(0)} y_\downarrow^{(1)} - y_\uparrow^{(1)} y_\downarrow^{(0)} \sim -2i\omega t y_\uparrow^{(0)} y_\uparrow^{(1)}. \tag{B7}$$

Plugging Eqs. (B6) and (B7) into Eq. (34), we get the large-time behaviors of the two eigenvalues of $\eta$ in Eqs. (40) and (41).

## APPENDIX C: THE FORM OF $\tau(t)$

In the discussion of dilation problem, $\tau = \sqrt{\eta - \mathbb{1}}$ is semipositive definite. Direct calculation from Eqs. (35) and (37) leads to

$$X = 2ad, \tag{C1}$$

$$Y = 2bd, \tag{C2}$$

$$Z = 2cd, \tag{C3}$$

$$W = a^2 + b^2 + c^2 + d^2. \tag{C4}$$

By substituting Eqs. (C1)–(C3) into Eq. (C4), we have

$$d^2 = \tfrac{1}{2}(W \pm \sqrt{W^2 - X^2 - Y^2 - Z^2}). \tag{C5}$$

Since $\tau$ is semipositive definite, its trace and determinant are both non-negative. This means that $d \geqslant 0$ and $d^2 \geqslant a^2 + b^2 + c^2$. Combining these with Eq. (C4), we know $d^2 \geqslant \frac{1}{2}W \geqslant 0$. Thus, we have Eq. (36) in the main text.


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