Note for *Quantum Optics*: Quantization of EM-fields

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[Reference:]

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I. NORMAL VARIABLES IN CLASSICAL ELECTRODYNAMICS

A. Maxwell-Lorentz Equations

The basic equations in classical electrodynamics are Maxwell's equations:

$$\nabla \cdot \vec{E}(\vec{r},t) = \frac{1}{\epsilon_0}\,\rho(\vec{r},t), \tag{1}$$
$$\nabla \cdot \vec{B}(\vec{r},t) = 0, \tag{2}$$
$$\nabla \times \vec{E}(\vec{r},t) = -\frac{\partial}{\partial t}\vec{B}(\vec{r},t), \tag{3}$$
$$\nabla \times \vec{B}(\vec{r},t) = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}(\vec{r},t) + \frac{1}{\epsilon_0 c^2}\vec{J}(\vec{r},t), \tag{4}$$

with the electric field $\vec{E}(\vec{r},t)$, the magnetic field $\vec{B}(\vec{r},t)$, the charge density $\rho(\vec{r},t)$, and the current $\vec{J}(\vec{r},t)$. Along with the Lorentz equation is used for the influence of electric and magnetic forces, *i.e.*, for the dynamics of each particle α, having the mass m_α, charge q_α, at the position $\vec{r}_\alpha$ with the velocity $\vec{v}_\alpha$:

$$m_\alpha \frac{d^2}{d\,t^2}\vec{r}_\alpha = q_\alpha \cdot \vec{E}(\vec{r}_\alpha(t),t) + q_\alpha \cdot \vec{v}_\alpha \times \vec{B}(\vec{r}_\alpha(t),t). \tag{5}$$

Here the notation $\vec{r}$ is used to denote a vector; r is used for a scalar, and $\hat{r}$ is used for a unit vector $\hat{r} \equiv \vec{r}/|r| = \vec{r}/r$.

B. Equation of Continuity

Based on the vector identities for arbitrary vector field $\vec{A}$ and scalar field ϕ:

$$\nabla \cdot (\nabla \times \vec{A}) = 0, \tag{6}$$
$$\nabla \times (\nabla \phi) = 0, \tag{7}$$

and with Eqs. (1) and (4), on can show the *equation of continuity* in Maxwell's equation,

$$\frac{\partial}{\partial t}\rho(\vec{r},t) + \nabla \cdot \vec{J}(\vec{r},t) = 0. \tag{8}$$

C. Vector identities

Some vector identities we would use:

$$\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}, \tag{9}$$
$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{D} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C}). \tag{10}$$

D. Vector and Scalar potentials

Based on the vector identities, the fields $\vec{E}$ and $\vec{B}$ in Eqs. (2) and (3) can always be written in the form:

$$\begin{aligned} \vec{B}(\vec{r},t) &= \nabla \times \vec{A}(\vec{r},t), & (11)\\ \vec{E}(\vec{r},t) &= -\frac{\partial}{\partial t}\vec{A}(\vec{r},t) - \nabla U, & (12)\end{aligned}$$

where $\vec{A}$ is called the *vector potential*, and U is called the *scalar potential*.

Substituting the fields $\vec{E}$ and $\vec{B}$ shown in Eqs. (11) and (12) back to the Maxwell's equations, we have the equations of motions for the vector and scalar potentials,

$$\begin{aligned} \triangle U(\vec{r},t) &= -\frac{1}{\epsilon_0}\rho(\vec{r},t) - \nabla\cdot\frac{\partial}{\partial t}\vec{A}(\vec{r},t), & (13)\\ \Box\vec{A}(\vec{r},t), &= \frac{1}{\epsilon_0\, c^2}\vec{J}(\vec{r},t) - \nabla[\nabla\cdot\vec{A}(\vec{r},t) + \frac{1}{c^2}\frac{\partial}{\partial t}U(\vec{r},t)], & (14)\end{aligned}$$

where the short-handed notations are

$$\triangle \equiv \nabla^2, \qquad (15)$$

$$\Box \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \triangle. \qquad (16)$$

E. Gauge Invariance

The fields $\vec{E}$ and $\vec{B}$ are invariants under the following *gauge transformation*:

$$\vec{A}(\vec{r},t) \to \vec{A}'(\vec{r},t) = \vec{A}(\vec{r},t) + \nabla F(\vec{r},t), \qquad (17)$$

$$U(\vec{r},t) \to U'(\vec{r},t) = U(\vec{r},t) - \frac{\partial}{\partial t}F(\vec{r},t), \qquad (18)$$

where $F(\vec{r},t)$ is an arbitrary function of $\vec{r}$ and t.

F. Lorentz gauge

The *Lorentz gauge* is defined as

$$\nabla\cdot\vec{A}(\vec{r},t) + \frac{1}{c^2}\frac{\partial}{\partial t}U(\vec{r},t) = 0, \qquad (19)$$

then the equations of motions for the vector and scalar potentials, Eqs. (13) and (14) can be written in a symmetric form:

$$\begin{aligned} \Box U(\vec{r},t) &= \frac{1}{\epsilon_0}\rho(\vec{r},t), & (20)\\ \Box\vec{A}(\vec{r},t) &= \frac{1}{\epsilon_0\, c^2}\vec{J}(\vec{r},t). & (21)\end{aligned}$$

Or we can have a *covariant notation*:

$$\sum_\mu \partial_\mu \vec{A}^\mu = 0, \qquad (22)$$

$$\sum_\nu \partial_\nu \partial^\nu \vec{A}^\mu = \frac{1}{\epsilon_0\, c^2}\vec{J}^\mu, \qquad (23)$$

where

$$\partial_\mu \equiv \{\frac{1}{c}\frac{\partial}{\partial t}, \nabla\}, \qquad (24)$$

$$\vec{A}^\mu \equiv \{\frac{U}{c}, \vec{A}\}, \qquad (25)$$

$$\vec{J}^\mu \equiv \{c\,\rho, \vec{J}\}. \qquad (26)$$

G. Coulomb gauge

The *Coulomb gauge*, or called the *radiation gauge* is defined as

$$\nabla \cdot \vec{A}(\vec{r}, t) = 0, \tag{27}$$

then the equations of motions for the vector and scalar potentials, Eqs. (13) and (14) can be written in a symmetric form:

$$\triangle U(\vec{r}, t) = -\frac{1}{\epsilon_0}\rho(\vec{r}, t), \tag{28}$$

$$\Box \vec{A}(\vec{r}, t) = \frac{1}{\epsilon_0 c^2}\vec{J}(\vec{r}, t) - \frac{1}{c^2}\nabla\frac{\partial}{\partial t}U(\vec{r}, t). \tag{29}$$

Eq. (28) is the *Poisson's equation* for U. Even though the covariance is lost, but In the following we would use the *Coulomb gauge*.

H. Fourier transform

We would use $\vec{\mathcal{E}}(\vec{k}, t)$ as the Fourier transform of $\vec{E}(\vec{r}, t)$, though the following definition:

$$\vec{\mathcal{E}}(\vec{k}, t) = \frac{1}{(2\pi)^{3/2}}\int d^3\vec{r}\, \vec{E}(\vec{r}, t)\, e^{-i\vec{k}\cdot\vec{r}}, \tag{30}$$

$$\vec{E}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}}\int d^3\vec{k}\, \vec{\mathcal{E}}(\vec{k}, t)\, e^{+i\vec{k}\cdot\vec{r}}. \tag{31}$$

In the following, we use the notation $\rightleftarrows$ to denote the Fourier transform pair, such as

$$\vec{B}(\vec{r}, t) \rightleftarrows \vec{\mathcal{B}}(\vec{k}, t), \tag{32}$$

$$\rho(\vec{r}, t) \rightleftarrows \rho(\vec{k}, t), \tag{33}$$

$$\vec{J}(\vec{r}, t) \rightleftarrows \vec{\mathcal{J}}(\vec{k}, t), \tag{34}$$

$$\vec{A}(\vec{r}, t) \rightleftarrows \vec{\mathcal{A}}(\vec{k}, t), \tag{35}$$

$$U(\vec{r}, t) \rightleftarrows \mathcal{U}(\vec{k}, t). \tag{36}$$

I. Identity in the Fourier transform

1. Parseval-Plancherel identity:

$$\int d^3\vec{r}\, F^*(\vec{r})\, G(\vec{r}) = \int d^3\vec{k}\, \mathcal{F}^*(\vec{k})\, \mathcal{G}(\vec{k}). \tag{37}$$

2. Convolution theorem:

$$\frac{1}{(2\pi)^{3/2}}\int d^3\vec{r}\,'\, F^*(\vec{r}\,')\, G(\vec{r} - \vec{r}\,') \rightleftarrows \mathcal{F}^*(\vec{k})\, \mathcal{G}(\vec{k}). \tag{38}$$

3. Delta function:

$$\delta(\vec{r} - \vec{r}_\alpha) \rightleftarrows \frac{1}{(2\pi)^{3/2}}\, e^{i\vec{k}\cdot\vec{r}_\alpha}. \tag{39}$$

4. $1/r$

$$\frac{1}{4\pi}\frac{1}{r} \rightleftarrows \frac{1}{(2\pi)^{3/2}}\frac{1}{k^2}. \tag{40}$$

5. $\vec{r}/r^3$ ($1/r^2$ along the direction of $\vec{r}$):

$$\nabla \cdot (\frac{1}{4\pi}\frac{-1}{r}) = \frac{1}{4\pi}\frac{\vec{r}}{r^3} \rightleftarrows \frac{1}{(2\pi)^{3/2}}\frac{-i\vec{k}}{k^2}. \tag{41}$$

J. Maxwell's equation in the Momentum space

By replacing ∇ by $i\vec{k}$ in the momentum space, the Maxwell's equations in Eqs. (1-4) become

$$
\begin{aligned}
i\vec{k}\cdot\vec{\mathcal{E}}(\vec{k},t) &= \frac{1}{\epsilon_0}\,\rho(\vec{k},t), && (42)\\
i\vec{k}\cdot\vec{\mathcal{B}}(\vec{k},t) &= 0, && (43)\\
i\vec{k}\times\vec{\mathcal{E}}(\vec{k},t) &= -\frac{\partial}{\partial t}\vec{\mathcal{B}}(\vec{k},t), && (44)\\
i\vec{k}\times\vec{\mathcal{B}}(\vec{k},t) &= \frac{1}{c^2}\frac{\partial}{\partial t}\vec{\mathcal{E}}(\vec{k},t)+\frac{1}{\epsilon_0 c^2}\vec{\mathcal{J}}(\vec{k},t). && (45)
\end{aligned}
$$

The original partial differential equations in the real space, now become *strictly local* in the momentum space.

K. Longitudinal and transverse Vector Fields

By definition, a longitudinal vector field $\vec{V}_{||}(\vec{r})$ is a vector field such that

$$
\begin{aligned}
\nabla\times\vec{V}_{||}(\vec{r}) &= 0, && (46)\\
i\vec{k}\times\vec{\mathcal{V}}_{||}(\vec{k}) &= 0. && (47)
\end{aligned}
$$

A transverse vector field $\vec{V}_\perp(\vec{r})$ is a vector field such that

$$
\begin{aligned}
\nabla\cdot\vec{V}_\perp(\vec{r}) &= 0, && (48)\\
i\vec{k}\cdot\vec{\mathcal{V}}_\perp(\vec{k}) &= 0. && (49)
\end{aligned}
$$

In the momentum space, all vector fields can be simply decomposed into *longitudinal* and *transverse* components:

$$
\begin{aligned}
\vec{\mathcal{V}}(\vec{k}) &= \vec{\mathcal{V}}_{||}(\vec{k})+\vec{\mathcal{V}}_\perp(\vec{k}), \qquad where && (50)\\
\vec{\mathcal{V}}_{||}(\vec{k}) &= \hat{k}[\hat{k}\cdot\vec{\mathcal{V}}(\vec{k})], && (51)\\
\vec{\mathcal{V}}_\perp(\vec{k}) &= \vec{\mathcal{V}}(\vec{k})-\vec{\mathcal{V}}_{||}(\vec{k}), && (52)
\end{aligned}
$$

where $\hat{k}\equiv\vec{k}/k$ is used.

L. Longitudinal Electric and Magnetic Fields

From Maxwell's equations, Eq. (2), it can be seen that the magnetic field is purely transverse, *i.e.*,

$$\vec{\mathcal{B}}(\vec{k}) = 0 = \vec{B}(\vec{r}). \qquad (53)$$

The longitudinal electric field in the momentum space is

$$\vec{\mathcal{E}}_{||}(\vec{k}) = -\frac{i}{\epsilon_0}\rho(\vec{k})\,\frac{\vec{k}}{k^2}, \qquad (54)$$

or in the real space

$$
\begin{aligned}
\vec{E}_{||}(\vec{r}) &= \frac{1}{4\pi\epsilon_0}\int d^3\vec{r}\,'\,\rho(\vec{r}\,',t)\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}, && (55)\\
&= \frac{1}{4\pi\epsilon_0}\sum_\alpha q_\alpha\frac{\vec{r}-\vec{r}_\alpha(t)}{|\vec{r}-\vec{r}_\alpha(t)|^3}, \qquad \rho(\vec{r},t)=\sum_\alpha q_\alpha\delta(\vec{r}-\vec{r}_\alpha) && (56)
\end{aligned}
$$

M. Total energy

The *total energy* for the electric and magnetic fields, plus the particles, is

$$H = \sum_\alpha \frac{1}{2} m_\alpha v_\alpha^2(t) + \frac{\epsilon_0}{2}\int d^3\vec{r}\,[E^2(\vec{r},t) + c^2\,B^2(\vec{r},t)], \tag{57}$$

$$= \sum_\alpha \frac{1}{2} m_\alpha v_\alpha^2(t) + H_\perp + H_\parallel, \tag{58}$$

where the energy contributed from the longitudinal electric field (the longitudinal magnetic field is zero) is,

$$H_\parallel = \frac{\epsilon_0}{2}\int d^3\vec{r}\,E_\parallel^2(\vec{r},t), \tag{59}$$

$$= \frac{\epsilon_0}{2}\int d^3\vec{k}\,\mathcal{E}_\parallel^2(\vec{k},t), \tag{60}$$

$$= \frac{1}{2\epsilon_0}\int d^3\vec{k}\,\rho^*(\vec{k},t)\frac{\rho(\vec{k})}{k^2}, \tag{61}$$

$$= \frac{1}{8\pi\epsilon_0}\int\int d^3\vec{r}\,d^3\vec{r}\,'\,\frac{\rho(\vec{r})\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}, \tag{62}$$

$$= \sum_\alpha \frac{q_\alpha^2}{2\epsilon_0\,(2\pi)^3}\int d^3\vec{k}\,\frac{1}{k^2} + \sum_{\alpha\neq\beta}\frac{q_\alpha q_\beta}{\epsilon_0\,(2\pi)^3}\int d^3\vec{k}\,\frac{e^{-\vec{k}\cdot(\vec{r}_\alpha-\vec{r}_\beta)}}{k^2}, \tag{63}$$

where the first and second terms in the last equation refer to the Coulomb self energy of the particle and Coulomb interaction energy between pairs of particles, respectively.

The corresponding energy contributed from the transverse electric and magnetic field is,

$$H_\perp \equiv \frac{\epsilon_0}{2}\int d^3\vec{r}\,[E_\perp^2(\vec{r},t) + c^2\,B_\perp^2(\vec{r},t)]. \tag{64}$$

In this way, the total energy for the whole system is completely fixed by giving

$$\{\vec{r}_\alpha(t), \dot{\vec{r}}_\alpha(t), \vec{\mathcal{E}}_\perp(\vec{k},t), \vec{\mathcal{B}}(\vec{k},t)\}, \tag{65}$$

where $\vec{\mathcal{B}}(\vec{k}) = \vec{\mathcal{B}}_\perp(\vec{k})$.

II. THE TRANSVERSE FIELDS AS THE NORMAL VARIABLES

The equations of motions for the transverse electric and magnetic fields are

$$\dot{\vec{\mathcal{B}}} = -i\vec{k}\times\vec{\mathcal{E}}_\perp, \tag{66}$$

$$\dot{\vec{\mathcal{E}}} = i\,c^2\vec{k}\times\vec{\mathcal{B}} - \frac{1}{\epsilon_0}\vec{\mathcal{J}}_\perp. \tag{67}$$

By setting the non-homogeneous part zero, *i.e.*, $\vec{\mathcal{J}}_\perp = 0$, one can find the corresponding eigenfunctions,

$$\vec{\mathcal{E}}_\perp \pm c\hat{k}\times\vec{\mathcal{B}}, \tag{68}$$

with the corresponding eigenvalue $\omega^2 = k^2\,c^2$.

In this way, one can define two new variables to replace the transverse electric and magnetic fields, they are

$$\vec{\alpha}(\vec{k},t) \equiv -\frac{i}{2\,\mathcal{N}(k)}[\vec{\mathcal{E}}_\perp(\vec{k},t) - c\hat{k}\times\vec{\mathcal{B}}(\vec{k},t)], \tag{69}$$

$$\vec{\beta}(\vec{k},t) \equiv -\frac{i}{2\,\mathcal{N}(k)}[\vec{\mathcal{E}}_\perp(\vec{k},t) + c\hat{k}\times\vec{\mathcal{B}}(\vec{k},t)]. \tag{70}$$

Moreover, the real character of $\vec{E}_\perp$ and $\vec{B}$ requires that

$$\vec{\beta}(\vec{k},t) = -\vec{\alpha}^*(-\vec{k},t). \tag{71}$$

Now, the total energy for the whole system is completely fixed by giving

$$\{\vec{r}_\alpha(t), \dot{\vec{r}}_\alpha(t), \vec{\alpha}(\vec{k},t)\}, \tag{72}$$

and the related transverse electric and magnetic fields are

$$\vec{\mathcal{E}}_\perp(\vec{k},t) \;=\; i\mathcal{N}(k)[\vec{\alpha}(\vec{k},t) - \vec{\alpha}^*(-\vec{k},t)], \tag{73}$$

$$\vec{\mathcal{B}}(\vec{k},t) \;=\; \frac{i\mathcal{N}(k)}{c}[\hat{k}\times\vec{\alpha}(\vec{k},t) + \hat{k}\times\vec{\alpha}^*(-\vec{k},t)]. \tag{74}$$

A. Evolution of the Normal Variables

From the Maxwell's equations, we have

$$\frac{d}{dt}\vec{\alpha}(\vec{k},t) + i\omega\,\vec{\alpha}(\vec{k},t) = \frac{i}{2\epsilon_0\mathcal{N}(k)}\vec{\mathcal{J}}_\perp(\vec{k},t). \tag{75}$$

It resembles the equation of motion of a fictitious harmonic oscillator, *i.e.*,

$$\frac{d}{dt}x(t) + i\omega\,x(t) = f_d. \tag{76}$$

B. Transverse polarizations

Since $\vec{\alpha}$ is a transverse vector field, as $\vec{\mathcal{E}}_\perp$ and $\vec{\mathcal{B}}$, one can expand $\vec{\alpha}(\vec{k})$ on two unit vectors $\hat{e}_1$ and $\hat{e}_2$, normal to on another and both localed in the plane normal to $\hat{k}$, that is

$$\vec{\alpha}(\vec{k},t) = \hat{e}_1\alpha_{e_1}(\vec{k},t) + \hat{e}_2\alpha_{e_2}(\vec{k},t), \tag{77}$$

where $\alpha_j(\vec{k},t) = \hat{e}_j\cdot\vec{\alpha}(\vec{k},t)$ for $j = 1,2$. We have two degrees of polarizations in the transverse.

C. Total Energy of the Transverse Fields

In terms of the normal variables $\vec{\alpha}(\vec{k})$, the total energy contributed from the transverse electric and magnetic field becomes,

$$\begin{aligned} H_\perp &\equiv \frac{\epsilon_0}{2}\int d^3\vec{r}\,[E_\perp^2(\vec{r},t) + c^2\,B_\perp^2(\vec{r},t)], \\ &= \epsilon_0\int d^3\vec{k}\,\mathcal{N}^2[\vec{\alpha}^*(\vec{k})\cdot\vec{\alpha}(\vec{k}) + \vec{\alpha}(-\vec{k})\cdot\vec{\alpha}^*(-\vec{k})]. \end{aligned} \tag{78}$$

Define the normalization coefficient $\mathcal{N}(k)$ as

$$\mathcal{N}(k) \equiv \sqrt{\frac{\hbar\omega_k}{2\epsilon_0}}, \tag{79}$$

then we have

$$H_\perp \;=\; \int d^3\vec{k}\sum_j \frac{\hbar\omega_k}{2}[\alpha_j^*(\vec{k},t)\cdot\alpha_j(\vec{k},t) + \alpha_j(\vec{k},t)\cdot\alpha_j^*(\vec{k},t)]. \tag{80}$$

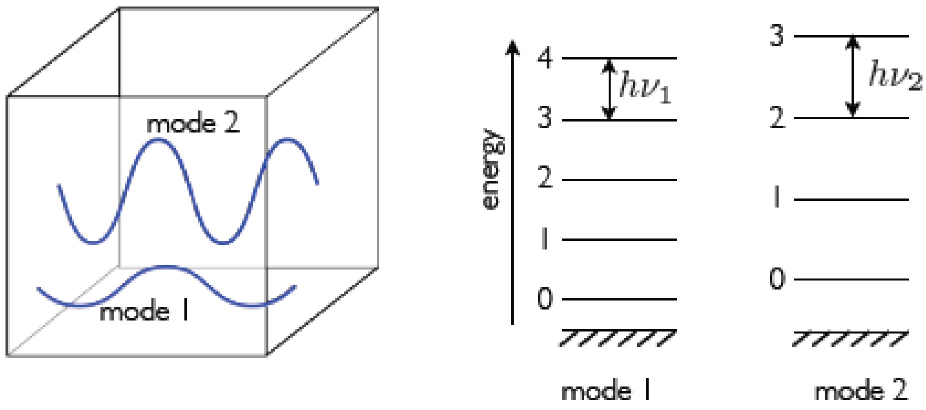

FIG. 1: Quantized modes for fields.

III. POSSIBLE QUANTIZATION SCHEMES

Here, we introduce the *annihilation* and *creation* operators, $\hat{a}_j$ and $\hat{a}_j^\dagger$, for the normal variables α_j and α_j^*, respectively,

$$\alpha_j \rightarrow \hat{a}_j, \tag{81}$$
$$\alpha_j^* \rightarrow \hat{a}_j^\dagger, \tag{82}$$

with the commutator equal to 1, from the quantum theory of *simple harmonic oscillators*,

$$[\hat{a}_i, \hat{a}_j] = 0, \tag{83}$$
$$[\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0, \tag{84}$$
$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}. \tag{85}$$
$$\tag{86}$$

In this way, the Hamiltonian contributed from the transverse fields in the Coulomb gauge is

$$H_\perp = \int d^3\vec{k} \sum_j \frac{\hbar\omega_k}{2}[\hat{a}_j^\dagger \hat{a}_j + \hat{a}_j \hat{a}_j^\dagger], \tag{87}$$

which has the same form as that of a quantized *simple harmonic oscillator*.

A. Analogy with Simple Harmonic Oscillator

The associate SHO Hamiltonian is

$$\hat{H} = \hbar\omega(\hat{a}^\dagger \hat{a} + \frac{1}{2}), \tag{88}$$
$$= \hbar\omega(\hat{N} + \frac{1}{2}). \tag{89}$$

Here we have introduced the *Number operator*, $\hat{N}$, *annihilation operator*, $\hat{a}$, and the *creation operator*, $\hat{a}^\dagger$:

$$\hat{N}|n\rangle = n|n\rangle, \tag{90}$$
$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \tag{91}$$
$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{92}$$

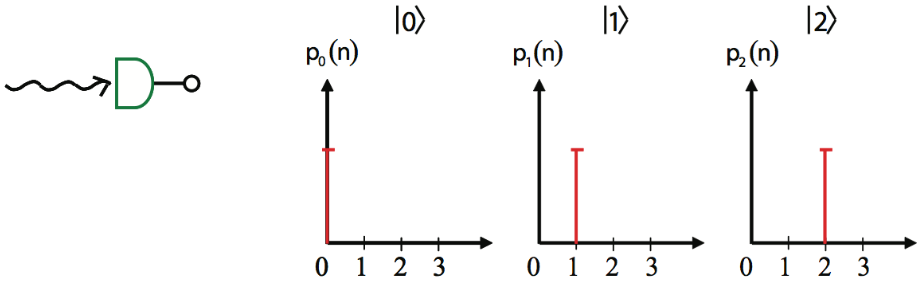

FIG. 2: Measurement of the $1, 2, 3$ photon number eigenstates for photon counting detection.

The dynamics of the operator is governed by the Heisenberg's equation:

$$\frac{d}{dt}\hat{O} = \frac{1}{i\hbar}[\hat{O}, \hat{H}]. \tag{93}$$

For the annihilation operator of SHO, $\hat{a}$, we have

$$\frac{d}{dt}\hat{a} = \frac{1}{i\hbar}[\hat{a}, \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})] = -i\omega\hat{a}, \tag{94}$$

with the solution

$$\hat{a}(t) = \hat{a}(t = 0)\exp[-i\omega t]. \tag{95}$$

B. Transverse Electrical fields

In terms of operators, we have the corresponding *quantized* transverse field (a single mode):

$$\hat{\mathcal{E}}_\perp(\vec{k}, t) \;=\; i\sqrt{\frac{\hbar\omega}{2\epsilon_0}}[\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}]. \tag{96}$$

Note that $\hat{a}$ and $\hat{a}^\dagger$ are not hermitian operators, but $(\hat{a}^\dagger)^\dagger = \hat{a}$. Moreover, the expectation value of the quantized transverse field is zero, *i.e.*

$$\langle n|\hat{\mathcal{E}}|n\rangle = 0. \tag{97}$$

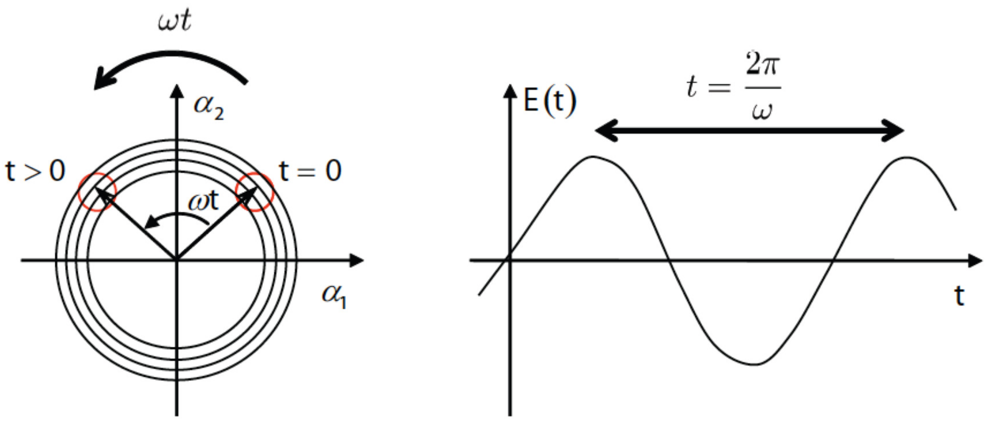

FIG. 3: Phase diagram for the electrical field.

C. Quadrature Operators

One can define two hermitian operators

$$\hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger), \qquad (98)$$

$$\hat{X}_2 = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger), \qquad (99)$$

which have the commutator relation,

$$[\hat{X}_1, \hat{X}_2] = \frac{i}{2}. \qquad (100)$$

In terms of the *quadrature operators*, the quantized transverse field becomes

$$\hat{\mathcal{E}}_\perp(\vec{k}, t) = \sqrt{\frac{\hbar\omega}{2\epsilon_0}}[2\hat{X}_1 \sin(\omega t) - 2\hat{X}_2 \cos(\omega t)] \equiv E_0[\hat{X}_1 \sin(\omega t) - \hat{X}_2 \cos(\omega t)], \qquad (101)$$

where the E_0 is the amplitude. In Fig. 3, we show the *phase diagram* in the coordinates of two *quadratures*, where one can define

$$\hat{X}_1, \hat{X}_2 \equiv \text{amplitude, phase quadratures.} \qquad (102)$$

IV. ROLE OF QUANTUM OPTICS

1. Photons occupy an *electromagnetic mode*: we will always refer to modes in quantum optics, typically a plane wave;

2. The energy in a mode is not continuous but *discrete in quanta* of $\hbar\omega$.

3. The observables are just represented by probabilities as usual in quantum mechanics.

4. There is a *zero point energy* inherent to each mode, which is equivalent with fluctuations of the electromagnetic field in vacuum, due to *the uncertainty principle*.

A few more words for the *Vacuum*, which is not just nothing, it is full of energy. Several interesting phenomena are related to the vacuum, such as

1. *Spontaneous emission* is actually stimulated by the vacuum fluctuation of the electromagnetic field.

2. One can modify vacuum fluctuations by resonators and photonic crystals, *Purcell effect*.

3. The electron does not crash into the core in the atomic structure, due to vacuum fluctuation of the electromagnetic field.

4. Gravity is not a fundamental force but a side effect matter modifies the vacuum fluctuations, by Sakharov.

5. *Casimir effect*: two charged metal plates repel, or attract, each other due to the potential force induced by the vacuum.

6. *Lamb shift*: the energy level difference between $2S_{1/2}$ and $2P_{1/2}$ in hydrogen.

7. ...