Simulating Broken $\mathcal{PT}$-Symmetric Hamiltonian Systems by Weak Measurement

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(Received 27 September 2018; published 23 August 2019)

By embedding a $\mathcal{PT}$-symmetric (pseudo-Hermitian) system into a large Hermitian one, we disclose the relations between $\mathcal{PT}$-symmetric quantum theory and weak measurement theory. We show that the weak measurement can give rise to the inner product structure of $\mathcal{PT}$-symmetric systems, with the preselected state and its postselected state resident in the dilated conventional system. Typically in quantum information theory, by projecting out the irrelevant degrees and projecting onto the subspace, even local broken $\mathcal{PT}$-symmetric Hamiltonian systems can be effectively simulated by this weak measurement paradigm.

DOI: 10.1103/PhysRevLett.123.080404

Introduction.—Generalizing the conventional Hermitian quantum mechanics, Bender and his colleagues established the parity-time ($\mathcal{PT}$) symmetric quantum mechanics in 1998 [1]. With the additional degree of freedom from a nonconservative Hamiltonian, as well as the existence of exceptional points between unbroken and broken $\mathcal{PT}$ symmetries, optical $\mathcal{PT}$-symmetric devices have been demonstrated with many useful applications [2–7]. Although calling for more caution on physical interpretations, especially on the consistency problem of local $\mathcal{PT}$-symmetric operation and the no-signaling principle [8], $\mathcal{PT}$-symmetric quantum mechanics has been stimulating our understanding on many interesting problems such as spectral equivalence [9], quantum brachistochrone [10] and the Riemann hypothesis [11].

Compared with the Dirac inner product in conventional quantum mechanics, $\mathcal{PT}$-symmetric quantum theory can be well manifested by the η-inner product [12,13]. In the broken $\mathcal{PT}$-symmetry case, the η-inner product of a state with itself can be negative, which makes the broken $\mathcal{PT}$-symmetric quantum system a complete departure from conventional quantum mechanics. While in the unbroken $\mathcal{PT}$-symmetry case, the η-inner product presents a completely analogous physical interpretation to the Dirac inner product, giving rise to many similar properties between $\mathcal{PT}$-symmetric and conventional quantum mechanics. Recent works also show that the η-inner product is tightly related to the properties of superposition and coherence in conventional quantum mechanics [14].

Despite the original motivation to build a new framework of quantum theory, researchers are aware of the importance of simulating $\mathcal{PT}$-symmetric systems with conventional quantum mechanics. It will help explore the properties and physical meaning of $\mathcal{PT}$-symmetric quantum systems. On this issue, one should answer the question in what sense a quantum system can be viewed as $\mathcal{PT}$ symmetric. One approach, initialized by Günther and Samsonov, is to embed unbroken $\mathcal{PT}$-symmetric Hamiltonians into higher dimensional Hermitian Hamiltonians [15–17]. By dilating the system to a large Hermitian one and projecting out the ancillary system, this paradigm successfully simulates the evolution of unbroken $\mathcal{PT}$-symmetric Hamiltonians. Such a way, inspired by Naimark dilation and typical ideas in quantum simulation, endows direct physical meaning of $\mathcal{PT}$-symmetric quantum systems in the sense of open systems. However, the simulation of broken $\mathcal{PT}$-symmetric systems is still in suspense, due to its essential distinctions with conventional quantum systems.

In this Letter, we illustrate the simulation for broken $\mathcal{PT}$-symmetric systems based on weak measurement [18]. For a system weakly coupled to the apparatus, the pointer state will be shifted by the weak value when a weak measurement is performed. The weak value, tightly related to the nonclassical features of quantum mechanics, such as the Hardy's paradox [19], three box paradox [20], and Leggett-Garg inequalities [21], can take values beyond the expected values of an observable, and even be a complex number. The weak measurement theory has provided new ways to measure geometric phases [22–25] and non-Hermitian systems [26,27], as well as to amplify signals as a sensitive estimation of small evolution parameters [28–30]. Our aim is to propose a concrete scenario in which the quantum system can be viewed as $\mathcal{PT}$ symmetric by utilizing the weak measurement. Our result reveals the connections between $\mathcal{PT}$ symmetry and the weak measurement theory, providing the important missing point for the simulation problem of broken $\mathcal{PT}$-symmetric quantum systems.

Generalized embedding of $\mathcal{PT}$-symmetric systems.—Consider n-dimensional discrete quantum systems, on which operators are denoted by calligraphy letters $\mathcal{P}$, $\mathcal{T}$, $\mathcal{H}$, while their matrices by capital letters P, T, H. A linear operator $\mathcal{P}$ is said to be a parity operator if $P^2 = I$, where I denotes the $n\times n$ identity matrix. An antilinear operator $\mathcal{T}$ is said to be a time reversal operator if $T\bar{T} = I$ and $PT = T\bar{P}$, where $\bar{T}$ ($\bar{P}$) stands for the complex conjugation of T (P). A Hamiltonian $\mathcal{H}$ (H) is said to be $\mathcal{PT}$ (PT) symmetric if $HPT = PT\bar{H}$ [31]. $\mathcal{H}$ is called unbroken $\mathcal{PT}$ symmetric if it is diagonalizable and all of its eigenvalues are real. Otherwise, $\mathcal{H}$ is called broken $\mathcal{PT}$ symmetric.

In quantum mechanics, a Hamiltonian H gives rise to a unitary evolution of the system. Let ϕ_1 and ϕ_2 be two states. One can introduce a Hermitian operator η to define the η-inner product by $\langle\phi_1|\phi_2\rangle_\eta = \langle\phi_1|\eta|\phi_2\rangle$. With respect to the η-inner product, H presents a unitary evolution if and only if $H^\dagger\eta = \eta H$ [12,13,32–34], where $H^\dagger$ denotes the conjugation and transpose of H. Here, η is said to be the metric operator of H. Moreover, for a generic $\mathcal{PT}$-symmetric operator H and its metric operator η, there always exists some matrix Ψ' such that $\Psi'^{-1}H\Psi' = J$ and $\Psi'^\dagger\eta\Psi' = S$, where

$$J = \mathrm{diag}[J_{n_1}(\lambda_1,\bar{\lambda}_1),\ldots,J_{n_p}(\lambda_p,\bar{\lambda}_p),J_{n_{p+1}}(\lambda_{p+1}),\ldots,J_r(\lambda_r)], \tag{1}$$

$J_{n_k}(\lambda_k,\bar{\lambda}_k) = \begin{pmatrix} J_{n_k}(\lambda_k) & 0 \\ 0 & J_{n_k}(\bar{\lambda}_k)\end{pmatrix}$, $J_{n_j}(\lambda_j)$ are the Jordan blocks, $\lambda_1,\ldots,\lambda_p$ are complex numbers, and $\lambda_{p+1},\ldots,\lambda_r$ are real numbers,

$$S = \mathrm{diag}(S_{2n_1},\ldots,S_{2n_p},\epsilon_{n_{p+1}}S_{n_{p+1}},\ldots,\epsilon_{n_r}S_{n_r}), \tag{2}$$

n_i denote the orders of Jordan blocks in Eq. (1), i.e.,

$$S_k = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}_{k\times k}$$

and $\epsilon_i = \pm 1$ is uniquely determined by η [16,35]. For convenience, we only consider the situations in which $\epsilon_i = 1$. In this case, S is a permutation matrix and $S^2 = I$. Note that S can be equal to I if and only if H is unbroken $\mathcal{PT}$ symmetric [16]. Henceforth we always assume $S = I$ in the unbroken case. The following theorem gives an important property of $\mathcal{PT}$-symmetric Hamiltonians.

Theorem 1.—Let H be an $n\times n$ $\mathcal{PT}$-symmetric matrix and η be the metric matrix of H. Let J and S be matrices in Eqs. (1) and (2). Then, there exist $n\times n$ invertible matrices Ψ, Ξ, Σ, and a $2n\times 2n$ Hermitian matrix $\tilde{H}$ such that for $\tilde{\Psi} = \binom{\Psi}{\Xi}$ and $\tilde{\Phi} = \binom{\Psi}{\Sigma}$, the following equations hold,

$$\tilde{\Phi}^\dagger\tilde{\Psi} = S, \qquad \tilde{\Phi}^\dagger\tilde{H}\tilde{\Psi} = SJ. \tag{3}$$

Proof.—As was discussed, there exist a matrix Ψ' such that $\Psi'^{-1}H\Psi' = J$ and $\Psi'^\dagger\eta\Psi' = S$ [16,35]. Since $\Psi'^\dagger\Psi' > 0$, there always exists a positive number c such that $c^2\Psi'^\dagger\Psi' > I$. Set $\Psi = c\Psi'$. Since $\Psi^\dagger\Psi > I \geq S$, $\Psi^\dagger\Psi - S$ is invertible.

Let Ξ be an $n\times n$ invertible matrix. Taking $\Sigma = (\Xi^{-1})^\dagger(S - \Psi^\dagger\Psi)$, $\eta = (\Psi^{-1})^\dagger S\Psi^{-1}$, $H_1 = \eta H$, $H_2 = (\Psi^\dagger)^{-1}(\Xi)^\dagger$ and $H_4 = -H_2^\dagger\Psi\Xi^{-1} - (\Sigma^\dagger)^{-1}\Psi^\dagger H_2$, one can directly verify that

$$\tilde{H} = \begin{pmatrix} H_1 & H_2 \\ H_2^\dagger & H_4 \end{pmatrix}$$

is Hermitian and Eq. (3) holds. ∎

Theorem 1 actually gives out the inner product structure of H in a subspace. Note that the matrix Ψ in Theorem 1 can be written as $\Psi = (|\psi_1\rangle,\ldots,|\psi_n\rangle)$, where the column vectors $\{|\psi_i\rangle\}$ form a linear basis of $\mathbb{C}^n$. Similarly, $\Xi = (|\xi_1\rangle,\ldots,|\xi_n\rangle)$ and $\Sigma = (|\sigma_1\rangle,\ldots,|\sigma_n\rangle)$. Correspondingly we have $\tilde{\Psi} = (|\tilde{\psi}_1\rangle,\ldots,|\tilde{\psi}_n\rangle)$ and $\tilde{\Phi} = (|\tilde{\phi}_1\rangle,\ldots,|\tilde{\phi}_n\rangle)$, where $|\tilde{\psi}_i\rangle = \binom{|\psi_i\rangle}{|\xi_i\rangle}$ and $|\tilde{\phi}_i\rangle = \binom{|\psi_i\rangle}{|\sigma_i\rangle}$. Moreover, $\tilde{\Phi}S = (|\tilde{\mu}_1\rangle,\ldots,|\tilde{\mu}_n\rangle) = (|\tilde{\phi}_{s(1)}\rangle,\ldots,|\tilde{\phi}_{s(n)}\rangle)$, where S is the permutation matrix in Theorem 1, and s is the permutation induced by S. Similarly, we can write $\Psi S = (|\mu_1\rangle,\ldots,|\mu_n\rangle)$, where $|\mu_i\rangle = |\psi_{s(i)}\rangle$. From the definition of $|\tilde{\mu}_i\rangle$, we have $\langle\tilde{\mu}_i|\tilde{\psi}_j\rangle = (S\tilde{\Phi}^\dagger\tilde{\Psi})_{ij}$ and $\langle\tilde{\mu}_i|\tilde{H}|\tilde{\psi}_j\rangle = (S\tilde{\Phi}^\dagger\tilde{H}\tilde{\Psi})_{ij}$. According to Eq. (3), we have

$$\langle\tilde{\mu}_i|\tilde{\psi}_j\rangle = \delta_{i,j}, \qquad \langle\tilde{\mu}_i|\tilde{H}|\tilde{\psi}_j\rangle = J_{i,j}, \tag{4}$$

where $J_{i,j}$ is the (i,j)th entry of J.

On the other hand, note that the metric matrix η of H is $(\Psi^\dagger)^{-1}S\Psi^{-1}$. Thus we have the following relations between the Dirac and η-inner products:

$$\langle\tilde{\mu}_i|\tilde{\psi}_j\rangle = \langle\mu_i|\psi_j\rangle_\eta, \tag{5}$$

$$\langle\tilde{\mu}_i|\tilde{H}|\tilde{\psi}_j\rangle = \langle\mu_i|H|\psi_j\rangle_\eta, \tag{6}$$

where $\langle\mu_i|H|\psi_j\rangle_\eta = \langle\mu_i|\eta H|\psi_j\rangle$. The results show that there exist two different bases with the same projections onto the subspace of the $\mathcal{PT}$-symmetric system, with respect to the η-inner product. When confined to the subspace, the Hermitian Hamiltonian $\tilde{H}$ in large space has the same effect as a $\mathcal{PT}$-symmetric Hamiltonian H, in the sense of this η-inner product.

Simulation of $\mathcal{PT}$-symmetric Hamiltonian systems.—To infer a quantum system is $\mathcal{PT}$ symmetric, it is sufficient to identify the Hamiltonian and its inner product structure. In the weak measurement formalism, one starts by preselecting an initial state $|\varphi_i\rangle$. The target system is coupled to the measurement apparatus, which is in a pointer state $|P\rangle$. Usually, $|P\rangle = (2\pi\Delta^2)^{-\frac{1}{4}}\exp(-Q^2/4\Delta^2)$, a Gaussian state with Δ its standard deviation. Let A be an observable of the system and M be that of the apparatus, conjugate to Q [18]. The interaction Hamiltonian between the system and apparatus is $H_{\mathrm{int}} = f(t)A \otimes M$, with interaction strength $g = \int f(t)dt$. The state evolves as $|\varphi_i\rangle \otimes |P\rangle \to e^{-igA\otimes M}|\varphi_i\rangle \otimes |P\rangle$. Now if the system satisfies the weak condition that g/Δ is sufficiently small, then for a postselected state $|\varphi_f\rangle$ that $\langle\varphi_f|\varphi_i\rangle \neq 0$, one has $\langle\varphi_f|e^{-igA\otimes M}|\varphi_i\rangle|P\rangle \approx \langle\varphi_f|\varphi_i\rangle e^{-ig\langle A\rangle_w M}|P\rangle = \langle\varphi_f|\varphi_i\rangle(2\pi\Delta^2)^{-\frac{1}{4}}\exp[-(Q-g\langle A\rangle_w)^2/4\Delta^2]$, where $\langle A\rangle_w = (\langle\varphi_f|A|\varphi_i\rangle/\langle\varphi_f|\varphi_i\rangle)$ is called the weak value. That is, the state is shifted by $g\langle A\rangle_w$. Thus the weak value $\langle A\rangle_w$ can be read out experimentally, as a generalization of the eigenvalues in Von Neumann measurement [36].

From Eq. (4), we have $\lambda_i = J_{i,i} = \langle\tilde{\mu}_i|\tilde{H}|\tilde{\psi}_i\rangle = (\langle\tilde{\mu}_i|\tilde{H}|\tilde{\psi}_i\rangle/\langle\tilde{\mu}_i|\tilde{\psi}_i\rangle)$. Therefore, the eigenvalues of H can be obtained via a weak measurement, by preselecting the vector $|\tilde{\psi}_i\rangle$ and postselecting the vector $|\tilde{\mu}_i\rangle$. This observation implies that one can use weak measurement to simulate the measurements on a $\mathcal{PT}$-symmetric system.

In conventional quantum mechanics, the expectation value of a Hermitian Hamiltonian $H_0 = \sum_i \lambda_i|u_i\rangle\langle u_i|$ with respect to a state $|\psi_0\rangle = \sum_i d_i|u_i\rangle$ is given by the inner product $\langle\psi_0|H_0|\psi_0\rangle$. For a $\mathcal{PT}$-symmetric Hamiltonian system with the metric matrix η, the expectation value of a Hamiltonian H with respect to a state $|u\rangle = \sum_i a_i|\psi_i\rangle$ is instead given by $\langle u|H|u\rangle_\eta$. Given two vectors $|v\rangle = \sum_i b_i|\mu_i\rangle$ and $|w\rangle = \sum_i c_i|\psi_i\rangle$ of the $\mathcal{PT}$-symmetric system. Let $|\tilde{v}\rangle = \sum_i b_i|\tilde{\mu}_i\rangle$ (unnormalized for convenience) and $|\tilde{w}\rangle = \sum_i c_i|\tilde{\psi}_i\rangle$ be two vectors in the extended system. It follows from Eq. (6) that $\langle v|H|w\rangle_\eta = \langle\tilde{v}|\tilde{H}|\tilde{w}\rangle$. Assume that $|u\rangle$ satisfies the condition $\langle u|u\rangle_\eta = \pm 1$. Now take two states $|\tilde{u}_1\rangle = \sum_i a_{s(i)}|\tilde{\mu}_i\rangle$ and $|\tilde{u}_2\rangle = \sum_i a_i|\tilde{\psi}_i\rangle$, whose projections to the $\mathcal{PT}$-symmetric subspace are both $|u\rangle$. Then we have

$$\frac{\langle u|H|u\rangle_\eta}{\langle u|u\rangle_\eta} = \frac{\langle\tilde{u}_1|\tilde{H}|\tilde{u}_2\rangle}{\langle\tilde{u}_1|\tilde{u}_2\rangle}. \tag{7}$$

Therefore, confined to the $\mathcal{PT}$-symmetric subspace, a weak measurement can completely describe the expectations of H.

In conventional quantum mechanics, when an eigenvalue is detected, the measured state collapses to the corresponding eigenstate. However, the problem in a $\mathcal{PT}$-symmetric system is subtle. According to Eq. (5), $\langle\psi_i|\psi_i\rangle_\eta \neq 0$ only if $i = s(i)$. This observation makes it reasonable to assume that for any vector $|u\rangle = \sum_i a_i|\psi_i\rangle$ satisfying $\langle u|u\rangle_\eta \neq 0$, if $a_i \neq 0$, then $a_{s(i)} \neq 0$. That is, if $\langle u|u\rangle_\eta \neq 0$, its vector components of $|\psi_i\rangle$ and $|\psi_{s(i)}\rangle$ take zero or nonzero values simultaneously, while the eigenvalues associated with ψ_i and $\psi_{s(i)}$ are either equal or complex conjugations. In this case, one can generalize the detection of an eigenvalue of λ_i in conventional quantum mechanics to the following. For $|u\rangle = \sum_i a_i|\psi_i\rangle$, if the value of

$$\frac{a_i\overline{a_{s(i)}}\lambda_i + \bar{a}_i a_{s(i)}\bar{\lambda}_i}{a_i\overline{a_{s(i)}} + \bar{a}_i a_{s(i)}}$$

is detected [37], the state $|u\rangle$ will collapse to

$$\frac{a_i|\psi_i\rangle + a_{s(i)}|\psi_{s(i)}\rangle}{|a_i\overline{a_{s(i)}} + a_{s(i)}\bar{a}_i|^{\frac{1}{2}}}.$$

Apparently, when $i = s(i)$, the state $|u\rangle$ will collapse to $|\psi_i\rangle$, similar to the case of conventional quantum mechanics. Note that $i = s(i)$ only if the system is unbroken $\mathcal{PT}$ symmetric, for which it is analogous to conventional quantum mechanics and such an analogy in state collapse is not unexpected.

By pre- and postselecting the states, we see that the weak measurements can successfully simulate an arbitrary η-inner product. Furthermore, when confined to the subspace, the measurement results actually extract the same information as a $\mathcal{PT}$-symmetric Hamiltonian system. Such information helps us eventually infer that the subsystem is $\mathcal{PT}$ symmetric.

Discussions and conclusion.—We further discuss the mechanism and physical implications related to the weak measurement paradigm, by comparing it with the embedding paradigm [15,16]. The essence of the embedding paradigm is to realize the evolution of a $\mathcal{PT}$-symmetric Hamiltonian, by evolving the state under the Hermitian Hamiltonian in the large space and then projecting it to the subspace. The key to this paradigm can be mathematically described as follows [16]: For a given $n \times n$ unbroken $\mathcal{PT}$-symmetric Hamiltonian H, find a $2n \times 2n$ Hermitian matrix $\tilde{H}$, $n \times n$ invertible matrices Ψ, Ξ so that $\tilde{\Psi}^\dagger\tilde{\Psi} = I$ and the following equations,

$$e^{-it\tilde{H}}\tilde{\Psi} = \tilde{\Psi}e^{-itJ}, \qquad e^{-itH}\Psi = \Psi e^{-itJ} \tag{8}$$

hold, where $\tilde{\Psi} = \binom{\Psi}{\Xi}$. The equations are actually equivalent to the following conditions [38]:

$$\tilde{\Psi}^\dagger\tilde{\Psi} = I, \qquad \tilde{H}\tilde{\Psi} = \tilde{\Psi}J, \qquad H\Psi = \Psi J. \tag{9}$$

Equation (8) ensures that the unitary evolution $\tilde{U}(t)$ gives the evolution $U(t)$ of a $\mathcal{PT}$-symmetric Hamiltonian H in a subspace. In this sense, the embedding paradigm gives a natural way of simulation. Nevertheless, in the broken $\mathcal{PT}$-symmetric case, the solutions do not exist [16]. In fact, Eq. (3) is mathematically a generalization of Eq. (9) [39]. Like the case of the embedding paradigm, it is natural to further require that $\tilde{\Phi}^\dagger e^{-it\tilde{H}}\tilde{\Psi} = Se^{-itJ}$, so that $e^{-it\tilde{H}}$ gives the same effect as e^{-itH} in the subspace. However, such a requirement cannot be satisfied for broken $\mathcal{PT}$ symmetry, which is obvious from the unboundedness of Se^{-itJ}.

However, consider sufficiently small time $t \in [0, \epsilon]$. We have $|\tilde{u}(t)\rangle = e^{-it\tilde{H}}|\tilde{u}\rangle \approx (I - it\tilde{H})|\tilde{u}\rangle$. On the other hand, $|u(t)\rangle = e^{-itH}|u\rangle \approx (I - itH)|u\rangle$. Now Eqs. (5) and (6) ensure that when confined to the subspace, $|\tilde{u}(t)\rangle$ is equivalent to $|u(t)\rangle$ in the sense of η-inner product [40]. This observation implies that $\mathcal{PT}$-symmetric quantum systems can be well approximated in a sufficiently small time evolution, by choosing two different sets of bases $\{|\tilde{\phi}_i\rangle\}$ and $\{|\tilde{\psi}_i\rangle\}$ with the same components in the subspace, which can be realized by weak measurement. Here instead of the small time interval, the weak condition that g/Δ is sufficiently small ensures the approximation. The weak measurement paradigm can be viewed as a generalization of the embedding paradigm, due to the fact that Eq. (9) is a special case of Eq. (3) in the $\mathcal{PT}$-symmetric unbroken case. Hence, the Hamiltonian $\tilde{H}$ in the embedding paradigm can also be utilized in the weak measurement approach, although the embedding paradigm itself does not work. Comparing our approach with that in Ref. [26], where one obtains the expected value of a Hamiltonian in the Dirac inner product by using the polar decomposition, our method lays emphasis on the properties of a $\mathcal{PT}$-symmetric Hamiltonian with respect to the η-inner product.

In summary, we have proposed a weak measurement paradigm to investigate the behaviors of broken $\mathcal{PT}$-symmetric Hamiltonian systems. By embedding the $\mathcal{PT}$-symmetric system into a large Hermitian system and utilizing weak measurements, we have shown how a $\mathcal{PT}$-symmetric Hamiltonian can be simulated. Our paradigm may shine new light on the study of $\mathcal{PT}$-symmetric quantum mechanics and its physical implications and applications.

This work is supported by National Natural Science Foundation of China (11171301, 11571307, 11690032, 61490711, 61877054 and 11675113), National Key R&D Program of China under Grant No. 2018 YFA0306202, Beijing Municipal Commission of Education (KZ201810028042), and Ministry of Science and Technology of Taiwan (105-2628-M-007-003-MY4).

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[1] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998).
[2] R. El-Ganainy, K. Makris, D. Christodoulides, and Z. H. Musslimani, Opt. Lett. **32**, 2632 (2007).
[3] K. G. Makris, R. El-Ganainy, D. N. Christodoulides, and Z. H. Musslimani, Phys. Rev. Lett. **100**, 103904 (2008).
[4] A. Guo, G. J. Salamo, D. Duchesne, R. Morandotti, M. Volatier-Ravat, V. Aimez, G. A. Siviloglou, and D. N. Christodoulides, Phys. Rev. Lett. **103**, 093902 (2009).
[5] C. E. Ruter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip, Nat. Phys. **6**, 192 (2010).
[6] L. Chang, X. Jiang, S. Hua, C. Yang, J. Wen, L. Jiang, G. Li, G. Wang, and M. Xiao, Nat. Photonics **8**, 524 (2014).
[7] J.-S. Tang, Y.-T. Wang, S. Yu, D.-Y. He, J.-S. Xu, B.-H. Liu, G. Chen, Y.-N. Sun, K. Sun, Y.-J. Han, C.-F. Li, and G.-C. Guo, Nat. Photonics **10**, 642 (2016).
[8] Y.-C. Lee, M.-H. Hsieh, S. T. Flammia, and R.-K. Lee, Phys. Rev. Lett. **112**, 130404 (2014).
[9] P. Dorey, C. Dunning, and R. Tateo, J. Phys. A **34**, 5679 (2001); **40**, R205 (2007).
[10] U. Günther and B. F. Samsonov, Phys. Rev. A **78**, 042115 (2008).
[11] C. M. Bender, D. C. Brody, and M. P. Muller, Phys. Rev. Lett. **118**, 130201 (2017).
[12] A. Mostafazadeh, J. Math. Phys. (N.Y.) **43**, 205, 2814, and 3944 (2002).
[13] A. Mostafazadeh, Int. J. Geom. Methods Mod. Phys. **07**, 1191 (2010).
[14] M. Huang, R.-K. Lee, and J. Wu, J. Phys. A **51**, 414004 (2018).
[15] U. Günther and B. F. Samsonov, Phys. Rev. Lett. **101**, 230404 (2008).
[16] M. Huang, A. Kumar, and J. Wu, Phys. Lett. A **382**, 2578 (2018).
[17] K. Kawabata, Y. Ashida, and M. Ueda, Phys. Rev. Lett. **119**, 190401 (2017).
[18] Y. Aharonov, D. Z. Albert, and L. Vaidman, Phys. Rev. Lett. **60**, 1351 (1988).
[19] Y. Aharonov, A. Botero, S. Popescu, B. Reznik, and J. Tollaksen, Phys. Lett. A **301**, 130 (2002).
[20] K. J. Resch, J. S. Lundeen, and A. M. Steinberg, Phys. Lett. A **324**, 125 (2004).
[21] A. Palacios-Laloy, F. Mallet, F. Nguyen, P. Bertet, D. Vion, D. Esteve, and A. N. Korotkov, Nat. Phys. **6**, 442 (2010).
[22] O. Hosten and P. Kwiat, Science **319**, 787 (2008).
[23] E. Sjoqvist, Phys. Lett. A **359**, 187 (2006).
[24] H. Kobayashi, S. Tamate, T. Nakanishi, K. Sugiyama, and M. Kitano, Phys. Rev. A **81**, 012104 (2010).
[25] L. Zhang, A. Datta, and I. A. Walmsley, Phys. Rev. Lett. **114**, 210801 (2015).
[26] A. K. Pati, U. Singh, and U. Sinha, Phys. Rev. A **92**, 052120 (2015).
[27] L. Vaidman, A. Ben-Israel, J. Dziewior, L. Knips, M. Weißl, J. Meinecke, C. Schwemmer, R. Ber, and H. Weinfurter, Phys. Rev. A **96**, 032114 (2017).
[28] J. S. Lundeen, B. Sutherland, A. Patel, C. Stewart, and C. Bamber, Nature (London) **474**, 188 (2011).
[29] D. J. Starling, P. B. Dixon, A. N. Jordan, and J. C. Howell, Phys. Rev. A **82**, 063822 (2010).
[30] N. Brunner and C. Simon, Phys. Rev. Lett. **105**, 010405 (2010).
[31] In this paper $\mathcal{PT}$ symmetry is the synonym of pseudo-Hermitian, between which we will not distinguish.
[32] J.-w. Deng, U. Guenther, and Q.-h. Wang, arXiv:1212.1861.
[33] P. D. Mannheim, Phil. Trans. R. Soc. A **371**, 20120060 (2013).
[34] R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University, New York, 2013).

[35] I. Gohberg, P. Lancaster, and L. Rodman, *Matrices and Indefinite Scalar Products* (Birkhäuser Verlag, Basel, 1983), Vol. 8.

[36] J. Dressel, M. Malik, F. M. Miatto, A. N. Jordan, and R. W. Boyd, Rev. Mod. Phys. **86**, 307 (2014).

[37] Actually, this value is $(\langle a_i\psi_i + a_{s(i)}\psi_{s(i)}|H|a_i\psi_i + a_{s(i)}\psi_{s(i)}\rangle_\eta / a_i\overline{a_{s(i)}} + \bar{a}_i a_{s(i)})$, which reduces to $\langle\psi_i|H|\psi_i\rangle_\eta = \lambda_i$ if $i = s(i)$ (only one vector ψ_i considered).

[38] Equation (8) is actually the matrix version of the embedding in [16]. Denote $\Xi = \tau\Psi$. Then (9) reduces to $H_1 + H_2\tau = H$ and $H_2^\dagger + H_4\tau = \tau H$, which gives the equivalent description of the embedding property. A concrete solution to (8) can also be found in [17].

[39] When unbroken $\mathcal{PT}$ symmetric, it is always possible to take $\Xi = \Sigma$ and $S = I$, Eq. (9) is a special case of Eq. (3).

[40] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.123.080404 for the brief discussion with an example, which also includes Ref. [41].

[41] C. M. Bender, D. C. Brody, H. F. Jones, and B. K. Meister, Phys. Rev. Lett. **98**, 040403 (2007).