Note for *Quantum Optics*: Photon-atom interaction

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(Dated: Spring, 2021)

Reference:
Ch. 5, 6 in *"Quantum Optics,"* by M. Scully and M. Zubairy.
Ch. 5, 6 in *"Mesoscopic Quantum Optics,"* by Y. Yamamoto and A. Imamoglu.
Ch. 5 in *"The Quantum Theory of Light,"* by R. Loudon.
Ch. 10 in *"Quantum Optics,"* by D. Wall and G. Milburn.
Ch. 13 in *"Elements of Quantum Optics,"* by P. Meystre and M. Sargent III.

A. Classical picture for the polarization response

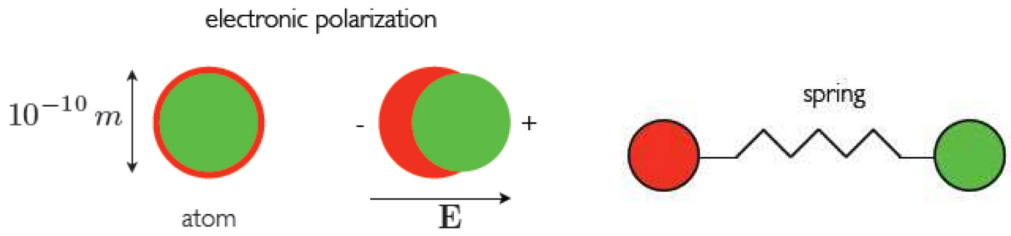

FIG. 1: Spring as the polarization response.

Consider the *bound electron cloud*, which is displaced by Δx when the electric field incidents. The corresponding atomic dipole is

$$p = q\Delta x. \tag{1}$$

For many atomic dipoles in a medium, one can sum up to a larger dipole, denoted by $\mathbf{P}$, as the dipole moment per volume,

$$\mathbf{P} = Np, \tag{2}$$

where N is the atomic number density.

I. LORENTZ MODEL

Consider the polarization response of the bound electron cloud as a spring, then, we can introduce a damped harmonic oscillator,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \sigma\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = \frac{q}{m}E(t). \tag{3}$$

For the harmonic wave, $E(t) = E_0\exp(-i\omega t)$ and $x(t) = x_0\exp(-i\omega t)$, we have

$$x(t) = \frac{1}{\omega_0^2 - i\omega\sigma - \omega^2}\frac{q}{m}E(t). \tag{4}$$

The corresponding atomic polarization is

$$\mathbf{P}(t) = Nq\Delta x(t) = \epsilon_0\chi\mathbf{E}(t), \tag{5}$$

where χ is a complex number with the form

$$\chi(\omega) = \frac{Nq^2}{\epsilon_0 m}\frac{1}{\omega_0^2 - i\omega\sigma - \omega^2}. \tag{6}$$

By redefining $\frac{Nq^2}{\epsilon_0 m\omega_0^2} \equiv \chi_0$ and $\Delta\nu \equiv \sigma$, we have

$$\chi(\omega) = \chi_0\frac{\omega_0^2}{\omega_0^2 - \omega^2 - i\omega\Delta\nu}, \tag{7}$$

which can be written as $\chi = \chi' + i\chi$", with the corresponding real and imaginary parts:

$$\chi'(\omega) = \chi_0\frac{\omega_0^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + (\omega\Delta\nu)^2}, \tag{8}$$

$$\chi"(\omega) = \chi_0\frac{\omega_0^2\omega\Delta\nu}{(\omega_0^2 - \omega^2)^2 + (\omega\Delta\nu)^2}. \tag{9}$$

It is known that the spectrum for the imaginary part has a *Lorentzian profile*.

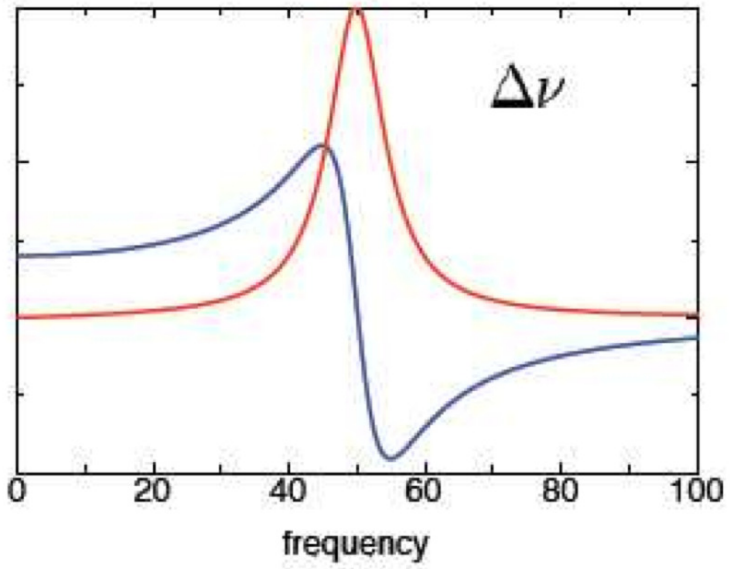

FIG. 2: Spectra for the real and imaginary parts of the polarization.

II. MINIMAL-COUPLING HAMILTONIAN

An electron of charge q and mass m interacting with an external EM field is described by the *minimal-coupling* Hamiltonian,

$$\hat{H} = \frac{1}{2m}[\mathbf{p} - e\mathbf{A}(r,t)]^2 + e\mathbf{U}(r,t), \tag{10}$$

where $\mathbf{p} = -i\hbar\nabla$ is the canonical momentum operator, $\mathbf{A}(r,t)$ and $\mathbf{U}(r,t)$ are the vector and scalar potentials of the external field, respectively. The electron is described by the wave function $\Psi(r,t)$; while the field is described by the vector and scalar potentials $\mathbf{A}$ and $\mathbf{U}$. In this way, the *photon* has been *derived* from the Schrödinger equation plus the local gauge invariance arguments. The gauge field theory leads to the unification of the weak and the electromagnetic interactions,

It should be noted that $A(r,t)$ and $U(r,t)$ are gauge-dependent potentials; while the gauge-independent quantities are the electric and magnetic fields:

$$\mathbf{E} = -\nabla\mathbf{U} - \frac{\partial\mathbf{A}}{\partial t}, \tag{11}$$

$$\mathbf{B} = \nabla\mathbf{A}. \tag{12}$$

A. Dipole approximation and $\mathbf{r}\cdot\mathbf{E}$ Hamiltonian

If the entire atom is immersed in a plane EM wave,

$$\mathbf{A}(r_0 + r, t) = \mathbf{A}(t)\exp[ik\cdot(r_0 + r)] \approx A(t)\exp(ik\cdot r_0), \tag{13}$$

where r_0 is the location of the electron. In this way, the dipole approximation, $\mathbf{A}(r,t) \approx \mathbf{A}(r_0,t)$, and the minimal-coupling Hamiltonian becomes,

$$\hat{H} = \frac{1}{2m}[\mathbf{p} - e\mathbf{A}(r_0,t)]^2 + e\mathbf{U}(r,t) + V(r), \tag{14}$$

where $V(r)$ is the atomic binding potential.

With the *radiation gauge*, we have

$$\mathbf{U}(r,t) = 0, \quad \text{and} \quad \nabla\cdot\mathbf{A}(r,t) = 0. \tag{15}$$

In this scenario, the minimal-coupling Hamiltonian becomes,

$$\hat{H} = \frac{\mathbf{p}^2}{2m} + V(r) + e\mathbf{r}\cdot\frac{\partial\mathbf{A}(r_0,t)}{\partial t}. \tag{16}$$

In terms of the gauge-independent field $\mathbf{E}$, the Hamiltonian for $\Psi(r,t)$ is

$$\hat{H} = \frac{\mathbf{p}^2}{2m} + V(r) - e\mathbf{r}\cdot\mathbf{E}(r_0,t), \tag{17}$$

$$\equiv \hat{H}_0 + \hat{H}_1. \tag{18}$$

Here, the Hamiltonian $\hat{H}_0$ corresponds to the system *without* interaction; while the Hamiltonian $\hat{H}_1$ corresponds to the system *under* interaction:

$$\hat{H}_0 = \frac{\mathbf{p}^2}{2m} + V(r), \tag{19}$$

$$\hat{H}_1 = -e\mathbf{r}\cdot\mathbf{E}(r_0,t). \tag{20}$$

This Hamiltonian is our starting point for the atom-field interaction.

III. INTERACTION OF A SINGLE TWO-LEVEL ATOM WITH A SINGLE-MODE FIELD

Consider the interaction of a single-mode radiation field of frequency ν, and a two-level atom with upper and lower level states $|a\rangle$ and $|b\rangle$, the unperturbed part of the Hamiltonian $\hat{H}_0$ has the eigenvalues $\hbar\omega_a$ and $\hbar\omega_b$ for the atom. The wave function of a two-level atom can be written in the form,

$$|\Psi t\rangle = C_a(t)|a\rangle + C_b(t)|b\rangle, \tag{21}$$

with the corresponding Schrödinger equation

$$i\hbar\frac{\partial\Psi(t)}{\partial t} = (\hat{H}_0 + \hat{H}_1)\Psi(t), \tag{22}$$

where

$$\begin{aligned}
\hat{H}_0 &= |a\rangle\langle a| + |b\rangle\langle b|)\hat{H}_0|a\rangle\langle a| + |b\rangle\langle b|) = \hbar\omega_a|a\rangle\langle a| + \hbar\omega_b|b\rangle\langle b|, &(23)\\
\hat{H}_1 &= -e\mathbf{r}\cdot\mathbf{E}(t) = -e(|a\rangle\langle a| + |b\rangle\langle b|)\mathbf{r}(|a\rangle\langle a| + |b\rangle\langle b|)\mathbf{E}, &(24)\\
&= -[\mathbf{p}_{ab}|a\rangle\langle b| + \mathbf{p}_{ba}|b\rangle\langle a|]\mathbf{E}(t), &(25)
\end{aligned}$$

with $\mathbf{p}_{ab} = \mathbf{p}^*_{ba} = e\langle a|\mathbf{r}|b\rangle$.

A. Semi-classical approach

Consider a *classical* single-mode field,

$$\mathbf{E}(t) = E_0\cos\nu t, \tag{26}$$

the corresponding equation of motion for the probability amplitude are

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}C_a &= -i\omega_a C_a + i\Omega_R\cos(\nu t)e^{-i\phi}C_b, &(27)\\
\frac{\mathrm{d}}{\mathrm{d}t}C_b &= -i\omega_b C_b + i\Omega_R\cos(\nu t)e^{+i\phi}C_a, &(28)
\end{aligned}$$

where $\Omega_R = \frac{|\mathbf{p}_{ab}|E_0}{\hbar}$ is the Rabi frequency which is proportional to the amplitude of the classical field, and ϕ is the phase of the dipole matrix element $\mathbf{p}_{ab} = |\mathbf{p}_{ab}|\exp(i\phi)$.

Define the slowly varying amplitudes,

$$c_a = C_a e^{i\omega_a t}, \quad \text{and} \quad c_b = C_b e^{i\omega_b t}, \tag{29}$$

then we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}c_a &= i\frac{\Omega_R}{2}e^{-i\phi}[e^{i(\omega-\nu)t} + e^{i(\omega+\nu)t}]c_b \approx i\frac{\Omega_R}{2}e^{-i\phi}e^{i(\omega-\nu)t}c_b, &(30)\\
\frac{\mathrm{d}}{\mathrm{d}t}c_b &= i\frac{\Omega_R}{2}e^{i\phi}[e^{-i(\omega-\nu)t} + e^{-i(\omega+\nu)t}]c_a \approx i\frac{\Omega_R}{2}e^{i\phi}e^{-i(\omega-\nu)t}c_a, &(31)
\end{aligned}$$

where $\omega = \omega_a - \omega_b$ is the atomic transition frequency. Here, we also apply the *rotating-wave approximation* by neglecting terms with $\exp[\pm i(\omega+\nu)t]$.

B. Rabi oscillation

The solutions for the coupled probability amplitude are

$$\begin{aligned}
c_a(t) &= \{[\cos(\frac{\Omega t}{2}) - i\frac{\Delta}{\Omega}\sin(\frac{\Omega t}{2})]c_a(0) + i\frac{\Omega_R}{\Omega}\sin(\frac{\Omega t}{2})e^{-i\phi}c_b(0)\}e^{i\Delta t/2}, &(32)\\
c_b(t) &= \{[\cos(\frac{\Omega t}{2}) + i\frac{\Delta}{\Omega}\sin(\frac{\Omega t}{2})]c_b(0) + i\frac{\Omega_R}{\Omega}\sin(\frac{\Omega t}{2})e^{i\phi}c_b(0)\}e^{-i\Delta t/2}, &(33)
\end{aligned}$$

where

$$\Delta = \omega - \nu, \quad \text{frequency detuning}, \tag{34}$$

$$\Omega = \sqrt{\Omega_R^2 + \Delta^2}. \tag{35}$$

It is easy to verify that

$$|c_a(t)|^2 + |c_b(t)|^2 = 1. \tag{36}$$

Moreover, assume that the atom is initially in the excited state $|a\rangle$, i.e $c_a(0) = 1$ and $c_b(0) = 0$, then the population inversion is

$$W(t) = |c_a(t)|^2 - |c_b(t)|^2 = \frac{\Delta^2 - \Omega_2^R}{\Omega^2} \sin^2(\frac{\Omega}{2}t) + \cos^2(\frac{\Omega}{2}t). \tag{37}$$

The population oscillates with the frequency $\Omega = \sqrt{\Omega_R^2 + \Delta^2}$. When the atom is at resonance with the incident field $\Delta = 0$, we get $\Omega = \Omega_R$, and

$$W(t) = \cos(\Omega_R t), \tag{38}$$

which gives the inversion oscillates between -1 and $+1$ at a frequency Ω_R.

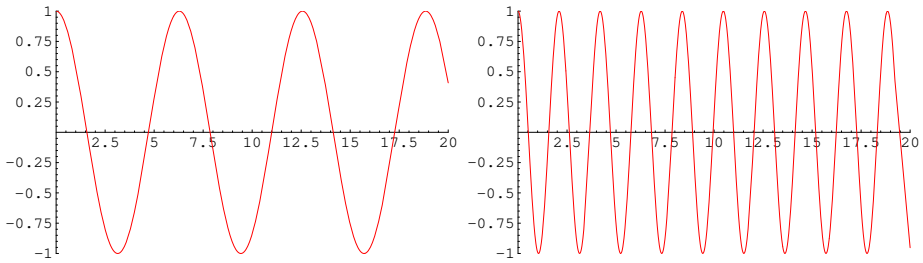

FIG. 3: Rabi oscillation for (Left) $\Omega_R = 1.0$, $\Delta = 0.0$; and (Right) $\Omega_R = 3.0$, $\Delta = 0.0$.

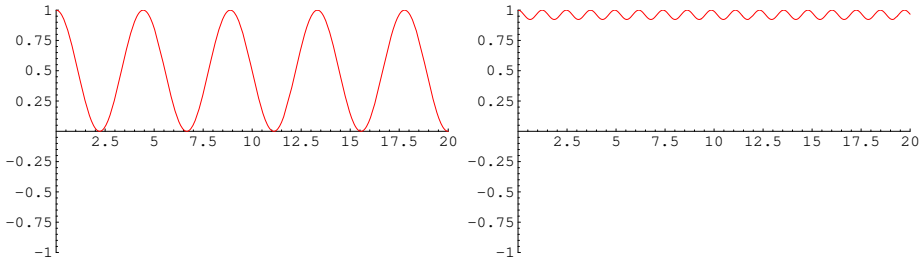

FIG. 4: Rabi oscillation for (Left) $\Omega_R = 1.0$, $\Delta = 1.0$; and (Right) $\Omega_R = 3.0$, $\Delta = 5.0$.

C. Interaction picture

Consider a system described by $|\Psi(t)\rangle$ evolving under the action of a Hamiltonian $\hat{H}(t)$ decomposable as,

$$\hat{H}(t) = \hat{H}_0 + \hat{H}_1(t),$$

where $\hat{H}_0$ is time-independent. If we define

$$|\Psi_I(t)\rangle = \exp(i\hat{H}_0 t/\hbar)|\Psi(t)\rangle, \tag{39}$$

then $|\Psi_I(t)\rangle$ evolves accords to

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\Psi_I(t)\rangle = \hat{H}_I(t)|\Psi_I(t)\rangle, \tag{40}$$

where

$$\hat{H}_I(t) = \exp(i\hat{H}_0 t/\hbar)\hat{H}_1(t)\exp(-i\hat{H}_0 t/\hbar). \tag{41}$$

The evolution is in the **interaction picture** generated by $\hat{H}_0$.

D. Dipole interaction with the interaction picture

For the dipolar interaction Hamiltonian,

$$\begin{aligned}
\hat{H}_0 &= \hbar\omega_a|a\rangle\langle a| + \hbar\omega_b|b\rangle\langle b|, &(42)\\
\hat{H}_1 &= -[\mathbf{p}_{ab}|a\rangle\langle b| + \mathbf{p}_{ba}|b\rangle\langle a|]\mathbf{E}(t) = -\hbar\Omega_R[e^{-i\phi}|a\rangle\langle b| + e^{i\phi}|b\rangle\langle a|]\cos\nu t, &(43)
\end{aligned}$$

where $\mathbf{p}_{ab} = \mathbf{p}^*_{ba} = e\langle a|\mathbf{r}|b\rangle$ and $\Omega_R = \frac{|\mathbf{p}_{ab}|E_0}{\hbar}$, the corresponding interaction picture Hamiltonian is

$$\begin{aligned}
\hat{H}_I(t) &= \exp(i\hat{H}_0 t/\hbar)\hat{H}_1(t)\exp(-i\hat{H}_0 t/\hbar), &(44)\\
&= -\frac{\hbar}{2}\Omega_R[e^{-i\phi}|a\rangle\langle b|e^{i(\omega-\nu)t} + e^{i\phi}|b\rangle\langle a|e^{-i(\omega-\nu)t} &(45)\\
&+ e^{-i\phi}|a\rangle\langle b|e^{i(\omega+\nu)t} + e^{i\phi}|b\rangle\langle a|e^{-i(\omega+\nu)t}]. &(46)
\end{aligned}$$

Furthermore, in the *rotating-wave approximation*, we have

$$\hat{H}_I(t) = -\frac{\hbar}{2}\Omega_R[e^{-i\phi}|a\rangle\langle b|e^{i(\omega-\nu)t} + e^{i\phi}|b\rangle\langle a|e^{-i(\omega-\nu)t}]. \tag{47}$$

On resonance $\omega - \nu = 0$, this interaction Hamiltonian becomes

$$\hat{H}_I(t) = -\frac{\hbar}{2}\Omega_R[e^{-i\phi}|a\rangle\langle b| + e^{i\phi}|b\rangle\langle a|]. \tag{48}$$

The time-evolution operator in the interaction picture $\hat{U}_I(t)$ is

$$\begin{aligned}
\hat{U}_I(t) &= \overleftarrow{T}\exp[-\frac{i}{\hbar}\int_{t_0}^{t}\mathrm{d}\tau\hat{H}_I(\tau)], &(49)\\
&= \cos(\frac{\Omega_R t}{2})(|a\rangle\langle a| + |b\rangle\langle b|) + i\sin(\frac{\Omega_R t}{2})(e^{-i\phi}|a\rangle\langle b| + e^{i\phi}|b\rangle\langle a|). &(50)
\end{aligned}$$

If the atom is initially in the excited state $|\Psi(t=0)\rangle = |a\rangle$, then

$$\begin{aligned}
|\Psi(t)\rangle &= \hat{U}_I(t)|a\rangle, &(51)\\
&= \cos(\frac{\Omega_R t}{2})|a\rangle + i\sin(\frac{\Omega_R t}{2})e^{i\phi}|b\rangle. &(52)
\end{aligned}$$

IV. DENSITY OPERATOR

For the quantum mechanical description, if we know that the system is in state $|\psi\rangle$, then an operator $\hat{O}$ has the expectation value,

$$\langle\hat{O}\rangle_{\rm qm} = \langle\psi|\hat{O}|\psi\rangle. \tag{53}$$

But, typically, we do not know that we are in state $|\psi\rangle$, then an ensemble average must be performed,

$$\langle\langle\hat{O}\rangle_{\rm qm}\rangle_{\rm ensemble} = \sum_{\psi} P_{\psi}\langle\psi|\hat{O}|\psi\rangle, \tag{54}$$

where the P_ψ is the probability of being in the state $|\psi\rangle$ and we introduce a density operator,

$$\hat{\rho} = \sum_{\psi} P_{\psi}|\psi\rangle\langle\psi|. \tag{55}$$

The expectation value of any operator $\hat{O}$ is given by,

$$\langle\hat{O}\rangle_{\rm qm} = {\rm Tr}[\hat{\rho}\hat{O}], \tag{56}$$

where Tr stands for trace.

A. Equation of motion for the density matrix

In the Schrödiner picture,

$$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = \hat{H}|\Psi\rangle, \tag{57}$$

then we have

$$i\hbar\frac{\partial}{\partial t}\hat{\rho} = \hat{H}\hat{\rho} - \hat{\rho}\hat{H} = [\hat{H},\hat{\rho}], \tag{58}$$

which is called the Liouville or Von Neumann equation of motion for the density matrix. Using density operator instead of a specific state vector can give statistical as well as quantum mechanical information.

Here, note the Heisenberg equation is, $i\hbar\frac{\rm d}{{\rm d}t}\hat{A}(t) = [\hat{A},\hat{H}(t)]$.

B. Decay processes in the density matrix

The excited atomic levels can also decay due to spontaneous emission or collisions and other phenomena. In this case, the decay rates can be incorporated by a relaxation matrix Γ,

$$\langle n|\Gamma|m\rangle = \gamma_n\delta_{nm}. \tag{59}$$

Then the density matrix equation of motion becomes,

$$\frac{\partial}{\partial t}\hat{\rho} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] - \frac{1}{2}\{\Gamma,\hat{\rho}\}, \tag{60}$$

where $\{\Gamma,\hat{\rho}\} = \Gamma\hat{\rho} + \hat{\rho}\Gamma$. And the ijth matrix element is,

$$\frac{\partial}{\partial t}\rho_{ij} = -\frac{i}{\hbar}\sum_k(H_{ik}\rho_{kj} - \rho_{ik}H_{kj}) - \frac{1}{2}\sum_k(\Gamma_{ik}\rho_{kj} + \rho_{ik}\Gamma_{kj}). \tag{61}$$

As an example, consider a two-level atom with upper and lower level states $|a\rangle$ and $|b\rangle$,

$$|\Psi(t)\rangle = C_a(t)|a\rangle + C_b(t)|b\rangle. \tag{62}$$

The corresponding density matrix operator is

$$\hat{\rho} = |\Psi\rangle\langle\Psi| = |C_a|^2|a\rangle\langle a| + C_aC_b^*|a\rangle\langle b| + C_bC_a^*|b\rangle\langle a| + |C_b|^2|b\rangle\langle b|, \quad (63)$$
$$= \rho_{aa}|a\rangle\langle a| + \rho_{ab}|a\rangle\langle b| + \rho_{ba}|b\rangle\langle a| + \rho_{bb}|b\rangle\langle b|. \quad (64)$$

Here, the diagonal elements, ρ_{aa} and ρ_{bb}, are the probabilities in the upper and lower states; while the off-diagonal elements, ρ_{ab} and ρ_{ba}, are the atomic polarizations. From the equation of motion for the two-level atom

$$\frac{\partial}{\partial t}\hat{\rho} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] - \frac{1}{2}\{\Gamma, \hat{\rho}\},$$

we have

$$\frac{\partial}{\partial t}\rho_{aa} = \frac{i}{\hbar}[\mathbf{p}_{ab}\mathbf{E}\rho_{ba} - \text{c.c}] - \gamma_a\rho_{aa}, \quad (65)$$
$$\frac{\partial}{\partial t}\rho_{bb} = -\frac{i}{\hbar}[\mathbf{p}_{ab}\mathbf{E}\rho_{ba} - \text{c.c}] - \gamma_b\rho_{bb}, \quad (66)$$
$$\frac{\partial}{\partial t}\rho_{ab} = -\frac{i}{\hbar}\mathbf{p}_{ab}\mathbf{E}(\rho_{aa} - \rho_{bb}) - (i\omega + \frac{\gamma_a + \gamma_b}{2})\rho_{ab}. \quad (67)$$

The physical interpretation of the elements of the density matrix allows us to include terms associated wither certain processes. For example, one can have *elastic* collision between atoms in a gas, during an atom-atom collision the energy levels experience random Stark shifts,

$$\frac{\partial}{\partial t}\rho_{ab} = -[i\omega + i\delta\omega(t) + \gamma_{ab}]\rho_{ab}, \quad (68)$$

after integration,

$$\rho_{ab} = \exp[-(i\omega + \gamma_{ab})t - i\int_0^t \mathrm{d}t'\delta\omega(t')]\rho_{ab}(0). \quad (69)$$

For a zero-mean random process, we have $\langle\delta\omega(t)\rangle = 0$, but the variations in $\delta\omega(t)$ are usually rapid compared to other changes which occur in times like γ_{ph},

$$\langle\delta\omega(t)\delta\omega(t')\rangle = 2\gamma_{\text{ph}}\delta(t - t'). \quad (70)$$

Assume that $\delta\omega(t)$ is described by a Gaussian random process, then

$$\langle\exp[-i\int_0^t \mathrm{d}t'\delta\omega(t')]\rangle = \exp[-\gamma_{\text{ph}}t], \quad (71)$$

which gives for the average of ρ_{ab},

$$\rho_{ab} = \exp[-(i\omega + \gamma_{ab} + \gamma_{\text{ph}})t]\rho_{ab}(0). \quad (72)$$

For the process of atom-atom collisions,

$$\frac{\partial}{\partial t}\rho_{ab} = -[i\omega + \gamma]\rho_{ab} + \frac{i}{\hbar}\mathbf{p}_{ab}\mathbf{E}(\rho_{aa} - \rho_{bb}), \quad (73)$$

where $\gamma = \gamma_{ab} + \gamma_{\text{ph}}$ is the new decay rate.

C. Maxwell-Schrödinger equations

The equations for the two-level atomic medium coupled to the field **E** are

$$\frac{\partial}{\partial t}\rho_{aa} = \frac{i}{\hbar}[\mathbf{p}_{ab}\mathbf{E}\rho_{ba} - \text{c.c}] - \gamma_a\rho_{aa}, \tag{74}$$

$$\frac{\partial}{\partial t}\rho_{bb} = -\frac{i}{\hbar}[\mathbf{p}_{ab}\mathbf{E}\rho_{ba} - \text{c.c}] - \gamma_b\rho_{bb}, \tag{75}$$

$$\frac{\partial}{\partial t}\rho_{ab} = -\frac{i}{\hbar}\mathbf{p}_{ab}\mathbf{E}(\rho_{aa} - \rho_{bb}) - (i\omega + \frac{\gamma_a + \gamma_b}{2})\rho_{ab}. \tag{76}$$

The condition of *self-consistency* requires that the equation of motion for the field **E** is driven by the atomic population matrix elements. Remember that the field is described by the Maxwell's equation,

$$\nabla \cdot \mathbf{D} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{77}$$

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = J + \frac{\partial \mathbf{D}}{\partial t}. \tag{78}$$

V. JAYNES-CUMMINGS HAMILTONIAN

In the dipole approximation, the semi-classical Hamiltonian is

$$\hat{H}_0 = \hbar\omega_a|a\rangle\langle a| + \hbar\omega_b|b\rangle\langle b|, \tag{79}$$
$$\hat{H}_1 = -(\mathbf{p}_{ab}|a\rangle\langle b| + \mathbf{p}_{ba}|a\rangle\langle b|)\mathbf{E}(t). \tag{80}$$

By including the quantized field,

$$\hat{H} = \hat{H}_A + \hat{H}_F - e\mathrm{r}\cdot\mathrm{E}, \tag{81}$$
$$= \sum_i \hbar\omega_i\hat{\sigma}_{ii} + \sum_k \hbar\nu_k(\hat{a}_k^\dagger\hat{a}_k + \frac{1}{2}) - \sum_{i,j}\mathbf{P}_{ij}\hat{\sigma}_{ij}\sum_k E_k(\hat{a}_k + \hat{a}_k^\dagger), \tag{82}$$
$$= \hbar\omega_i\hat{\sigma}_{ii} + \sum_k \hbar\nu_k(\hat{a}_k^\dagger\hat{a}_k + \frac{1}{2}) + \hbar\sum_{i,j}\sum_k g_k^{ij}\hat{\sigma}_{ij}(\hat{a}_k + \hat{a}_k^\dagger), \tag{83}$$

where

$$g_k^{ij} = -\frac{\mathbf{P}_{ij}\cdot E_k}{\hbar} \tag{84}$$

is the coupling constant.

For a two-level atom, $\mathbf{P}_{ab} = \mathbf{P}_{ba}$, we have $g_k = g_k^{ab} = g_k^{ba}$, then

$$\hat{H} = \hbar\omega_a\hat{\sigma}_{aa} + \hbar\omega_b\hat{\sigma}_{bb} + \sum_k \hbar\nu_k(\hat{a}_k^\dagger\hat{a}_k + \hbar\frac{1}{2}) + \hbar\sum_k g_k(\hat{\sigma}_{ab} + \hat{\sigma}_{ba})(\hat{a}_k + \hat{a}_k^\dagger). \tag{85}$$

By defining new operators,

$$\hat{\sigma}_z = \hat{\sigma}_{aa} - \hat{\sigma}_{bb} = |a\rangle\langle a| - |b\rangle\langle b|, \tag{86}$$
$$\hat{\sigma}_+ = \hat{\sigma}_{ab} = |a\rangle\langle b|, \tag{87}$$
$$\hat{\sigma}_- = \hat{\sigma}_{ba} = |b\rangle\langle a|, \tag{88}$$

and the new energy level

$$\hbar\omega_a\hat{\sigma}_{aa} + \hbar\omega_b\hat{\sigma}_{bb} = \frac{1}{2}\hbar\omega\hat{\sigma}_z + \frac{1}{2}(\omega_a + \omega_b), \tag{89}$$

where $\omega = \omega_a - \omega_b$, the Hamiltonian for a two-level atom interaction with quantized fields becomes

$$\hat{H} = \frac{1}{2}\hbar\omega\hat{\sigma}_z + \sum_k \hbar\nu_k(\hat{a}_k^\dagger\hat{a}_k + \frac{1}{2}) + \hbar\sum_k g_k(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a}_k + \hat{a}_k^\dagger), \tag{90}$$

where the atomic operators satisfy the spin-1/2 algebra of the Pauli matrices, *i.e.*,

$$[\hat{\sigma}_-, \hat{\sigma}_+] = -\hat{\sigma}_z, \quad \text{and} \quad [\hat{\sigma}_-, \hat{\sigma}_z] = 2\hat{\sigma}_-. \tag{91}$$

In the *rotating-wave approximation*, we drop terms $\hat{a}_k\hat{\sigma}_-$ and $\hat{a}_k^\dagger\hat{\sigma}_+$, then we have **Jaynes-Cummings** Hamiltonian

$$\hat{H} = \frac{1}{2}\hbar\omega\hat{\sigma}_z + \sum_k \hbar\nu_k(\hat{a}_k^\dagger\hat{a}_k + \frac{1}{2}) + \hbar\sum_k g_k(\hat{\sigma}_+\hat{a}_k + \hat{a}_k^\dagger\hat{\sigma}_-), \tag{92}$$

A. Interaction of a single two-level atom with a single-mode field

The interaction Hamiltonian for the **Jaynes-Cummings** model is,

$$\hat{V} = \exp[i\hat{H}_0 t/\hbar]\hat{H}_1\exp[-i\hat{H}_0 t/\hbar], \tag{93}$$
$$= \hbar g(\hat{\sigma}_+\hat{a}e^{i\Delta t} + \hat{a}^\dagger\hat{\sigma}_- e^{-i\Delta t}), \tag{94}$$

where $\Delta = \omega - \nu$. Based this , the the equation of motion for the state $|\Psi$ is

$$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = \hat{V}|\Psi\rangle, \tag{95}$$

where the state $|\Psi\rangle$ is the superposition of

$$|\Psi(t)\rangle = \sum_n [c_{a,n}(t)|a,n\rangle + b_{a,n}(t)|b,n\rangle]. \tag{96}$$

It can be seen that only transitions between the states $|a,n\rangle$ and $|b,n+1\rangle$ allowed, for which the dynamics follows

$$\frac{\mathrm{d}}{\mathrm{d}t}c_{a,n} = -ig\sqrt{n+1}e^{i\Delta t}c_{b,n+1}, \tag{97}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}c_{b,n+1} = -ig\sqrt{n+1}e^{-i\Delta t}c_{a,n}. \tag{98}$$

Here, we give a comparison with respect to the semi-classical equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}c_a = i\frac{\Omega_R}{2}e^{-i\phi}e^{i(\omega-\nu)t}c_b, \tag{99}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}c_b = i\frac{\Omega_R}{2}e^{i\phi}e^{-i(\omega-\nu)t}c_a. \tag{100}$$

For the initially excited state, $c_{a,n}(0) = c_n(0)$ and $c_{b,n+1}(0) = 0$, and here $c_n(0)$ is the probability amplitude for the field along, the solutions are

$$c_{a,n}(t) = c_n(0)[\cos(\frac{\Omega_n t}{2}) - \frac{i\Delta}{\Omega_n}\sin(\frac{\Omega_n t}{2})]e^{i\Delta t/2}, \tag{101}$$

$$c_{b,n+1}(t) = -c_n(0)\frac{2ig\sqrt{n+1}}{\Omega_n}\sin(\frac{\Omega_n t}{2})e^{i\Delta t/2}. \tag{102}$$

Here, the Rabi frequency is

$$\Omega_n = \Delta^2 + 4g^2(n+1), \tag{103}$$

which is proportional to the photon number of the field. Moreover, the probability $p(n)$ that there are n photons in the field at time t is,

$$p(n) = |c_{a,n}(t)|^2 + |c_{b,n}(t)|^2, \tag{104}$$

$$= |c_n(0)|^2[\cos^2(\frac{\Omega_n t}{2}) + (\frac{\Delta}{\Omega_n})^2\sin^2(\frac{\Omega_n t}{2})] + |c_{n-1}(0)|^2(\frac{4g^2 n}{\Omega_{n-1}^2})\sin^2(\frac{\Omega_{n-1}t}{2})]. \tag{105}$$

B. Revival and Collapse of the population inversion

Take n photons in the field at time $t = 0$ with a coherent state as an example, that is

$$|c_n(0)|^2 = \frac{\langle n\rangle^n e^{-\langle n\rangle}}{n!}. \tag{106}$$

The corresponding population inversion is,

$$W(t) = \sum_n |c_{a,n}(t)|^2 - |c_{b,n}(t)|^2 = \sum_0^\infty |c_n(0)|^2[\frac{\Delta^2}{\Omega_n^2} + \frac{4g^2(n+1)}{\Omega_n^2}\cos(\Omega_n t)]. \tag{107}$$

Each term in the summation represents Rabi oscillation for a definite value of n. At the initial time $t = 0$, the atom is prepared in a definite state and therefore all the terms in the summation are *correlated*. As times increases, the Rabi oscillations associated with different frequent excitations have different frequencies and there fore become *uncorrelated*, leading to a collapse of inversion. As time is further increased, the correlation is restored and revival occurs.

In the semi-classical theory, the population inversion evolves with sinusoidal Rabi oscillations, and collapses to zero when on resonance. For the quantized fields, the collapse and revival of inversion is repeated with increasing time, but the amplitude of Rabi oscillations decreasing and the time duration in which revival takes place increasing,

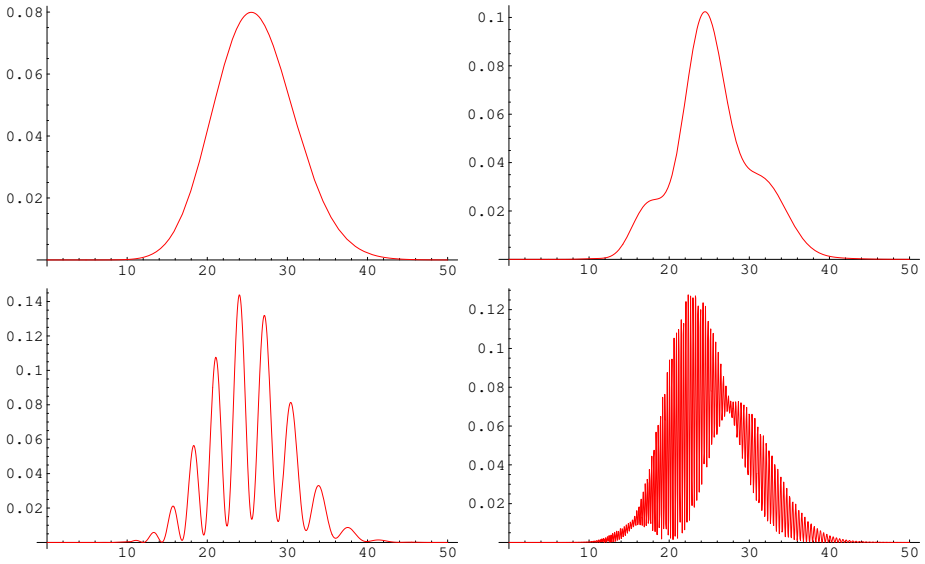

FIG. 5: Time evolution for a coherent state interacting under the Janey-Cummings Hamiltonian, with the parameters: $\Delta = 0$ and $\langle n \rangle = 25$. Here, the time are choose as $gt = 0$, $gt = 3.0$, $gt = 10$, and $gt = 100$, from Left to Right (Up to Down), respectively.

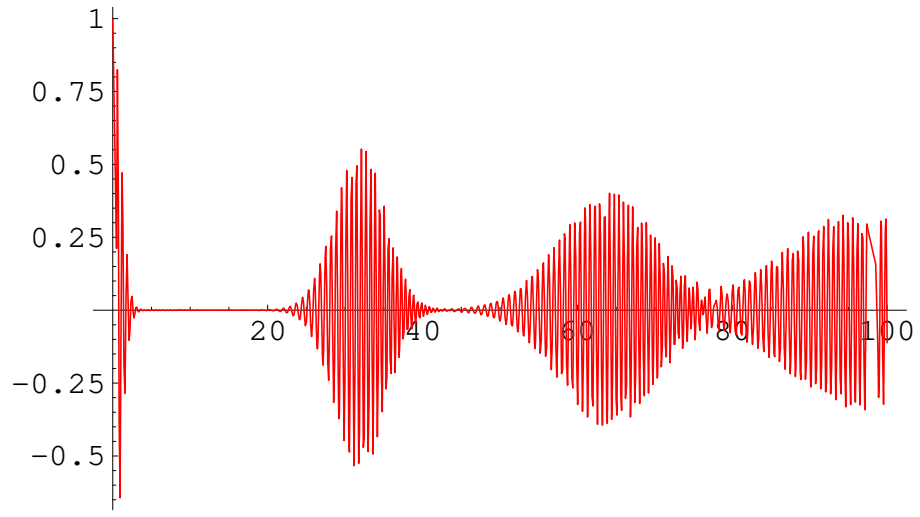

FIG. 6: Temporal dynamics fo the population inversion, $W(t)$, with the parameters $\Delta = 0$ and $\langle n \rangle = 25$.

C. Vacuum Rabi Oscillation

The revivals occur only because of the quantized photon distribution. However, for a continuous photon distribution, like a classical random field, there is only a collapse but no revivals. A comparison between the Fourier transform and Discrete Fourier transform can be illustrated. Nevertheless, even for initial vacuum field, $|c_n(0)|^2 = \delta_{n0}$, the inversion is

$$W(t) = \frac{1}{\Delta^2 + 4g^2}[\Delta^2 + 4g^2 \cos(\sqrt{\Delta^2 + 4g^2}t)]. \tag{108}$$

That means the Rabi oscillation takes place due to the vacuum state. The transition from the upper level to the lower level in the *vacuum* becomes possible due to spontaneous emission.

VI. COLLECTIVE ANGULAR MOMENTUM OPERATORS

For a two-level atom, one can use Pauli spin operator to describe,

$$\hat{s} = \frac{1}{2}\hbar\sigma, \tag{109}$$

where

$$\begin{aligned} \hat{\sigma}_z &= |a\rangle\langle a| - |b\rangle\langle b|, \quad \hat{\sigma}_+ = |a\rangle\langle b|, \quad \hat{\sigma}_- = |b\rangle\langle a|, \\ \hat{\sigma}_x &= |a\rangle\langle b| + |b\rangle\langle a|, \quad \text{and} \quad \hat{\sigma}_y = -i(|a\rangle\langle b| - |b\rangle\langle a|). \end{aligned}$$

Now, for an assembly of N two-level atoms, the corresponding Hilbert space is spanned by the set of 2^N product states,

$$|\Phi\rangle = \prod_{n=1}^{N} |\Psi_n\rangle. \tag{110}$$

Then, we can define the collective angular momentum operators,

$$\hat{J}_\mu = \frac{1}{2}\hat{\sigma}_{n\mu}, \quad (\mu = x, y, z). \tag{111}$$

A. Analogs between $\hat{J}$ and $\hat{a}$, $\hat{a}^\dagger$

The analogies between the free-field quantization, $\hat{a}$ and $\hat{a}^\dagger$, and the free atom quantization are,

$$[\hat{J}_x, \hat{J}_y] = i\hat{J}_z \;\leftrightarrow\; [\hat{q}, \hat{p}] = i\hbar, \tag{112}$$

$$\hat{J}_- = \hat{J}_x - i\hat{J}_y \;\leftrightarrow\; \hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + i\hat{p}), \tag{113}$$

$$\hat{J}_+ = \hat{J}_x + i\hat{J}_y \;\leftrightarrow\; \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} - i\hat{p}), \tag{114}$$

$$\hat{J}_z = \frac{1}{2}(\hat{J}_+\hat{J}_- \hat{J}_-\hat{J}_+) \;\leftrightarrow\; \hat{n} = \hat{a}^\dagger\hat{a}. \tag{115}$$

The related commutation relations are,

$$[\hat{J}_-, \hat{J}_+] = -2\hat{J}_z \;\leftrightarrow\; [\hat{a}, \hat{a}^\dagger] = 1, \tag{116}$$

$$[\hat{J}_-, \hat{J}_z] = \hat{J}_- \;\leftrightarrow\; [\hat{a}, \hat{n}] = \hat{a}, \tag{117}$$

$$[\hat{J}_+, \hat{J}_z] = -\hat{J}_+ \;\leftrightarrow\; [\hat{a}^\dagger, \hat{n}] = -\hat{a}^\dagger. \tag{118}$$

When all the atoms are in the ground state, the eigenvalue of $\hat{J}_z$ is $-J = -\frac{N}{2}$, and the commutation relation is reduced to a *bosonlike* one, $[\hat{J}_-, \hat{J}_+] = N \leftrightarrow [\hat{a}, \hat{a}^\dagger] = 1$.

B. Angular momentum eigenstates (Dicke states)

The Dicke states are defined as the simultaneous eigenstates of the Hermitian operators $\hat{J}_z$ and $\hat{J}^2$, i.e.

$$\hat{J}_z|M, J\rangle = M|M, J\rangle, \quad \text{and} \quad , \hat{J}^2|M, J\rangle = J(J+1)|M, J\rangle, \tag{119}$$

where $(M = -J, -J+1, \ldots, J-1, J)$ and

$$\hat{J}_+|M, J\rangle = \sqrt{J(J+1) - M(M+1)}|M+1, J\rangle \;\leftrightarrow\; \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \tag{120}$$

$$\hat{J}_-|M, J\rangle = \sqrt{J(J+1) - M(M-1)}|M-1, J\rangle \;\leftrightarrow\; \hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \tag{121}$$

$$\hat{J}_-|-J, J\rangle = 0 \;\leftrightarrow\; \hat{a}|0\rangle = 0, \tag{122}$$

$$|M, J\rangle = \frac{1}{(M+J)!}\begin{pmatrix} 2J \\ M+J \end{pmatrix}^{-1/2} \hat{J}_+^{(M+J)}|-J, J\rangle \;\leftrightarrow\; |n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle. \tag{123}$$

The Dicke states is the counterpart of the Fock state, the state $|M, J\rangle$ denotes an atomic ensemble where exactly $J + M$ atoms are in the excited state out of $N = 2J$ atoms. The Dicke states $|-J, J\rangle$ corresponds to the case in which all the atoms are in the ground state, $J = N/2$; while the Dicke states $|-J+1, J\rangle$ corresponds to the case in which only one atom is in the excited state; and the Dicke states $|J, J\rangle$ corresponds to the case in which all the atom are in the excited state.

C. Interaction between N two-level atoms and a single-mode field

The total Hamiltonian for N two-level atoms with a single-mode field is,

$$\hat{H} = \frac{1}{2}\hbar\omega\hat{J}_z + \hbar\nu(\hat{a}^\dagger\hat{a} + \frac{1}{2}) + \hbar g(\hat{J}_+\hat{a} + \hat{a}^\dagger\hat{J}_-). \tag{124}$$

VII. SPONTANEOUS EMISSION OF A TWO-LEVEL ATOM

The interaction Hamiltonian, in the rotating-wave approximation, for a two-level atom is,

$$\hat{V} = \hbar \sum_k (g_k(r_0)^* \hat{\sigma}_+ \hat{a}_k e^{i(\omega-\nu_k)t} + g_k(r_0)\hat{a}_k^\dagger \hat{\sigma}_- e^{-i(\omega-\nu_k)t}), \tag{125}$$

where $g_k(r_0) = g_k \exp(-ik \cdot r_0)$ is the spatial dependent coupling coefficient. Assume at $t = 0$ the atom is in the excited state $|a\rangle$ and the field modes are in the vacuum state $|0\rangle$,

$$|\Psi(t)\rangle = c_a(t)|a, 0\rangle + \sum_k c_{b,k}|b, 1_k\rangle, \tag{126}$$

with $c_a(0) = 1$ and $c_{b,k}(0) = 0$. In the interaction picture, $|\dot{\Psi}(t)\rangle = -\frac{i}{\hbar}|\Psi(t)\rangle$, we have

$$\dot{c}_a(t) = -i \sum_k g_k^*(r_0) e^{i(\omega-\nu_k)t} c_{b,k}(t), \tag{127}$$

$$\dot{c}_b(t) = -i g_k(r_0) e^{-i(\omega-\nu_k)t} c_a(t). \tag{128}$$

A. Weisskopf-Wigner theory of spontaneous emission

The exact solutions are

$$c_b(t) = -i g_k(r_0) \int_0^t \mathrm{d}t' e^{-i(\omega-\nu_k)t'} c_a(t'), \tag{129}$$

$$\dot{c}_a(t) = -\sum_k |g_k(r_0)|^2 \int_0^t \mathrm{d}t' e^{i(\omega-\nu_k)(t-t')} c_a(t'). \tag{130}$$

Now, we assume that the filed modes are closely spaced in frequency,

$$\sum_k \to 2\frac{V}{(2\pi)^3} \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \sin\theta \int_0^{\infty} \mathrm{d}k k^2, \tag{131}$$

where V is the quantization volume. Then the coupling coefficient becomes,

$$|g_k(r_0)|^2 = |\frac{\mathbf{P} \cdot E_k}{\hbar}|^2 = \frac{\nu_k}{2\hbar\epsilon_0 V} \mathbf{P}_{ab}^2 \cos^2\theta, \tag{132}$$

where θ is the angle between the atomic dipole moment $\mathbf{P}_{ab}$ and the electric field polarization vector $\hat{\epsilon}_k$, i.e. $\hat{E}_k(r,t) = \hat{\epsilon}_k(\frac{\hbar\nu_k}{\epsilon_0 V})^{1/2}[\hat{a}_k + \hat{a}_k^\dagger]$.

Under this assumption, the equation for $c_a(t)$ becomes

$$\dot{c}_a(t) = -\frac{4\mathbf{P}_{ab}^2}{(2\pi)^2 6\hbar\epsilon_0 c^3} \int_0^{\infty} \mathrm{d}\nu_k \int_0^t \mathrm{d}t' \nu_k^3 e^{i(\omega-\nu_k)(t-t')} c_a(t'), \tag{133}$$

where we have use $k = \nu_k/c$.

For most of the optical problems, ν_k varies little around the atomic transition frequency ω. Then, we can safely replace ν_k^3 by ω^3 and the lower limit in the ν_k integration by $-\infty$, that is

$$\dot{c}_a(t) = -\frac{4\mathbf{P}_{ab}^2\omega^3}{(2\pi)^2 6\hbar\epsilon_0 c^3} \int_{-\infty}^{\infty} \mathrm{d}\nu_k \int_0^t \mathrm{d}t' e^{i(\omega-\nu_k)(t-t')} c_a(t'), \tag{134}$$

$$= -\frac{4\mathbf{P}_{ab}^2\omega^3}{(2\pi)^2 6\hbar\epsilon_0 c^3} \int_0^t \mathrm{d}t' 2\pi\delta(t-t') c_a(t'), \tag{135}$$

$$\equiv -\frac{\Gamma}{2} c_a(t), \tag{136}$$

where $\Gamma = \frac{4\mathbf{P}_{ab}^2\omega^3}{12\pi^2\hbar\epsilon_0 c^3}$ is the decay rate of the excited state.