Note for *Quantum Optics*:
Master equations

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Reference:
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Ch. 13 in *"Elements of Quantum Optics,"* by P. Meystre and M. Sargent III.
Ch. 10 in *"Introductory Quantum Optics,"* by C. Gerry and P. Knight.
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I. LANGEVIN NOISE

Heisenberg-Langevin equation is a directly correspondence to classical description of stochastic system. In many cases, Heisenberg-Langevin equation is *nonlinear*, and in general it is extremely difficult to deal with the Heisenberg-Langevin equation. Another method is to develop a Schrödinger or interaction picture analysis, in this way we want to use a linear deterministic differential equation for the *reduced system density operator*. Naturally, as the quantum system is open, there is statistical as well as quantum uncertainty and a true wave function description is no longer possible.

II. MASTER EQUATION

We consider a system S interacting with a reservoir R via the interaction Hamiltonian $\hat{V}$. The combined density operator is denoted by $\hat{\rho}(t)$. Assume that at an initial time $t = 0$, the two systems are uncorrelated,

$$\hat{\rho}(t=0) = \hat{\rho}_S(0) \otimes \hat{\rho}_R(0). \tag{1}$$

In the interaction picture, the dynamics of $\hat{\rho}(t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}(t) = \frac{1}{i\hbar}[\hat{H}_I(t), \hat{\rho}(t)]. \tag{2}$$

Since the number of degrees of freedom of the reservoir is very large, it is impossible to keep track of its quantum evolution. We can only focus on the system with a reduced density operator, by tracing over the reservoir degrees of freedom,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_S(t) = \frac{1}{i\hbar}\mathrm{Tr}_R([\hat{H}_I(t), \hat{\rho}(t)]), \tag{3}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}(t) = \frac{1}{i\hbar}[\hat{H}_I(t), \hat{\rho}(t)]. \tag{4}$$

Without any approximation, the master equation for the reduced density operator is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_S(t) = (\frac{1}{i\hbar})^2 \int_0^t \mathrm{d}t' \mathrm{Tr}_R([\hat{H}_I(t), [\hat{H}_I(t'), \hat{\rho}(t')]]) + \frac{1}{i\hbar}\mathrm{Tr}_R([\hat{H}_I(t), \hat{\rho}(0)]). \tag{5}$$

Since $\mathrm{Tr}_R([\hat{H}_I(t), \hat{\rho}(0)])$ vanish for all the interaction Hamiltonians of interest in quantum optics, we have the master equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_S(t) = (\frac{1}{i\hbar})^2 \int_0^t \mathrm{d}t' \mathrm{Tr}_R([\hat{H}_I(t), [\hat{H}_I(t'), \hat{\rho}(t')]]), \tag{6}$$

III. BORN-MARKOV APPROXIMATION

Four key approximations are to be used in the following,

1. Rotating-wave approximation,
2. Born approximation,

$$\hat{\rho}(t') = \hat{\rho}_S(t') \otimes \hat{\rho}_R(t'),$$

3. The initial radiation field density operator commutes with the free Hamiltonian and the reservoir is not affected by the interaction with the system,

$$\hat{\rho}_R(t) = \mathrm{Tr}_S[\hat{\rho}(t)] = \hat{\rho}_R(0),$$

4. Markov approximation,

$$\hat{\rho}_S(t') \approx \hat{\rho}_S(t),$$

With the Born-Markov approximation, the master equation becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_S(t) = (\frac{1}{i\hbar})^2 \int_0^t \mathrm{d}t' \mathrm{Tr}_R([\hat{H}_I(t), [\hat{H}_I(t'), \hat{\rho}_S(t) \otimes \hat{\rho}_R(0)]]). \tag{7}$$

IV. EXAMPLES

- atom damping by field reservoirs,
- field damping by field reservoirs,
- field damping by atomic reservoirs,

V. ATOM DAMPING BY FIELD RESERVOIRS

Consider a two-level atom damped by a field reservoir in free space, the interaction Hamiltonian is

$$\hat{H}_I = \sum_k \hbar(g_k \hat{\sigma}_- \hat{a}_k^\dagger e^{-i(\omega-\omega_k)t} + \mathrm{H.\ C}). \tag{8}$$

Assume the reservoir density operator is a multimode thermal field,

$$\hat{\rho}_R = \prod_k \sum_n \frac{\exp(-\frac{\hbar\omega_k n}{k_B T})}{1-\exp(-\frac{\hbar\omega_k n}{k_B T})} |n\rangle_{kk}\langle n|. \tag{9}$$

The equation of motion for the reduced density operator $\mathrm{Tr}_R[\hat{\rho}(t)] \equiv \hat{\rho}_a(t)$ is,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_a(t) &= (\frac{1}{i\hbar})^2 \int_0^t \mathrm{d}t' \mathrm{Tr}_R([\hat{H}_I(t), [\hat{H}_I(t'), \hat{\rho}_a(t) \otimes \hat{\rho}_R(0)]]), \quad (10)\\ &= -\frac{1}{2}\Gamma\{n_{th}[\hat{\sigma}_-\hat{\sigma}_+\hat{\rho}_a - \hat{\sigma}_+\hat{\rho}_a\hat{\sigma}_-] + (n_{th}+1)[\hat{\sigma}_+\hat{\sigma}_-\hat{\rho}_a - \hat{\sigma}_-\hat{\rho}_a\hat{\sigma}_+])\} + \mathrm{H.\ C}, \quad (11)\end{aligned}$$

VI. FIELD DAMPING BY FIELD RESERVOIRS

Consider a single-mode field in a cavity with a finite leakage rate, and assume the reservoir density operator is a multimode thermal field,

$$\hat{\rho}_R = \prod_k \sum_n \frac{\exp(-\frac{\hbar\omega_k n}{k_B T})}{1-\exp(-\frac{\hbar\omega_k n}{k_B T})} |n\rangle_{kk}\langle n|. \tag{12}$$

The equation of motion for the reduced density operator $\mathrm{Tr}_R[\hat{\rho}(t)] \equiv \hat{\rho}_f(t)$ is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_f(t) = (\frac{1}{i\hbar})^2 \int_0^t \mathrm{d}t' \mathrm{Tr}_R([\hat{H}_I(t),[\hat{H}_I(t'),\hat{\rho}_f(t)\otimes\hat{\rho}_R(0)]]), \tag{13}$$

$$= -\int_{t_0}^t \mathrm{d}t' \sum_k g_k^2 \{n_{th}[\hat{a}\hat{a}^\dagger\hat{\rho}_f(t') - \hat{a}^\dagger\hat{\rho}_f(t')\hat{a}]e^{-i(\omega-\omega_k)(t-t')} \tag{14}$$

$$+(n_{th}+1)[\hat{a}^\dagger\hat{a}\hat{\rho}_f(t') - \hat{a}\hat{\rho}_f(t')\hat{a}^\dagger])e^{i(\omega-\omega_k)(t-t')}\} + \mathrm{H.~C.} \tag{15}$$

By replacing $\sum_k g_k^2$ with the integral $\int \mathrm{d}\omega_k D(\omega_k) g(\omega_k)^2$, we have

$$\int_{t_0}^t \mathrm{d}t' \sum_k g_k^2 e^{\pm i(\omega-\omega_k)(t-t')} = \int_{t_0}^t \mathrm{d}t' \int \mathrm{d}\omega_k D(\omega_k) g(\omega_k)^2 e^{\pm i(\omega-\omega_k)(t-t')}, \tag{16}$$

$$\approx \int \mathrm{d}\omega_k D(\omega_k) g(\omega_k)^2 \pi\delta(\omega-\omega_k), \tag{17}$$

$$\approx \pi D(\omega) g(\omega)^2 \equiv \frac{1}{2}(\frac{\omega}{Q_e}), \tag{18}$$

where ω/Q_e is the cavity photon decay rate due to leakage (output coupling) via a partially reflecting mirror.

The equation of motion for the reduced density operator $\mathrm{Tr}_R[\hat{\rho}(t)] \equiv \hat{\rho}_f(t)$ is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_f(t) = -\frac{1}{2}(\frac{\omega}{Q_e})\{n_{th}[\hat{a}\hat{a}^\dagger\hat{\rho}_f(t') - \hat{a}^\dagger\hat{\rho}_f(t')\hat{a}] + (n_{th}+1)[\hat{a}^\dagger\hat{a}\hat{\rho}_f(t') - \hat{a}\hat{\rho}_f(t')\hat{a}^\dagger])\}, \tag{19}$$

$$+\mathrm{H.~C.} \tag{20}$$

VII. FIELD DAMPING BY ATOMIC RESERVOIRS

Consider the damping of an optical cavity mode by a two-level atomic beam reservoir. This is the *reverse problem of a laser.* The statistics of the atomic reservoir is determined by the Boltzmann distribution,

$$\hat{\rho}_{R=atom}(t=0) = \begin{pmatrix} \rho_{aa} & 0 \\ 0 & \rho_{bb} \end{pmatrix} = \rho_{aa}|a\rangle\langle a| + \rho_{bb}|b\rangle\langle b|, \tag{21}$$

where

$$\rho_{aa} = \frac{1}{1+\exp(\hbar\omega_0/k_B T)}, \quad \text{and} \quad \rho_{aa} = \frac{\exp(\hbar\omega_0/k_B T)}{1+\exp(\hbar\omega_0/k_B T)}. \tag{22}$$

Assume there is no quantum coherence between the upper and lower states, $\rho_{ab} = \rho_{ba} = 0$.

The interaction Hamiltonian for a single atom is

$$\hat{H}_I = \hbar g(\hat{\sigma}_-\hat{a}^\dagger + \hat{\sigma}_+\hat{a}) = \hbar g \begin{pmatrix} 0 & \hat{a} \\ \hat{a}^\dagger & 0 \end{pmatrix}. \tag{23}$$

At $t = 0$, the atom-field density operator is,

$$\hat{\rho}(t) = \hat{\rho}_f(t) \otimes \hat{\rho}_R = \begin{pmatrix} \rho_{aa}\hat{\rho}_f(t) & 0 \\ 0 & \rho_{bb}\hat{\rho}_f(t) \end{pmatrix}. \tag{24}$$

The terms for the commutator are

$$[\hat{H}_I, [\hat{H}_I, \hat{\rho}(t)]] = \hbar g^2 \begin{pmatrix} \hat{a}\hat{a}^\dagger \rho_{aa}\hat{\rho}_f(t) - \hat{a}\rho_{bb}\hat{\rho}_f(t)\hat{a}^\dagger & 0 \\ 0 & \hat{a}^\dagger\hat{a}\rho_{bb}\hat{\rho}_f(t) - \hat{a}^\dagger\rho_{ee}\hat{\rho}_f(t)\hat{a} \end{pmatrix} \tag{25}$$

$$+\text{H. C.} \tag{26}$$

The equation of motion for the reduced density operator $\text{Tr}_R[\hat{\rho}(t)] \equiv \hat{\rho}_f(t)$ is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_f(t) = (\frac{1}{i\hbar})^2 \int_0^t \mathrm{d}t'\text{Tr}_R([\hat{H}_I(t), [\hat{H}_I(t'), \hat{\rho}_f(t) \otimes \hat{\rho}_R(0)]]). \tag{27}$$

Assume that r atoms are injected into the cavity per second, and they spend an average time of τ seconds inside the cavity, i.e.

$$\int_0^\tau \mathrm{d}t' r t' = \frac{1}{2} r\tau^2, \tag{28}$$

then,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_f(t) = -\frac{1}{2}R_e[\hat{a}\hat{a}^\dagger\hat{\rho}_f - \hat{a}^\dagger\hat{\rho}_f\hat{a}] - \frac{1}{2}R_g[\hat{a}^\dagger\hat{a}\hat{\rho}_f - \hat{a}\hat{\rho}_f(t)\hat{a}^\dagger] + \text{H.C}, \tag{29}$$

where

$$R_e = r\rho_{aa}g^2\tau^2, \quad \text{and} \quad R_g = r\rho_{bb}g^2\tau^2. \tag{30}$$

Here, R_e is the rate coefficient for photon emission by atoms per second, R_g is the rate coefficient for photon absorption by atoms per second, and the cavity photon decay rate $\frac{\nu}{Q_0}$ and the thermal equilibrium photon number n_{th} are defined by

$$\frac{\nu}{Q_0} \equiv R_g - R_e, \quad \text{and} \quad R_e(1+n_{th}) = R_g n_{th} \to n_{th} = \frac{R_e}{R_g - R_e} = \frac{1}{\exp(\hbar\omega/k_b T) - 1}. \tag{31}$$

The later one condition gives the thermal equilibrium photon number n_{th}. We note that $R_e(1+n_{th})$ is the sum of *spontaneous* and *stimulated* emission rate per second; while $R_g n_{th}$ is (stimulated) *absorption* rate per second.

The diagonal elements of the reduced density matrix $\text{Tr}_R[\hat{\rho}(t)]$ are

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{n,n}(t) = -\frac{\nu}{Q_0}\{[n_{th}(n+1) - (n_{th}+1)n]\rho_{n,n} \tag{32}$$

$$-n_{th}\rho_{n-1,n-1} - (n_{th}+1)(n+1)\rho_{n+1,n+1}\}, \tag{33}$$

$$= [-R_e(n+1) - R_g n]\rho_{nn} + R_e n\rho_{n-1,n-1} + R_g(n+1)\rho_{n+1,n+1}, \tag{34}$$

Equilibrium is obtain when the net flow between all pairs of level vanishes,

$$R_g n\rho_{n,n} = R_e n\rho_{n-1,n-1}, \quad \text{or} \quad \rho_{n,n} = \frac{n_{th}}{n_{th}+1}\rho_{n-1,n-1}. \tag{35}$$

This condition is referred to as *detailed balance*. The solution for detailed balance is

$$\rho_{n,n} = [1 - \exp(-\hbar\omega/k_B T)]\exp(-n\hbar\omega/k_B T), \tag{36}$$

with $n_{th} = \frac{1}{\exp(\hbar\omega/k_b T) - 1}$. Detailed balance in this case gives the *thermal (Bose-Einstein)* distribution with an average photon number,

$$\langle n \rangle = \sum_n \rho_{n,n} n = \frac{1}{\exp(\hbar\omega/k_B T) - 1} = n_{th}, \tag{37}$$

this is the result we use for the thermal radiation field. Although the filed $\hat{\rho}_f$ may initially be in a pure state, the process of tracing over the (unobserved) atomic states leads to a field in a mixed state, $\hat{\rho}_f = \sum_n \rho_{n,n}|n\rangle\langle n|$. The effect of the atomic beam is to bring the field to the *same temperature* as that of atoms.

VIII. RESERVOIR, DECOHERENCE, AND MEASUREMENT

The reservoir theory lies in the process of tracing over the reservoir coordinates, which induces *dissipation* and *decoherence* of the system. At the same time, this is an *irreversible* dynamics for the system. This process corresponds to the lack of measurement as to whether the atom is in the upper level or in the lower level after interaction with the field. If the initial and final states of the atom are know, *i.e.*, if the information concerning the atomic beam is not discarded, the field remains in a *pure state*. The primary difference between the reservoir and quantum measurement theories is whether information stored in the environment (reservoir) that interacts with system is discarded or read out.