Note for *Quantum Optics*: Coherent States

Ray-Kuang Lee[1]
Institute of Photonics Technologies, National Tsing Hua University, Hsinchu, 300, Taiwan
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I. COHERENT STATES

We introduce the eigenstate of annihilation operator, called the *coherent state*,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \tag{1}$$

which in the basis of number states has the form,

$$|\alpha\rangle = \sum_n |n\rangle\langle n|\alpha\rangle, \qquad \text{for} \qquad \sum_n |n\rangle\langle n| = 1, \tag{2}$$

$$= \sum_n |n\rangle\langle 0|\frac{\hat{a}^n}{\sqrt{n!}}|\alpha\rangle, \qquad \text{for} \qquad |n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle, \tag{3}$$

$$= \sum_n \frac{\alpha^n}{\sqrt{n!}}\langle 0|\alpha\rangle|n\rangle, \tag{4}$$

By imposing the normalization condition, $\langle\alpha|\alpha\rangle = 1$, we obtain,

$$1 = \langle\alpha|\alpha\rangle = \sum_n\sum_m \langle m|n\rangle\frac{(\alpha^*)^m\alpha^n}{\sqrt{m!}\sqrt{n!}} = e^{|\alpha|^2}|\langle 0|\alpha\rangle|^2. \tag{5}$$

Now, the coherent state $|\alpha\rangle$ has the Poisson distribution in the photon number,

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{6}$$

A. Displacement operator

Coherent states can be generated by translating the vacuum state $|0\rangle$ to have a finite excitation amplitude α,

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{(\alpha\hat{a}^\dagger)^n}{n!}|0\rangle, \tag{7}$$

$$= e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\hat{a}^\dagger}|0\rangle. \tag{8}$$

Since $\hat{a}|0\rangle = 0$, we have $e^{-\alpha^* \hat{a}}|0\rangle = 0$ and

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha \hat{a}^\dagger} e^{-\alpha^* \hat{a}}|0\rangle. \tag{9}$$

Furthermore, for any two non-commuting operators $\hat{A}$ and $\hat{B}$, we have the Baker-Hausdorff relation,

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A},\hat{B}]}, \qquad \text{provided} \qquad [\hat{A},[\hat{A},\hat{B}]] = 0, \tag{10}$$

By using $\hat{A} = \alpha \hat{a}^\dagger$, $\hat{B} = -\alpha^* \hat{a}$, and $[\hat{A},\hat{B}] = |\alpha|^2$, we have,

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle = e^{+\alpha \hat{a}^\dagger - \alpha^* \hat{a}}|0\rangle, \tag{11}$$

where $\hat{D}(\alpha)$ is the *displacement operator*, which is physically realized by a classical oscillating current. In this way, the coherent state is displaced from the ground state of a simple harmonic oscillator. That is

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle = e^{+\alpha \hat{a}^\dagger - \alpha^* \hat{a}}|0\rangle. \tag{12}$$

The displacement operator $\hat{D}(\alpha)$ is a unitary operator, *i.e.*,

$$\hat{D}^\dagger(\alpha) = \hat{D}(-\alpha) = [\hat{D}(\alpha)]^{-1}, \tag{13}$$

and $\hat{D}(\alpha)$ acts as a displacement operator upon the amplitudes $\hat{a}$ and $\hat{a}^\dagger$, *i.e.*,

$$\hat{D}^{-1}(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a} + \alpha, \tag{14}$$
$$\hat{D}^{-1}(\alpha)\hat{a}^\dagger\hat{D}(\alpha) = \hat{a}^\dagger + \alpha^*. \tag{15}$$

B. Properties of Coherent States

1. The probability of finding n photons in $|\alpha\rangle$ is given by a Poisson distribution.
2. The coherent state is a minimum-uncertainty states,
3. The set of all coherent states $|\alpha\rangle$ is a complete set,

$$\int |\alpha\rangle\langle\alpha| \mathrm{d}^2\alpha = \pi \sum_n |n\rangle\langle n|, \quad \text{or} \quad \frac{1}{\pi}\int |\alpha\rangle\langle\alpha| \mathrm{d}^2\alpha = 1. \tag{16}$$

4. Two coherent states corresponding to different eigenstates α and β are not orthogonal,

$$\langle\alpha|\beta\rangle = \exp(-\frac{1}{2}|\alpha|^2 + \alpha^*\beta - \frac{1}{2}|\beta|^2) = \exp(-\frac{1}{2}|\alpha - \beta|^2). \tag{17}$$

5. Coherent states are *approximately* orthogonal only in the limit of large separation of the two eigenvalues, $|\alpha - \beta| \to \infty$. Therefore, any coherent state can be expanded using other coherent state,

$$|\alpha\rangle = \frac{1}{\pi}\int \mathrm{d}^2\beta |\beta\rangle\langle\beta|\alpha\rangle = \frac{1}{\pi}\int \mathrm{d}^2\beta e^{-\frac{1}{2}|\beta-\alpha|^2}|\beta\rangle. \tag{18}$$

This means that a coherent state forms an *overcomplete* set.

6. The simultaneous measurement of $\hat{a}_1$ and $\hat{a}_2$, represented by the projection operator $|\alpha\rangle\langle\alpha|$, is not an exact measurement but instead an approximate measurement with a finite measurement error.

C. q-representation of the coherent state

Since the coherent state is defined as the eigenstate of the annihilation operator,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle,$$

where $\hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + i\hat{p})$, then the q-representation of the coherent state is governed by,

$$(\omega q + \hbar\frac{\partial}{\partial q})\langle q|\alpha\rangle = \sqrt{2\hbar\omega}\alpha\langle q|\alpha\rangle, \tag{19}$$

with the solution,

$$\langle q|\alpha\rangle = (\frac{\omega}{\pi\hbar})^{1/4}\exp[-\frac{\omega}{2\hbar}(q - \langle q\rangle)^2 + i\frac{\langle p\rangle}{\hbar}q + i\theta], \tag{20}$$

where θ is an arbitrary real phase,

II. EXPECTATION VALUE OF THE ELECTRIC FIELD

For a single mode electric field, polarized in the x-direction,

$$\hat{E}_x = E_0[\hat{a}(t) + \hat{a}^\dagger(t)] \sin kz, \tag{21}$$

the expectation value of the electric field operator is

$$\langle\alpha|\hat{E}(t)|\alpha\rangle = E_0[\alpha e^{-i\omega t} + \alpha^* e^{i\omega t}] \sin kz = 2E_0|\alpha| \cos(\omega t + \phi) \sin kz. \tag{22}$$

Similar, we have

$$\langle\alpha|\hat{E}(t)^2|\alpha\rangle = E_0^2[4|\alpha|^2 \cos^2(\omega t + \phi) + 1] \sin^2 kz, \tag{23}$$

and the corresponding variance, the root-mean-square deviation, in the electric field is,

$$\langle\Delta\hat{E}(t)^2\rangle^{1/2} = \sqrt{\frac{\hbar\omega}{2\epsilon_0 V}} |\sin kz|, \tag{24}$$

We note that the variance, $\langle\Delta\hat{E}(t)^2\rangle^{1/2}$, is independent of the field strength $|\alpha|$. That means the quantum noise becomes less important as $|\alpha|^2$ increases, and why a highly excited coherent state $|\alpha| \gg 1$ can be treated as a *classical* EM field.

A. Generation of Coherent States

In classical mechanics we can excite a SHO into motion by, e.g. stretching the spring to a new equilibrium position,

$$\hat{H} = \frac{p^2}{2m} + \frac{1}{2}kx^2 - eE_0x, \tag{25}$$

$$= \frac{p^2}{2m} + \frac{1}{2}k(x - \frac{eE_0}{k})^2 - \frac{1}{2}(\frac{eE_0}{k})^2, \tag{26}$$

$$\tag{27}$$

By turning off the DC field, *i.e.*, $E_0 = 0$, we will have a coherent state $|\alpha\rangle$ oscillating without changing its shape. In analogy, applying the DC field to the SHO is mathematically equivalent to applying the displacement operator to the state $|0\rangle$.