Note for *Quantum Optics*: Quantum SHO

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[Reference:]

- Chapter I, in C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *"Photons & Atoms"*, John Wiley & Sons (1989).
- Chapter 2, in J. J. Sakurai, *"Modern Quantum Mechanics,"* Addison Wesley (1994).
- Chapter 7, in A. Goswami, *"Quantum Mechanics,"* WCB Publishers (1992).

I. CLASSICAL SPRING-MASS SYSTEM

The motion of a spring, with the mass m and the Hooke's constant k, can be described by the Newton's second law,

$$\vec{F} = m\vec{a} = m\frac{d^2\,\vec{x}}{dt^2} = -k\,\vec{x}. \tag{1}$$

The corresponding solution is

$$\vec{x}(t) = A\,\cos(\omega_0 t + \phi_0), \qquad \omega_0^2 = \frac{k}{m}, \tag{2}$$

with the kinetic energy (K.E.) and potential energy (P.E.):

$$\text{K.E.} : T = \frac{1}{2}m\,v^2 = \frac{1}{2}k\,A^2\,\sin^2(\omega_0 t + \phi_0), \tag{3}$$

$$\text{P.E.} : U = -\int \vec{F}\cdot d\vec{x} = \frac{1}{2}k\,x^2 = \frac{1}{2}k\,A^2\,\cos^2(\omega_0 t + \phi_0). \tag{4}$$

The total energy, $E = \text{K.E.} + \text{P.E.}$, of a simple harmonic oscillator (SHO) is a constant, that is

$$E = \frac{1}{2}k\,A^2 = \text{constant}. \tag{5}$$

The Hamiltonian of a SHO is

$$H = \text{K.E.} + \text{P.E.} = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}k\,x^2. \tag{6}$$

II. QUANTUM SHO

By introducing the position and momentum operators, $\hat{x}$ and $\hat{p}$, respectively, we have the Hamiltonian for a quantum SHO,

$$\hat{H} = \frac{1}{2}\frac{\hat{p}^2}{m} + \frac{1}{2}k\,\hat{x}^2, \tag{7}$$

where the $\hat{x}$ and $\hat{p}$ are non-commute operators, *i.e.*,

$$[\hat{x}, \hat{p}] = i\hbar. \tag{8}$$

A. Momentum operator

If one take the momentum as a generator of translation, *i.e.*,

$$p \equiv m\,v = m\frac{dx}{dt},$$

and for an *infinitesimal* translation, we have

$$\hat{\mathcal{T}}(dx)|x\rangle = |x + dx\rangle,$$

where $\hat{T}$ denotes the operator, and $|x\rangle$ is the eigen-state vector in the x-representation, with the eigen-value x, *i.e.*,

$$\hat{x}|x\rangle = x\,|x\rangle.$$

For a generator of translation, we have

$$\hat{\mathcal{T}}(dx)|x\rangle = |x + dx\rangle. \tag{9}$$

Following 4 properties are required to be satisfied for the generator of translation as a linear operator:

1. Unitary:

$$\langle x'|x'\rangle = \langle x|\hat{\mathcal{T}}^\dagger(dx)\hat{\mathcal{T}}(dx)|x\rangle = \langle x + dx|x + dx\rangle, \tag{10}$$

   where $\langle x|$ and $|x\rangle$ correspond to the *bra* and *ket* states in Dirac's notation, respectively. $\hat{\mathcal{T}}^\dagger$ denotes the *adjoint* of the operator $\hat{\mathcal{T}}$. An operator is *hermitian* if it is self-adjoint, *i.e.*,

$$\hat{O} = \hat{O}^\dagger.$$

2. Addition: The operations are additive, *i.e.*,

$$\hat{\mathcal{T}}(dx_1)\hat{\mathcal{T}}(dx_2)|x\rangle = \hat{\mathcal{T}}(dx_1 + dx_2)|x\rangle. \tag{11}$$

3. Inverse: There should be an inverse operator, denoted as $\hat{\mathcal{T}}^{-1}$ to satisfy

$$\hat{\mathcal{T}}^{-1}(dx) = \hat{\mathcal{T}}(-dx). \tag{12}$$

4. Identity: As $dx \to 0$, we should have

$$\lim_{dx\to 0}\hat{\mathcal{T}}(dx) = \hat{\mathcal{I}}. \tag{13}$$

To satisfy these four properties, we assume the generator of translation has the form:

$$\hat{\mathcal{T}}(dx) = \hat{\mathcal{I}} - i\hat{k}\cdot dx = \hat{\mathcal{I}} - i\frac{\hat{p}}{\hbar}\cdot dx, \tag{14}$$

where *de Broglie*'s relation is used $\hbar\hat{k} = \hat{p}$ and $k = 2\pi/\lambda$. Note that

$$\hat{x}\hat{\mathcal{T}}(dx')|x'\rangle = (x' + dx')|x' + dx'\rangle, \tag{15}$$
$$\hat{\mathcal{T}}(dx')\hat{x}|x'\rangle = x'|x' + dx'\rangle, \tag{16}$$

hence, we have the commutator $[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$ as

$$[\hat{x}, \hat{\mathcal{T}}(dx')] = dx', \tag{17}$$

or

$$[\hat{x}, \hat{p}] = i\hbar. \tag{18}$$

By extending to a general form, we have the commutation relation for the position and momentum operators,

$$[\hat{x}_i, \hat{p}_j] = i\hbar\,\delta_{ij}. \tag{19}$$

B. x- and p-representations

We use $|x\rangle$ and $|p\rangle$ for the states representations in the position and momentum spaces, respectively,

$$\hat{x}|x\rangle = x|x\rangle, \tag{20}$$
$$\hat{p}|p\rangle = p|p\rangle. \tag{21}$$

The transformation between these two spaces is

$$\hat{p}|x\rangle = -i\hbar\frac{\partial}{\partial x}|x\rangle, \tag{22}$$
$$\hat{x}|p\rangle = i\hbar\frac{\partial}{\partial p}|p\rangle, \tag{23}$$
$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp(\frac{i\,p\,x}{\hbar}). \tag{24}$$

There is a *Fourier transform* between two representations:

$$\psi(x) = \langle x|\phi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int dp\,\exp(\frac{i\,p\,x}{\hbar})\Psi(p), \tag{25}$$
$$\Psi(p) = \langle p|\phi\rangle = \frac{1}{\sqrt{2\pi\hbar}}\int dx\,\exp(\frac{-i\,p\,x}{\hbar})\psi(x). \tag{26}$$

C. Schrödinger's picture

The equation of motion of a quantum state is described by the Schrödinger's equation:

$$i\hbar\frac{\partial}{\partial t}|\phi\rangle = \hat{H}|\phi\rangle. \tag{27}$$

For the eigen-state of SHO Hamiltonian,

$$\hat{H} = \frac{1}{2}\frac{\hat{p}^2}{m} + \frac{1}{2}k\,\hat{x}^2, \tag{28}$$

we have

$$[\frac{d^2}{dx^2} + \frac{2m}{\hbar^2}(E - \frac{1}{2}kx^2)]\psi(x) = 0, \tag{29}$$

where $\Phi(x,t) = \langle x|\phi\rangle \exp(-iEt/\hbar) = \psi(x)\exp(-iEt/\hbar)$ denotes the state in the x-representation. By defining the new variable $\xi = \sqrt{\frac{m\omega}{\hbar}}\, x$, we have the *Hermite equation*,

$$[\frac{d^2}{d\xi^2} + (\epsilon - \xi^2)]\psi(\xi) = 0, \qquad \epsilon = \frac{2E}{\hbar\omega}, \tag{30}$$

which gives the *Hermite-Gaussian* solutions associated with Hermite polynomials H_n

$$\psi(\xi) = H_n(\xi)\exp[-\xi^2/2], \qquad \epsilon = 2n+1, \qquad n = 0,1,2,3\ldots \tag{31}$$

For the corresponding eigen-energy:

$$E = \frac{\hbar\omega}{2}\,\epsilon = \hbar\omega(n+\frac{1}{2}), \qquad n = 0,1,2,3,\ldots. \tag{32}$$

The *ground state* of a SHO is denoted as $|0\rangle$, which in the x-representation is

$$\Psi(x) = \langle x|0\rangle = c_0\exp[-\xi^2/2], \qquad \text{where } c_0 \text{ is a normalization constant}, \tag{33}$$

and the eigen-energy of the ground state is $E_0 = \hbar\omega/2$.

D. Heisenburg's picture

In terms of the operators, $[\hat{x},\hat{p}] = i\hbar$, we can introduce the *creation* ($\hat{a}^\dagger$)and *annihilation* ($\hat{a}$) operators, respectively,

$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega}}[m\omega\hat{x} + i\hat{p}], \tag{34}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}[m\omega\hat{x} - i\hat{p}]. \tag{35}$$

Or

$$\hat{x} = \frac{\hbar}{\sqrt{2m\hbar\omega}}(\hat{a} + \hat{a}^\dagger), \tag{36}$$

$$\hat{p} = \frac{\sqrt{2m\hbar\omega}}{2i}(\hat{a} - \hat{a}^\dagger). \tag{37}$$

The commutation relation for the creation and annihilation operators becomes

$$[\hat{a},\hat{a}^\dagger] = 1, \tag{38}$$

and the associate SHO Hamiltonian is

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}). \tag{39}$$

E. Properties of the creation and annihilation operators

1. $\hat{a}$ and $\hat{a}^\dagger$ are *NOT* Hermitian operators. That is

   $$\hat{a} \neq \hat{a}^\dagger,$$

   for which no real eigenvalue are generated.

2. Number operator: An Hermitian operator can be defined as

   $$\hat{N} \equiv \hat{a}^\dagger\hat{a}. \tag{40}$$

3. The commutation relations with the SHO Hamiltonian are:

$$[\hat{H}, \hat{a}] = -\hbar\omega\hat{a}, \tag{41}$$
$$[\hat{H}, \hat{a}^\dagger] = \hbar\omega\hat{a}^\dagger, \tag{42}$$
$$\tag{43}$$

4. Raising (step-up) and Lowering (step-down) operators: Consider the eigen-state of SHO Hamiltonian, $\hat{H}|\phi\rangle = E|\phi\rangle$, then

$$\hat{H}\,\hat{a}|\phi\rangle = (E - \hbar\omega)\hat{a}|\phi\rangle, \tag{44}$$
$$\hat{H}\,\hat{a}^\dagger|\phi\rangle = (E + \hbar\omega)\hat{a}^\dagger|\phi\rangle, \tag{45}$$

where $\hat{a}|\phi\rangle$ and $\hat{a}^\dagger|\phi\rangle$ are also eigen-states of SHO, but with the eigen-values $E - \hbar\omega$ and $E + \hbar\omega$, respectively. $|\phi\rangle$ is a *discrete* states, denoted as $|n\rangle$ in the following.

5. Ground state:

$$\begin{aligned}\langle\phi|\hat{H}|\phi\rangle &= \hbar\omega\langle\phi|\hat{a}^\dagger\hat{a} + \frac{1}{2}|\phi\rangle, \\ &= \hbar\omega\langle\hat{a}\phi|\hat{a}\phi\rangle + \frac{\hbar\omega}{2} \geq \frac{\hbar\omega}{2}.\end{aligned} \tag{46}$$

Based on above, we denote the ground state as $|n = 0\rangle = |0\rangle$, with the energy $E_0 = \frac{1}{2}\hbar\omega$, which is the eigen-state of

$$\hat{a}|0\rangle = 0, \qquad \text{the lowest energy state.} \tag{47}$$

6. Exited state:

$$|n\rangle = (\hat{a}^\dagger)^n|0\rangle, \tag{48}$$

with the energy

$$E_n = \hbar\omega(n + \frac{1}{2}).$$

7. Normalization constants:

$$\hat{N}|n\rangle = n|n\rangle, \tag{49}$$
$$\hat{a}|n\rangle = C_n|n-1\rangle = \sqrt{n}|n-1\rangle, \tag{50}$$
$$\hat{a}^\dagger|n\rangle = C_{n+1}|n+1\rangle = \sqrt{n+1}|n+1\rangle, \tag{51}$$

where the normalization constant C_n (C_{n+1}) can be derived by

$$\langle n|\hat{a}^\dagger\hat{a}|n\rangle = \langle n|\hat{N}|n\rangle = n = |C_n|^2\langle n-1|n-1\rangle. \tag{52}$$

8. Heisenberg's equation: The dynamics of the operator is governed by the Heisenberg's equation:

$$\frac{d}{dt}\hat{O} = \frac{1}{i\hbar}[\hat{O}, \hat{H}]. \tag{53}$$

For the annihilation operator of SHO, $\hat{a}$, we have

$$\frac{d}{dt}\hat{a} = \frac{1}{i\hbar}[\hat{a}, \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})] = -i\omega\hat{a}, \tag{54}$$

with the solution

$$\hat{a}(t) = \hat{a}(t = 0)\exp[-i\omega t]. \tag{55}$$

F. Quadrature Operators

One can define two hermitian operators

$$\hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger), \tag{56}$$

$$\hat{X}_2 = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger), \tag{57}$$

which have the commutator relation,

$$[\hat{X}_1, \hat{X}_2] = \frac{i}{2}. \tag{58}$$

G. Related Topics:

- Abelian and non-Abelian Operators
- Interaction Picture
- Uncertainty Relation