Note for *Quantum Optics*: Correlations

Ray-Kuang Lee[1]
Institute of Photonics Technologies, National Tsing Hua University, Hsinchu, 300, Taiwan
(Dated: Spring, 2021)

Reference:
Ch. 2, 4, 16, in *"Quantum Optics,"* by M. Scully and M. Zubairy.
Ch. 3, 4, in *"Mesoscopic Quantum Optics,"* by Y. Yamamoto and A. Imamoglu.
Ch. 6, in *"The Quantum Theory of Light,"* by R. Loudon.
Ch. 5, 7, in *"Introductory Quantum Optics,"* by C. Gerry and P. Knight.
Ch. 5, 8, in *"Quantum Optics,"* by D. Wall and G. Milburn.

I. CLASSICAL CORRELATION FUNCTIONS

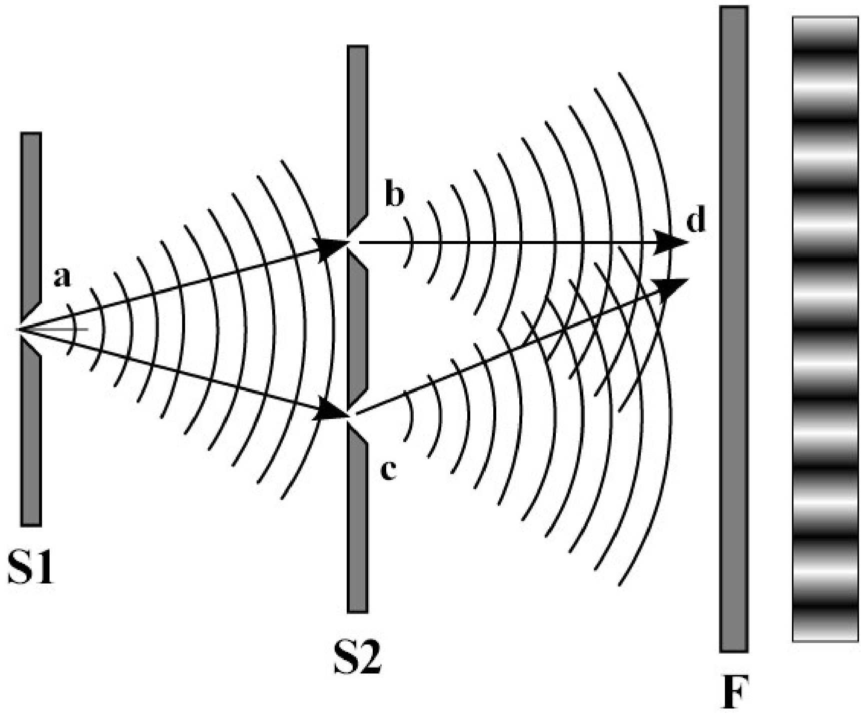

FIG. 1: Young's interferometer.

Consider Young's two-slit interference, the intensity at the output is described by

$$I(r) = \langle |E(r,t)|^2\rangle = \langle |K_1 E(r_1,t_1) + K_2 E(r_2,t_2)|^2\rangle, \tag{1}$$

where K_1 and K_2 are two constants, and time-average is defined as

$$\langle f(t)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T f(t)\,dt, \tag{2}$$

a stationary average. The *interference* pattern has the form

$$I(r) = I_1 + I_2 + 2\sqrt{I_1 I_2}\,\mathrm{Re}[\gamma^{(1)}(x_1,x_2)], \tag{3}$$

where $I_1 = |K_1|^2\langle|E(r_1,t_1)|^2\rangle$ and $I_2 = |K_2|^2\langle|E(r_2,t_2)|^2\rangle$ are the background (D.C) intensities. Here, the mutual correlation function, $\gamma^{(1)}(x_1,x_2)$ with $x_i = r_i, t_i$ is introduced:

$$\gamma^{(1)}(x_1,x_2) = \frac{\langle E^*(x_1)E(x_2)\rangle}{\sqrt{\langle|E(x_1)|^2\rangle\langle|E(x_2)|^2\rangle}}. \tag{4}$$

With such a correlation function, one can define the *degree of coherence* as

- complete coherence:

$$|\gamma^{(1)}(x_1,x_2)| = 1.$$

- partial coherence,

$$0 < |\gamma^{(1)}(x_1,x_2)| < 1.$$

- complete incoherence:

$$|\gamma^{(1)}(x_1,x_2)| = 0.$$

A. Chaotic light

If we apply a *Lorentzian* correlation function, that is

$$\langle E^*(t)E(t+\tau)\rangle = E_0^2 e^{-i\omega_0 t} e^{-|\tau|/\tau_0}, \tag{5}$$

then the corresponding classical correlation function

$$|\gamma(\tau)| = e^{-2|\tau|/\tau_0}. \tag{6}$$

This is a typical model for chaotic light, with a given correlation time τ_0.

II. QUANTUM PHOTODETECTORS

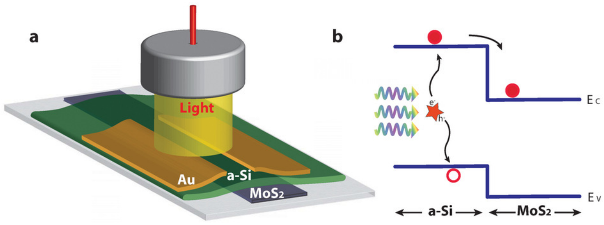

FIG. 2: A illustration of semiconductor photodetector.

With the same analogy, we consider the single-atom detector couples to the quantized field through the dipole interaction,

$$\hat{H}_I = -\hat{d} \cdot \hat{E}(r,t).$$

Here, we assume that the atom is initially in a ground state $|g\rangle$ and the field is in some state $|ß\rangle$. Upon the absorption of radiation, the atom makes a transition to the state $|e\rangle$ and the field to the state $|f\rangle$, then

$$\langle f|\langle e|\hat{H}_I|g\rangle|i\rangle \propto -\langle e|\hat{d}|g\rangle \langle f|\hat{a}|i\rangle, \tag{7}$$

where $\hat{E}(r,t) = \sum_j c_j[\hat{a}_j(t) + \hat{a}_j^\dagger(t)] = \hat{E}^{(+)}(r,t) + \hat{E}^{(-)}(r,t)$. The term associated with $|\langle e|\hat{d}|g\rangle\|^2$ is the *quantum efficiency* of a photodetector.

The probability that the detector measures all the possible final states,

$$\sum_f |\langle f|\hat{a}|i\rangle|^2 = \langle i|\hat{E}^{(-)}(r,t) \cdot \hat{E}^{(+)}(r,t)|i\rangle. \tag{8}$$

If we define a density operator,

$$\hat{\rho} = \sum_i P_i|i\rangle\langle i|, \tag{9}$$

then the expectation value can be replaced by the ensemble average,

$$\mathrm{Tr}\{\hat{\rho}\hat{E}^{(-)}(r,t) \cdot \hat{E}^{(+)}(r,t)\} = \sum_i P_i\langle i|\hat{E}^{(-)}(r,t) \cdot \hat{E}^{(+)}(r,t)|i\rangle. \tag{10}$$

III. FIRST-ORDER QUANTUM CORRELATION FUNCTION

With the analogy to the *classical* correlation function, we have the first-order quantum correlation function by normalizing it to one:

$$g^{(1)}(x_1, x_2) = \frac{G^{(1)}(x_1, x_2)}{[G^{(1)}(x_1, x_1)G^{(1)}(x_2, x_2)]^{1/2}}, \tag{11}$$

where $G^{(1)}(x_1, x_2) = \mathrm{Tr}\{\hat{\rho}\hat{E}^{(-)}(x_1) \cdot \hat{E}^{(+)}(x_2)\}$. Again, as the classical one, we have the degree of coherence, if

- complete coherence:

$$|g^{(1)}(x_1, x_2)| = 1.$$

- partial coherence,

$$0 < |g^{(1)}(x_1, x_2)| < 1.$$

- complete incoherence:

$$|g^{(1)}(x_1, x_2)| = 0.$$

Consider a single-mode plane wave, $\hat{E}^{(+)}(x) = iK\hat{a}e^{i(k\cdot r - \omega t)}$. Then

1. If the field is in a number state $|n\rangle$, we have

$$G^{(1)}(x, x) = K^2 n, \quad G^{(1)}(x_1, x_2) = K^2 n e^{i[k(r_1 - r_2) - \omega(t_1 - t_2)]}, \tag{12}$$

and

$$|g^{(1)}(x_1, x_2)| = 1. \tag{13}$$

2. If the field is a coherent state $|\alpha\rangle$, we have

$$G^{(1)}(x, x) = K^2|\alpha|^2, \quad G^{(1)}(x_1, x_2) = K^2|\alpha|^2 e^{i[k(r_1 - r_2) - \omega(t_1 - t_2)]}, \tag{14}$$

and

$$|g^{(1)}(x_1, x_2)| = 1, \tag{15}$$

IV. CLASSICAL SECOND-ORDER CORRELATION FUNCTION

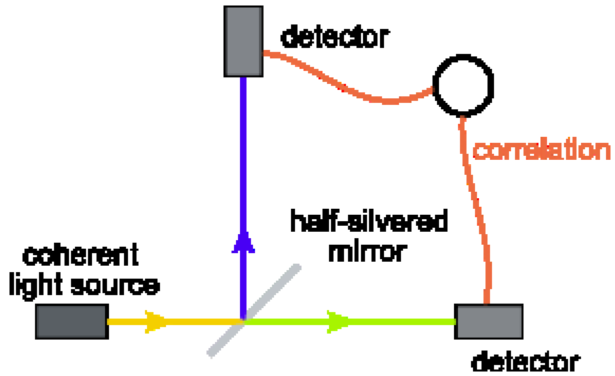

FIG. 3: Hanbury Brown and Twiss (HBT) interferometer.

We can extend the concept about correlation function to higher-orders. The classical second-order coherence function is defined as

$$\gamma^{(2)}(\tau) = \frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle^2} = \frac{\langle E^*(t)E^*(t+\tau)E(t+\tau)E(t)\rangle}{\langle E^*(t)E(t)\rangle^2}. \tag{16}$$

If the detectors are at different distances from the beam splitter,

$$\gamma^{(2)}(x_1,x_2) = \frac{\langle I(x_1)I(x_2)\rangle}{\langle I(x_1)\rangle\langle I(x_2)\rangle} = \frac{\langle E^*(x_1)E^*(x_2)E(x_2)E(x_1)\rangle}{\langle |E(x_1)|^2\rangle\langle |E(x_2)|^2\rangle}. \tag{17}$$

The field is said to be classical coherence to second order if $|\gamma^{(1)}(x_1,x_2)| = 1$ and $\gamma^{(2)}(x_1,x_2) = 1$, with the factorization,

$$\langle E^*(x_1)E^*(x_2)E(x_2)E(x_1)\rangle = \langle |E(x_1)|^2\rangle\langle |E(x_2)|^2\rangle. \tag{18}$$

For zero time-delay coherence function, we have

$$\gamma^{(2)}(0) = \frac{\langle I(t)^2\rangle}{\langle I(t)\rangle^2}. \tag{19}$$

For a sequence of N measurements taken at times $t_1, t_2, \ldots, t_N$,

$$\langle I(t)\rangle = \frac{I(t_1) + I(t_2) + \cdots I(t_N)}{N}, \quad \text{and} \quad \langle I(t)^2\rangle = \frac{I(t_1)^2 + I(t_2)^2 + \cdots I(t_N)^2}{N}. \tag{20}$$

From Cauchy's inequality, $(I_1 - I_2)^2 \geq 0$,

$$2I(t_1)I(t_2) \leq I(t_1)^2 + I(t_2)^2,$$

we have

$$\langle I(t)^2\rangle \geq \langle I(t)\rangle^2, \quad \text{or} \quad 1 \leq \gamma^{(2)}(0) < \infty. \tag{21}$$

For non-zero delay, we have

$$[I(t_1)I(t_1+\tau) + \cdots I(t_N)I(t_n+\tau)]^2 \leq [I(t_1)^2 + \cdots I(t_N)^2][I(t_1+\tau)^2 + \cdots I(t_N+\tau)^2], \tag{22}$$

then

$$\langle I(t)I(t+\tau)\rangle \leq \langle I(t)\rangle^2, \quad \text{or} \quad 1 \leq \gamma^{(2)}(\tau) \leq \gamma^{(2)}(0), \tag{23}$$

where $1 \leq \gamma^{(2)}(0) < \infty$.

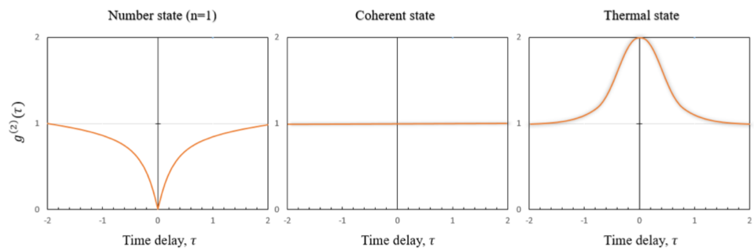

FIG. 4: Left: Photon anti-bunching effect from single photon source; Right: Photon bunching effect from thermal states.

1. For a light source containing a large number of independently photons,
$$\gamma^{(2)}(\tau) = 1 + |\gamma^{(1)}(\tau)|^2. \tag{24}$$
2. Since $0 \leq |\gamma^{(1)}(\tau)|^2 \leq 1$, it follows that for all kinds of chaotic light.
$$1 \leq \gamma^{(2)}(\tau) \leq 2. \tag{25}$$
3. For light source with a Lorentzian spectrum,
$$\gamma^{(2)}(\tau) = 1 + e^{-2|\tau|/\tau_0}. \tag{26}$$
4. For $\tau \to \infty$,
$$\gamma^{(2)}(\tau) \to 1. \tag{27}$$
5. For zero delay, $\tau \to 0$,
$$\gamma^{(2)}(\tau) \to 2. \tag{28}$$
6. Hanbury Brown and Twiss experiment shows that if the photon are emitted independently by the source, then the photons arrive in pairs at zero time delay. This is called the *photon bunching effect*.

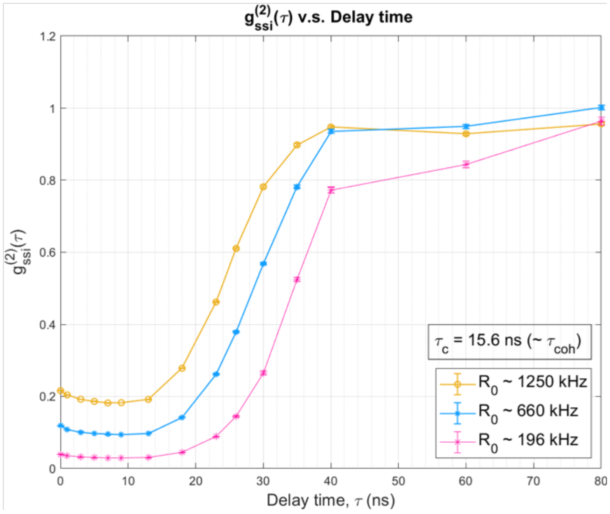

FIG. 5: Experimental data for the photon anti-bunching effect from single photon source.

V. QUANTUM SECOND-ORDER CORRELATION FUNCTION

With the same concept, we define the normalized second-order quantum correlation function,

$$g^{(2)}(x_1, x_2) = \frac{G^{(2)}(x_1, x_2)}{[G^{(1)}(x_1, x_1)G^{(1)}(x_2, x_2)]}, \tag{29}$$

where $g^{(2)}(x_1, x_2)$, is the joint probability of detecting one photon at (r_1, t_1) and (r_2, t_2). At a fixed position, $g^{(2)}$ depends only on the time difference,

$$g^{(2)}(\tau) = \frac{\langle \hat{E}^{(-)}(t)\hat{E}^{(-)}(t+\tau)\hat{E}^{(+)}(t+\tau)\hat{E}^{(+)}(t)\rangle}{\langle \hat{E}^{(-)}(t)\hat{E}^{(-)}(t)\rangle\langle \hat{E}^{(-)}(t+\tau)\hat{E}^{(-)}(t+\tau)\rangle}. \tag{30}$$

- For a single-mode field,

$$g^{(2)}(\tau) = \frac{\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a}\hat{a}\rangle}{\langle \hat{a}^\dagger \hat{a}\rangle^2} = \frac{\langle \hat{n}(\hat{n}-1)\rangle}{\langle \hat{n}\rangle^2} = 1 + \frac{\langle \Delta\hat{n}^2\rangle - \langle \hat{n}\rangle}{\langle \hat{n}\rangle^2}. \tag{31}$$

- For a coherent state $|\alpha\rangle$,

$$g^{(2)}(\tau) = 1, \tag{32}$$

  which has a *Poisson distribution*, *i.e.*, $\Delta\hat{n}^2\rangle = \langle \hat{n}\rangle$.

- For a single-mode thermal state, $\hat{\rho}_{\text{th}} = \frac{1}{Z}\sum \exp(-E_n/k_BT)|n\rangle\langle n|$,

$$g^{(2)}(\tau) = 2. \tag{33}$$

- For a non-classical state, with *sub-Poisson* photon number distribution, *i.e.*, $\langle \Delta\hat{n}^2\rangle < \langle \hat{n}\rangle$,

$$g^{(2)}(\tau) = g^{(2)}(0) < 1, \tag{34}$$

- For a Fock state $|n\rangle$,

$$g^{(2)}(0) = 1 - \frac{1}{n}. \tag{35}$$

- For a single photon source, $n = 1$,

$$g^{(2)}(0) = 0. \tag{36}$$