Note for *Quantum Optics*: Quantum theory of Damping

Ray-Kuang Lee[1]
Institute of Photonics Technologies, National Tsing-Hua University, Hsinchu, 300, Taiwan
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Reference:
Ch. 8, 9, 10 in *"Quantum Optics,"* by M. Scully and M. Zubairy.
Ch. 7 in *"Mesoscopic Quantum Optics,"* by Y. Yamamoto and A. Imamoglu.
Ch. 8 in *"The Quantum Theory of Light,"* by R. Loudon.
Ch. 14, 15 in *"Elements of Quantum Optics,"* by P. Meystre and M. Sargent III.
Ch. 8 in *"Introductory Quantum Optics,"* by C. Gerry and P. Knight.

I. DAMPING VIA OSCILLATOR RESERVOIR

Consider a single-mode field, with frequency ω, $\hat{a}$ and $\hat{a}^\dagger$ operators. Here, we introduce the concept of *reservoir*, which may be taken as any large collection of systems with many degrees (e.g. phonons, other photon modes, etc), with closely spaced frequencies ν_k, $\hat{b}_k$ and $\hat{b}_k^\dagger$ operators. The Hamiltonian for the field-reservoir system is

$$\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a} + \sum_k \hbar\nu_k\hat{b}^\dagger\hat{b} + \sum_k \hbar g_k(\hat{b}_k^\dagger\hat{a} + \hat{a}^\dagger\hat{b}_k). \tag{1}$$

The corresponding Heisenberg equations of motion for the operators are

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}(t) &= \frac{i}{\hbar}[\hat{H},\hat{a}] = -i\omega\hat{a}(t) - i\sum_k g_k\hat{b}_k(t), &(2)\\ \frac{\mathrm{d}}{\mathrm{d}t}\hat{b}_k(t) &= -i\nu_k\hat{b}(t) - ig_k\hat{a}(t). &(3)\end{aligned}$$

The closed equation for the field operator $\hat{a}(t)$ is,

$$\begin{aligned}\hat{b}_k(t) &= \hat{b}_k(0)e^{-i\nu_k t} - ig_k\int_0^\infty \mathrm{d}t' e^{-i\nu_k(t-t')}\hat{a}(t'), &(4)\\ \frac{\mathrm{d}}{\mathrm{d}t}\hat{a}(t) &= -i\omega\hat{a}(t) - \sum_k |g_k|^2\int_0^\infty \mathrm{d}t' e^{-i\nu_k(t-t')}\hat{a}(t') + \hat{f}_a(t), &(5)\end{aligned}$$

where

$$\hat{f}_a(t) = -i\sum_k g_k\hat{b}_k(0)e^{-i\nu_k t}.$$

If we define the slowly varying operator $\hat{a}'(t) = \hat{a}e^{i\omega t}$, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}'(t) &= -\sum_k |g_k|^2\int_0^\infty \mathrm{d}t' e^{-i(\nu_k-\omega)(t-t')}\hat{a}'(t') + \hat{F}_a(t), &(6)\\ \hat{F}_a(t) &= -i\sum_k g_k\hat{b}_k(0)e^{-i(\nu_k-\omega)t}. &(7)\end{aligned}$$

In the following formulations, we use the notation $\hat{a}(t)$ to replace $\hat{a}'(t)$.

II. MARKOVIAN WHITE NOISE

For a single-mode field interacting with reservoir, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}(t) &= -\sum_k |g_k|^2 \int_0^\infty \mathrm{d}t' e^{-i(\nu_k-\omega)(t-t')}\hat{a}(t') + \hat{F}_a(t), & (8)\\ \hat{F}_a(t) &= -i\sum_k g_k \hat{b}_k(0) e^{-i(\nu_k-\omega)t}. & (9)\end{aligned}$$

With the *Weisskopf-Wigner theory*, $G(t) = \frac{\Gamma}{2}\delta(t)$,

$$\sum_k |g_k|^2 \int_0^\infty \mathrm{d}t' e^{-i(\nu_k-\omega)(t-t')}\hat{a}(t') \equiv \int_0^\infty \mathrm{d}t' G(t-t')\hat{a}(t') \approx \frac{\Gamma}{2}\hat{a}(t), \tag{10}$$

where $G(t-t') = \sum_k |g_k|^2 e^{-i(\nu_k-\omega)(t-t')} = \int_\nu D(\nu)|g(\nu)|^2 e^{-i(\nu_k-\omega)(t-t')}$, $D(\nu)$ is the density of state for the reservoir, and the equation of motion for the field interacting with the reservoir is

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}(t) = -\frac{\Gamma}{2}\hat{a}(t) + \hat{F}_a(t). \tag{11}$$

These are *dissipation* and *fluctuation* terms.

For a Markovian process, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}(t) = -\frac{\Gamma}{2}\hat{a}(t) + \hat{F}_a(t), \quad \hat{F}_a(t) = -i\sum_k g_k \hat{b}_k(0) e^{-i(\nu_k-\omega)t}. \tag{12}$$

If one dismiss the fluctuation term $\hat{F}_a(t)$, *i.e.*,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}(t) = -\frac{\Gamma}{2}\hat{a}(t), \tag{13}$$

we have the solution $\hat{a}(t) = \hat{a}(0)e^{-\Gamma/2t}$.

For the non-interacting field, the commutation relation at $t = 0$ is $[\hat{a}(0), \hat{a}^\dagger(0)] = 1$, but as time evolves to $t \neq 0$, the commutation relation is not satisfied,

$$[\hat{a}(t), \hat{a}^\dagger(t)] = \exp(-\Gamma t). \tag{14}$$

The noise operator with appropriate correlation properties helps to maintain the commutation relation at all time.

III. THERMAL RESERVOIR

For thermal reservoir, the noise operator is defined as,

$$\hat{F}_a(t) = -i\sum_k g_k \hat{b}_k(0) e^{-i(\nu_k-\omega)t}. \tag{15}$$

Suppose the reservoir is in thermal equilibrium, we have

$$\begin{aligned}&< b_k(0) >_R = < b_k^\dagger(0) >_R = 0 & (16)\\ &< b_k(0) b_{k'}(0) >_R = 0 < b_k^\dagger(0) b_{k'}^\dagger(0) >_R = 0 & (17)\\ &< b_k^\dagger(0) b_{k'}(0) >_R = \bar{n}_{th}\delta_{kk'} & (18)\\ &< b_k(0) b_{k'}^\dagger(0) >_R = (\bar{n}_{th}+1)\delta_{kk'} & (19)\\ & & (20)\end{aligned}$$

where

$$\bar{n}_{th} = \langle n \rangle = \sum_n n\rho_{nn} = \frac{1}{\exp(\hbar\nu_k/k_B T) - 1}.$$

The noise operator for thermal reservoir is defined as,

$$\hat{F}_a(t) = -i\sum_k g_k \hat{b}_k(0) e^{-i(\nu_k-\omega)t}. \tag{21}$$

Here, in general, we call it as the *Langevin noise operator*, which has *zero means*

$$\langle \hat{F}_a(t)\rangle_R = \langle \hat{F}_a^\dagger(t)\rangle_R = 0, \tag{22}$$

but non-zero variances,

$$\begin{aligned}
\langle \hat{F}_a^\dagger(t)\hat{F}_a(t')\rangle_R &= \sum_k\sum_{k'} g_k g_{k'} \langle \hat{b}_k^\dagger \hat{b}_{k'}\rangle \exp[i(\nu_k-\omega)t - i(\nu_{k'}-\omega)t'], &(23)\\
&= \sum_k |g_k|^2 \bar{n}_k \exp[i(\nu_k-\omega)(t-t')], &(24)\\
&= \Gamma \bar{n}_{th}\delta(t-t'), &(25)
\end{aligned}$$

where $\bar{n}_{th} = \bar{n}_{(\nu_k-\omega)}$.

IV. FLUCTUATION-DISSIPATION THEORY

For a single-mode field interacting with thermal reservoir, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}(t) = -\frac{\Gamma}{2}\hat{a}(t) + \hat{F}_a(t). \tag{26}$$

The Langevin noise operators have *zero means* but non-zero variances,

$$\begin{aligned}
&\langle \hat{F}_a(t)\rangle_R = \langle \hat{F}_a^\dagger(t)\rangle_R = 0, &(27)\\
&\langle \hat{F}_a^\dagger(t)\hat{F}_a^\dagger(t')\rangle_R = \langle \hat{F}_a(t)\hat{F}_a(t')\rangle_R = 0, &(28)\\
&\langle \hat{F}_a^\dagger(t)\hat{F}_a(t')\rangle_R = \Gamma\bar{n}_{th}\delta(t-t'), &(29)\\
&\langle \hat{F}_a(t)\hat{F}_a^\dagger(t')\rangle_R = \Gamma(\bar{n}_{th}+1)\delta(t-t'). &(30)
\end{aligned}$$

The damping of the system is determined from the fluctuating forces of the reservoir, in other words, the fluctuations induced by the reservoir give rise to the dissipation in the system,

$$\Gamma = \frac{1}{\bar{n}_{th}}\int_{-\infty}^{\infty} \mathrm{d}t' \langle \hat{F}_a^\dagger(t)\hat{F}_a(t')\rangle_R. \tag{31}$$

This is one formulation of the fluctuation-dissipation theorem.