Note for *Quantum Optics*: Quantum Distribution Theory

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Reference:
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Ch. 6 in *"Quantum Optics,"* by D. Wall and G. Milburn.
Ch. 8 in *"Mesoscopic Quantum Optics,"* by Y. Yamamoto and A. Imamoglu.
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"Quantum Optics in Phase Space," by W. Schleich.

I. PHASE SPACE PROBABILITY DISTRIBUTION FUNCTION

A classical dynamical system may be described by a phase space probability distribution function,

$$f(\{q\},\{p\}),$$

where

$$\{q\}\equiv q_1,q_2,\ldots,q_N;\quad \text{and}\quad \{p\}\equiv p_1,p_2,\ldots,p_N,$$

The probability

$$f(\{q\},\{p\})\mathrm{d}^N q\,\mathrm{d}^N p,$$

gives the description about the system in a volume element $\mathrm{d}^N q\,\mathrm{d}^N p$. In quantum mechanics, the phase coordinates q_i and p_i can not described definite values simultaneously. Hence the concept of phase space distribution function does not exist for a quantum system, However, it's possible to construct a *quantum quasi-probability distribution* resembling the classical phase space distribution functions. Let us consider a one dimensional dynamical system, described classically by a phase space distribution function $f(q,p,t)$,

$$\langle A(q,p)\rangle_{\rm cl}=\int \mathrm{d}q\,\mathrm{d}p A(q,p)f(q,p,t),$$

for the quantum mechanical description, if we know that the system is in state $|\psi\rangle$, then an operator $\hat{O}$ has the expectation value,

$$\langle\hat{O}\rangle_{\rm qm}=\langle\psi|\hat{O}|\psi\rangle,$$

but we typically do not know that we are in state $|\psi\rangle$, then an ensemble average must be performed,

$$\langle\langle\hat{O}\rangle_{\rm qm}\rangle_{\rm ensemble}=\sum_\psi P_\psi\langle\psi|\hat{O}|\psi\rangle.$$

With the completeness $\sum_n |n\rangle\langle n|=1$,

$$\langle\langle\hat{O}\rangle_{\rm qm}\rangle_{\rm ensemble}=\sum_n\langle n|\hat{\rho}\hat{O}|n\rangle,$$

where the P_ψ is the probability of being in the state $|\psi\rangle$, we can introduce a density operator,

$$\hat{\rho}=\sum_\psi P_\psi|\psi\rangle\langle\psi|,$$

the expectation value of any operator $\hat{A}$ is given by,

$$\langle \hat{A}(\hat{q},\hat{p})\rangle_{\rm qm} = {\rm Tr}[\hat{\rho}\hat{A}(\hat{q},\hat{p})],$$

where Tr stands for trace.The density operator $\hat{\rho}$ can be expanded in terms of the number states,

$$\hat{\rho} = \sum_n \sum_m |n\rangle\langle n|\hat{\rho}|m\rangle\langle m| = \sum_n \sum_m \rho_{nm}|n\rangle\langle m|,$$

the expansion coefficients ρ_{nm} are complex and there is an infinite number of them. For problems where the phase-dependent properties of EM field are important, this make the general expansion rather less useful. In certain case where only the photon number distribution is of interest, one may use

$$\hat{\rho} = \sum_n P_n |n\rangle\langle n|.$$

For a chaotic field, $P_n = \frac{1}{1+\bar{n}}(\frac{\bar{n}}{1+\bar{n}})^n$; while for a Poisson distribution of photons, $P_n = \frac{e^{-\bar{n}}}{n!}\bar{n}^n$.

II. EXPANSION IN COHERENT STATES

Likewise the expansion may be in terms of coherent states,

$$\hat{\rho} = \frac{1}{\pi^2}\int\int {\rm d}^2\alpha {\rm d}^2\beta |\alpha\rangle\langle\alpha|\hat{\rho}|\beta\rangle\langle\beta|,$$

where $\frac{1}{\pi}\int|\alpha\rangle\langle\alpha|{\rm d}^2\alpha = 1$, the expectation value of any operator $\hat{A}$ is given by, $\langle\hat{A}(\hat{a},\hat{a}^\dagger)\rangle_{\rm qm} = {\rm Tr}[\hat{\rho}\hat{A}(\hat{a},\hat{a}^\dagger)]$, quasi-probability distribution,

$$\begin{aligned}
\langle\hat{O}(\hat{a},\hat{a}^\dagger)\rangle &= \int {\rm d}^2\alpha P(\alpha,\alpha^*)O_N(\alpha,\alpha^*), \quad \text{for normally ordering operators,} && (1)\\
&= \int {\rm d}^2\alpha Q(\alpha,\alpha^*)O_A(\alpha,\alpha^*), \quad \text{for antinormally ordering operators,} && (2)\\
&= \int {\rm d}^2\alpha W(\alpha,\alpha^*)O_S(\alpha,\alpha^*), \quad \text{for symmetric ordering operators,} && (3)
\end{aligned}$$

We can rewrite classical distribution as,

$$\begin{aligned}
f(q,p,t) &= \int {\rm d}q'\,{\rm d}p'\delta(q-q')\delta(p-p')f(q',p',t), && (4)\\
&= \frac{1}{4\pi^2}\int {\rm d}q'\,{\rm d}p'\,{\rm d}k\,{\rm d}l\exp\{i[k(q-q')+l(p-p')]\}f(q',p',t), && (5)\\
&= \frac{1}{4\pi^2}\int {\rm d}k\,{\rm d}l\exp(ikq)\exp(ilp)\int {\rm d}q'\,{\rm d}p'\exp(-ikq')\exp(-ilp')f(q',p',t), && (6)\\
&= \frac{1}{4\pi^2}\int {\rm d}k\,{\rm d}l\exp(ikq)\exp(ilp)\langle\exp(-ikq)\exp(-ilp)\rangle_{cl}, && (7)
\end{aligned}$$

with $\delta(x) = \frac{1}{2\pi}\int {\rm d}k\exp(ikx)$. For the quantum analog of $f(q,p,t)$,

1. replace the c-numbers q, p by the operators $\hat{q}$, $\hat{p}$,
2. replace the classical average by the quantum average,
3. express the exponential under the average as a sum of products of the form $q^m p^n$,

Due to non-commutativity of $\hat{q}$ and $\hat{p}$, there are several different operator forms of a c-number product $q^m p^n$, if $m,n \neq 0$. For example, $q^2 p$ may be represented by an of the forms: $\hat{q}^2\hat{p}$, $\hat{q}\hat{p}\hat{q}$, $\hat{p}\hat{q}^2$ or by their linear combination $c_1\hat{q}^2\hat{p}+c_2\hat{q}\hat{p}\hat{q}+c_3\hat{p}\hat{q}^2$, where x_i are arbitrary subject to the condition $c_1+c_2+c_3 = 1$. In general, we formally represent a c-number product as an operator as,

$$q^m p^n \to \Omega(\hat{q}^m\hat{p}^n),$$

which defines a linear combination of m $\hat{q}$'s and n $\hat{p}$'s. For example,

$$\begin{aligned}\exp[\alpha_1\hat{a}+\alpha_2\hat{a}^\dagger] &= \exp[\alpha_2\hat{a}^\dagger]\exp[\alpha_1\hat{a}]\exp[\frac{1}{2}\alpha_1\alpha_2], \quad \text{normally ordering,} && (8)\\ &= \exp[\alpha_1\hat{a}]\exp[\alpha_2\hat{a}^\dagger]\exp[-\frac{1}{2}\alpha_1\alpha_2], \quad \text{antinormally ordering,} && (9)\end{aligned}$$

with the Baker-Hausdorff relation, $e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}[\hat{A},\hat{B}]} = e^{+\frac{1}{2}[\hat{A},\hat{B}]}e^{\hat{B}}e^{\hat{A}}$, provided $[\hat{A},[\hat{A},\hat{B}]] = [[\hat{A},\hat{B}],\hat{B}] = 0$. Then, the quantum analog of the classical phase space distribution function is then,

$$f^{\Omega}(q,p,t) = \frac{1}{4\pi^2}\int \mathrm{d}k\,\mathrm{d}l\exp(ikq)\exp(ilp)\langle\Omega[\exp(-ik\hat{q})\exp(-il\hat{p})]\rangle_{qm}.$$

Note that different choices of the correspondence Ω lead to different $f^{\Omega}(q,p,t)$, each called a *quasi-probability distribution* function to emphasize that it is a mathematical construct and not a true phase space distribution function.

The quantum analog of the classical phase space distribution function in terms of the creation and annihilation operators $\hat{a}$ and $\hat{a}^\dagger$ is,

$$f^{\Omega}(\alpha,\alpha^*) = \frac{1}{\pi^2}\int \mathrm{d}^2\xi\exp[i(\alpha\xi+\alpha^*\xi^*)]\mathrm{Tr}[\Omega\{\exp(-i\hat{\alpha}\xi)\exp(-i\hat{\alpha}^\dagger\xi^*)\}\hat{\rho}], \quad (10)$$

Now, let

$$\begin{aligned}\Omega\{\exp(-i\hat{\alpha}\xi)\exp(-i\hat{\alpha}^\dagger\xi^*)\} &= \prod_{j=1}^{N}[\exp(-i\alpha_j\xi\hat{\alpha})\exp(-i\beta\xi^*\hat{\alpha}^\dagger], && (11)\\ &= \exp(-\frac{s}{2}|\xi|^2)\exp[-i(\xi\hat{\alpha}+\xi^*\hat{\alpha}^\dagger)], && (12)\end{aligned}$$

where s is a complex number related with products of the α_j and β_j. Although, the exact expression of s in terms of the α_j and β_j may be derived, it is inessential. The ordering for $s=0$ is called the *Weyl ordering*, or the *symmetric ordering*, The exponential operator may be put in the anitnormal or the normal ordering,

$$\begin{aligned}\exp[-i(\xi\hat{a}+\xi^*\hat{a}^\dagger)] &= \exp(-i\xi^*\hat{a}^\dagger)\exp(-i\xi\hat{a})\exp(-\frac{1}{2}|\xi|^2), \quad \text{normally ordering,} && (13)\\ &= \exp(-i\xi\hat{a})\exp(-i\xi^*\hat{a}^\dagger)\exp(\frac{1}{2}|\xi|^2), \quad \text{antinormally ordering,} && (14)\end{aligned}$$

The quantum analog of the classical phase space distribution function in the s-ordering is,

$$f^{(s)}(\alpha,\alpha^*) = \frac{1}{\pi^2}\int \mathrm{d}^2\xi\exp[i(\alpha\xi+\alpha^*\xi^*)]\exp(-\frac{s}{2}|\xi|^2)\mathrm{Tr}\{\exp[-i(\xi\hat{a}+\xi^*\hat{a}^\dagger)]\hat{\rho}\}.$$

This is some kind of two-dimensional Fourier transformation, if we define

$$\mathrm{Tr}\{\exp[-i(\xi\hat{a}+\xi^*\hat{a}^\dagger)]\hat{\rho}\} \equiv G(\xi,\xi^*)\exp(\frac{s}{2}|\xi|^2),$$

then

$$f^{(s)}(\alpha,\alpha^*) = \frac{1}{\pi^2}\int \mathrm{d}^2\xi G(\xi,\xi^*)\exp[i(\alpha\xi+\alpha^*\xi^*)],$$

and by the inverse Fourier transformation,

$$G(\xi,\xi^*) = \int \mathrm{d}^2\alpha f^{(s)}(\alpha,\alpha^*)\exp[-i(\alpha\xi+\alpha^*\xi^*)]$$

For antinormal form of the exponential:

$$\begin{aligned}\mathrm{Tr}[\exp[-i(\xi\hat{a}+\xi^*\hat{a}^\dagger)\hat{\rho}] &= \mathrm{Tr}[\exp(-i\xi\hat{a})\exp(-i\xi^*\hat{a}^\dagger)\hat{\rho}](\frac{1}{2}|\xi|^2), && (15)\\ &= \mathrm{Tr}[\exp(-i\xi^*\hat{a}^\dagger)\hat{\rho}\exp(-i\xi\hat{a})](\frac{1}{2}|\xi|^2), && (16)\\ &= \frac{1}{\pi}\int \mathrm{d}^2\alpha\exp[-i(\alpha\xi+\alpha^*\xi^*)](\frac{1}{2}|\xi|^2)\langle\alpha|\hat{\rho}|\alpha\rangle, && (17)\\ &= G(\xi,\xi^*)\exp(\frac{s}{2}|\xi|^2). && (18)\end{aligned}$$

For the density matrix in the coherent state representation,

$$\langle\alpha|\hat{\rho}|\alpha\rangle = \frac{1}{\pi}\int \mathrm{d}^2\xi G(\xi,\xi^*)\exp(\frac{s-1}{2}|\xi|^2)\exp[i(\alpha\xi+\alpha^*\xi^*)]$$

and the relationship between the density operator and its various phase space representation through $G(\xi,\xi^*)$ is,

$$\hat{\rho} = \frac{1}{\pi}\int \mathrm{d}^2\xi G(\xi,\xi^*)\exp(\frac{s-1}{2}|\xi|^2)\exp(i\xi^*\hat{a}^\dagger)\exp(i\xi\hat{a}), \quad \text{for antinormally ordering,} \tag{19}$$

$$= \frac{1}{\pi}\int \mathrm{d}^2\xi G(\xi,\xi^*)\exp(\frac{s}{2}|\xi|^2)\exp[i(\xi\hat{a}+\xi^*\hat{a}^\dagger)], \quad \text{for symmetric ordering,} \tag{20}$$

$$= \frac{1}{\pi}\int \mathrm{d}^2\xi G(\xi,\xi^*)\exp(\frac{s+1}{2}|\xi|^2)\exp(i\xi\hat{a})\exp(i\xi^*\hat{a}^\dagger), \quad \text{for normally ordering.} \tag{21}$$

The relation between different phase space representation $f^{(s)}$ and $f^{(t)}$ is,

$$f^{(s)}(\alpha,\alpha^*) = \frac{2}{\pi(s-t)}\int \mathrm{d}^2\beta\exp[-\frac{2|\alpha-\beta|^2}{s-t}]f^{(t)}(\beta,\beta^*),$$

The phase space distribution function in the s-ordering is,

$$f^{(s)}(\alpha,\alpha^*) = \frac{1}{\pi^2}\int \mathrm{d}^2\xi\exp[i(\alpha\xi+\alpha^*\xi^*)]\exp(-\frac{s}{2}|\xi|^2)\mathrm{Tr}[\exp[-i(\xi\hat{a}+\xi^*\hat{a}^\dagger)\hat{\rho}].$$

The phase space representation of any operator $\hat{A}$ is similar,

$$A^{(s)}(\alpha,\alpha^*) = \frac{1}{\pi^2}\int \mathrm{d}^2\xi\exp[i(\alpha\xi+\alpha^*\xi^*)]\exp(-\frac{s}{2}|\xi|^2)\mathrm{Tr}[\exp[-i(\xi\hat{a}+\xi^*\hat{a}^\dagger)\hat{A}],$$

and the expectation value of $\hat{A}$ is,

$$\mathrm{Tr}[\hat{A}\hat{\rho}] = \frac{1}{\pi}\int \mathrm{d}^2\xi G(\xi,\xi^*)\exp(\frac{s}{2}|\xi|^2)\mathrm{Tr}\{\hat{A}\exp[i(\xi\hat{a}+\xi^*\hat{a}^\dagger)]\}, \tag{22}$$

$$= \frac{1}{\pi}\int\int \mathrm{d}^2\xi\mathrm{d}^2\alpha f^{(s)}(\alpha,\alpha^*)\exp[-i(\xi\alpha+\xi^*\alpha^*)]\exp(\frac{s}{2}|\xi|^2)\mathrm{Tr}\{\hat{A}\exp[i(\xi\hat{a}+\xi^*\hat{a}^\dagger)]\}, \tag{23}$$

$$= \pi\int \mathrm{d}^2\alpha f^{(s)}(\alpha,\alpha^*)A^{(-s)}(\alpha,\alpha^*), \tag{24}$$

The expectation value of an operator is TE phase space integral of the product of its phase space function with its conjugate representation of the density operator.

III. P-REPRESENTATION, NORMALLY ORDERING

The density operator $\hat{\rho}$ can be expanded in terms of the number states,

$$\hat{\rho} = \sum_n\sum_m |n\rangle\langle n|\hat{\rho}|m\rangle\langle m| = \sum_n\sum_m \rho_{nm}|n\rangle\langle m|,$$

likewise the expansion may be in terms of coherent states,

$$\hat{\rho} = \frac{1}{\pi^2}\int\int \mathrm{d}^2\alpha\mathrm{d}^2\beta|\alpha\rangle\langle\alpha|\hat{\rho}|\beta\rangle\langle\beta|,$$

as only the photon number distribution is of interest, one may use

$$\hat{\rho} = \sum_n P_n|n\rangle\langle n|,$$

P-representation of a density operator,

$$\hat{\rho} = \int \mathrm{d}^2\alpha P(\alpha,\alpha^*)|\alpha\rangle\langle\alpha|,$$

Now, we have the P-representation of a density operator,

$$\hat{\rho} = \int \mathrm{d}^2\alpha P(\alpha,\alpha^*)|\alpha\rangle\langle\alpha|,$$

and substitute into

$$f^{(s)}(\alpha,\alpha^*) = \frac{1}{\pi^2}\int \mathrm{d}^2\xi \exp[i(\alpha\xi+\alpha^*\xi^*)]\exp(-\frac{s}{2}|\xi|^2)\mathrm{Tr}[\exp[-i(\xi\hat{a}+\xi^*\hat{a}^\dagger)]\hat{\rho}],$$

with $s=-1$ and the exponential operator in the normal-ordering, we have

$$\begin{aligned} f^{(-1)}(\alpha,\alpha^*) &= \frac{1}{\pi^2}\int \mathrm{d}^2\xi \int \mathrm{d}^2\beta P(\beta,\beta^*)\exp[i(\alpha\xi+\alpha^*\xi^*)]\mathrm{Tr}[e^{(-i\xi\hat{a})}|\beta\rangle\langle\beta|e^{(-i\xi^*\hat{a}^\dagger)}], && (25)\\ &= \frac{1}{\pi^2}\int \mathrm{d}^2\xi \int \mathrm{d}^2\beta P(\beta,\beta^*)\exp\{i[(\alpha-\beta)\xi+(\alpha^*-\beta^*)\xi^*]\}, && (26)\\ &= P(\alpha,\alpha^*). && (27)\end{aligned}$$

The phase space representation for $s=-1$ is thus the P-function, Equivalent, one can define

$$\begin{aligned} P(\alpha,\alpha^*) &= \mathrm{Tr}[\hat{\rho}\delta(\alpha^*-\hat{a}^\dagger)\delta(\alpha-\hat{a})], && (28)\\ &= \mathrm{Tr}[\int \mathrm{d}^2\beta P(\beta,\beta^*)|\beta\rangle\langle\beta|\delta(\alpha^*-\hat{a}^\dagger)\delta(\alpha-\hat{a})], && (29)\\ &= \int \mathrm{d}^2\alpha \int \mathrm{d}^2\beta P(\beta,\beta^*)\langle\alpha|\beta\rangle\langle\beta|\delta(\alpha^*-\hat{a}^\dagger)\delta(\alpha-\hat{a})|\alpha\rangle. && (30)\end{aligned}$$

Note it is normally ordering in the trace,

$$\delta(\alpha^*-\hat{a}^\dagger)\delta(\alpha-\hat{a}),$$

The function $P(\alpha,\alpha^*)$ can be used to evaluate the expectation values of any normal ordered function of $\hat{a}$ and $\hat{a}^\dagger$ using the methods of classical statistical mechanics,

$$\begin{aligned} \langle\hat{A}_N\rangle &= \mathrm{Tr}(\hat{A}_N) = \frac{1}{\pi}\int \mathrm{d}^2\xi G(\xi,\xi^*)\exp(\frac{-1}{2}|\xi|^2)\mathrm{Tr}\{\hat{A}\exp[i(\xi\hat{a}+\xi^*\hat{a}^\dagger)]\}, && (31)\\ &= \pi\int \mathrm{d}^2\alpha f^{(-1)}(\alpha,\alpha^*)A^{(1)}(\alpha,\alpha^*), && (32)\\ &= \int \mathrm{d}^2\alpha P(\alpha,\alpha^*)A_N(\alpha,\alpha^*). && (33)\end{aligned}$$

Since $\mathrm{Tr}(\hat{\rho})=1$,

$$\int \mathrm{d}^2\alpha P(\alpha,\alpha^*) = 1,$$

the function $P(\alpha,\alpha^*)$ is referred to as the P-representation or the coherent state representation,

$$\hat{\rho} = \int \mathrm{d}^2\alpha P(\alpha,\alpha^*)|\alpha\rangle\langle\alpha|.$$

The function $P(\alpha,\alpha^*)$ forms a connection between the classical and quantum coherence theory. Let $|\beta\rangle$ and $|-\beta\rangle$ be the coherent states, then

$$\begin{aligned} \langle-\beta|\hat{\rho}|\beta\rangle &= \int \mathrm{d}^2\alpha P(\alpha,\alpha^*)\langle-\beta|\alpha\rangle\langle\alpha|\beta\rangle, && (34)\\ &= e^{-|\beta|^2}\int \mathrm{d}^2\alpha P(\alpha,\alpha^*)e^{-|\alpha|^2}e^{\beta\alpha^*-\beta^*\alpha}, && (35)\\ &= e^{-|\beta|^2}\int \mathrm{d}x_\alpha \int \mathrm{d}y_\alpha P(x_\alpha,y_\alpha)e^{-(x_\alpha^2+y_\alpha^2)}e^{2i(y_\beta x_\alpha - x_\beta y_\alpha)}, && (36)\end{aligned}$$

with

$$\langle\alpha|\beta\rangle = \exp(-\frac{1}{2}|\alpha|^2 + \alpha^*\beta - \frac{1}{2}|\beta|^2) = \exp(-\frac{1}{2}|\alpha-\beta|^2),$$

where $\alpha = x_\alpha + iy_\alpha$ and $\beta = x_\beta + iy_\beta$ and this is the two-dimensional Fourier transform,

$$P(\alpha,\alpha^*) = \frac{e^{x_\alpha^2+y_\alpha^2}}{\pi^2}\int \mathrm{d}x_\beta \int \mathrm{d}y_\beta \langle-\beta|\hat{\rho}|\beta\rangle e^{(x_\beta^2+y_\beta^2)} e^{-2i(y_\beta x_\alpha - x_\beta y_\alpha)}, \quad (37)$$

$$= \frac{e^{|\alpha|^2}}{\pi^2}\int \mathrm{d}^2\beta \langle-\beta|\hat{\rho}|\beta\rangle e^{|\beta|^2} e^{-\beta\alpha^*+\beta^*\alpha}, \quad (38)$$

For the thermal field,

$$\hat{\rho} = \frac{\exp(-\hat{H}/k_BT)}{\mathrm{Tr}[\exp(-\hat{H}/k_BT)]},$$

where k_B is the Boltzmann constant and $\hat{H}$ is the free-field Hamiltonian, $\hat{H} = \hbar\omega(\hat{a}^\dagger + \hat{a} + 1/2)$,

$$\hat{\rho} = \sum_n [1 - \exp(\frac{-\hbar\omega}{k_BT})]\exp(-\frac{n\hbar\omega}{k_BT})|n\rangle\langle n|,$$

the expectation value of the photon number, $\langle\bar{n}\rangle = \mathrm{Tr}(\hat{a}^\dagger\hat{a}\hat{\rho}) = \frac{1}{\exp(\hbar\omega/k_BT)-1}$. The photon distribution in a thermal field becomes:

$$\hat{\rho} = \sum_n \frac{\langle n\rangle^n}{(1+\langle n\rangle)^{n+1}}|n\rangle\langle n|,$$

which is the Bose-Einstein distribution.

The P-representation of the thermal field is

$$P(\alpha,\alpha^*) = \frac{e^{|\alpha|^2}}{\pi^2}\int \mathrm{d}^2\beta \langle-\beta|\hat{\rho}|\beta\rangle e^{|\beta|^2} e^{-\beta\alpha^*+\beta^*\alpha}, \quad (39)$$

$$= \frac{e^{|\alpha|^2}}{\pi^2(1+\frac{1}{\langle n\rangle})}\int \mathrm{d}^2\beta \exp[\frac{-|\beta|^2}{1+\frac{1}{\langle n\rangle}}]e^{-\beta\alpha^*+\beta^*\alpha}, \quad (40)$$

$$= \frac{1}{\pi\langle n\rangle}e^{-|\alpha|^2/\langle n\rangle}, \quad (41)$$

which is a Gaussian distribution with the width of $\langle n\rangle$ in phase space,

For the coherent field, $\hat{\rho} = |\alpha_0\rangle\langle\alpha_0|$, we have the corresponding P-representation of the coherent field

$$P(\alpha,\alpha^*) = \frac{1}{\pi^2}e^{|\alpha|^2-|\alpha_0|^2}\int \mathrm{d}^2\beta \exp[-\beta(\alpha^*-\alpha_0^*) + \beta^*(\alpha-\alpha_0)], \quad (42)$$

$$= \delta^{(2)}(\alpha-\alpha_0), \quad (43)$$

which is a two-dimensional delta function in phase space, i.e.

$$f^{(s)}(\alpha,\alpha^*) = \frac{1}{\pi^2}\int \mathrm{d}^2\xi G(\xi,\xi^*)\exp[i(\alpha\xi+\alpha^*\xi^*)], \quad (44)$$

$$G(\xi,\xi^*) = \int \mathrm{d}^2\alpha f^{(s)}(\alpha,\alpha^*)\exp[-i(\alpha\xi+\alpha^*\xi^*)] \quad (45)$$

where $\mathrm{Tr}\{\exp[-i(\xi\hat{a}+\xi^*\hat{a}^\dagger)]\hat{\rho}\} \equiv G(\xi,\xi^*)\exp(\frac{s}{2}|\xi|^2)$.

For thermal field, its P-representation is a Gaussian function in phase space; while for coherent state, its P-representation is a 2D delta function in phase space. For a number state, $\hat{\rho} = |n\rangle\langle n|$, then

$$\langle-\beta|\hat{\rho}|\beta\rangle = \langle-\beta|n\rangle\langle n|\beta\rangle = \exp(-|\beta|^2)\frac{(-1)^n|\beta|^{2n}}{n!},$$

and the corresponding P-representation is, $P(\alpha,\alpha^*) = \frac{e^{|\alpha|^2}}{n!}\frac{\partial^{2n}}{\partial\alpha^n\partial\alpha^{*n}}\delta^{(2)}(\alpha)$, which is not a *non-negative* definite function for $n > 0$, whenever the photon distribution ρ_{nn} is narrower than the Poisson distribution, $P(\alpha,\alpha^*)$ becomes badly behaved.

1. Glauber-Sudarshan *P*-representation

Consider a single electromagnetic field mode in a cavity with finite leakage rate, the time evolution of the field density is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_f(t) = \frac{-1}{2}[R_e(\hat{a}\hat{a}^\dagger\hat{\rho}_f - \hat{a}^\dagger\hat{\rho}_f\hat{a}) + R_g(\hat{a}^\dagger\hat{a}\hat{\rho}_f - \hat{a}\hat{\rho}_f\hat{a}^\dagger)] + \text{adjoint},$$

where R_e and R_g are the photon emission and absorption rate coefficients. With the P-representation for the density operator, $\hat{\rho} = \int \mathrm{d}^2\alpha P(\alpha,\alpha^*)|\alpha\rangle\langle\alpha|$, we have

$$\int \mathrm{d}^2\alpha \dot{P}|\alpha\rangle\langle\alpha| = \frac{-1}{2}\int \mathrm{d}^2\alpha P[R_e(\hat{a}\hat{a}^\dagger|\alpha\rangle\langle\alpha| - \hat{a}^\dagger|\alpha\rangle\langle\alpha|\hat{a}) + R_g(\hat{a}^\dagger\hat{a}|\alpha\rangle\langle\alpha| - \hat{a}|\alpha\rangle\langle\alpha|\hat{a}^\dagger)] \tag{46}$$

$$+ \text{ adjoint}. \tag{47}$$

A. Fokker-Planck equation

With

$$|\alpha\rangle\langle\alpha|\hat{a} = (\frac{\partial}{\partial\alpha^*} + \alpha)|\alpha\rangle\langle\alpha|, \tag{48}$$

$$\hat{a}^\dagger|\alpha\rangle\langle\alpha| = (\frac{\partial}{\partial\alpha} + \alpha^*)|\alpha\rangle\langle\alpha|, \tag{49}$$

we have the Fokker-Planck equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}P(\alpha,\alpha^*) = \frac{-1}{2}(R_e - R_g)\{\frac{\partial}{\partial\alpha}[\alpha P(\alpha,\alpha^*)] + \frac{\partial}{\partial\alpha^*}[\alpha^* P(\alpha,\alpha^*)]\} + R_e\frac{\partial^2}{\partial\alpha\partial\alpha^*}P(\alpha,\alpha^*),$$

compared with,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_f(t) = \frac{-1}{2}[R_e(\hat{a}\hat{a}^\dagger\hat{\rho}_f - \hat{a}^\dagger\hat{\rho}_f\hat{a}) + R_g(\hat{a}^\dagger\hat{a}\hat{\rho}_f - \hat{a}\hat{\rho}_f\hat{a}^\dagger)] + \text{adjoint}.$$

The advantage of the Fokker-Planck equation is that it significantly simplifies the calculation process for the fields that are approximately coherent states. When the fields become nonclassical, the P-representation is no longer well-behaved, such as the squeezed and photon number states. In order to map an arbitrary nonclassical state into a classical probability density, the dimension of the phase space must at least be doubled. One may use off-diagonal or *positive-P-representation* for nonclassical states.

IV. Q-REPRESENTATION, ANTI-NORMALLY ORDERING

For $s = 1$, the density matrix in the coherent state representation is,

$$\langle\alpha|\hat{\rho}|\alpha\rangle = \frac{1}{\pi}\int d^2\xi G(\xi,\xi^*)\exp[i(\alpha\xi+\alpha^*\xi^*))], \tag{50}$$

$$= \pi f^{(1)}(\alpha,\alpha^*) \equiv Q(\alpha,\alpha^*), \tag{51}$$

where $f^{(1)}(\alpha,\alpha^*)$ is simply the matrix element of the operator in the coherent states representation, known as the Q-function. The expectation value, $\mathrm{Tr}[\hat{A}\hat{\rho}] = \frac{1}{\pi}\int d^2\alpha f^{(s)}(\alpha,\alpha^*)A^{(-s)}(\alpha,\alpha^*)$. If the density operator is represented by P-function, then

$$\langle\hat{a}^{\dagger m}\hat{a}^n\rangle = \int d^2\alpha P(\alpha,\alpha^*)\alpha^{*m}\alpha^m.$$

If the density operator is represented by P-function, then

$$\langle\hat{a}^n\hat{a}^{\dagger m}\rangle = \int d^2\alpha Q(\alpha,\alpha^*)\alpha^{*m}\alpha^m.$$

Q-representation is defined as the antinormally ordering in the trace,

$$Q(\alpha,\alpha^*) = \mathrm{Tr}[\hat{\rho}\delta(\alpha-\hat{a})\delta(\alpha^*-\hat{a}^\dagger)], \tag{52}$$

$$= \frac{1}{\pi}\mathrm{Tr}\int d^2\beta[\hat{\rho}\delta(\alpha-\hat{a})|\beta\rangle\langle\beta|\delta(\alpha^*-\hat{a}^\dagger)], \tag{53}$$

$$= \frac{1}{\pi}\mathrm{Tr}[\hat{\rho}|\alpha\rangle\langle\alpha|], \tag{54}$$

$$= \frac{1}{\pi}\langle\alpha|\hat{\rho}|\alpha\rangle, \tag{55}$$

i.e. $Q(\alpha,\alpha^*)$ is proportional to the diagonal element of the density operator in the coherent state representation. Unlike P-representation, $Q(\alpha,\alpha^*)$ is non-negative definite and bounded, i.e.

$$Q(\alpha,\alpha^*) = \frac{1}{\pi}\sum_\psi P_\psi|\langle\psi|\alpha\rangle|^2,$$

Since $|\langle\psi|\alpha\rangle|^2 \leq 1$, we have

$$Q(\alpha,\alpha^*) \leq \frac{1}{\pi},$$

Q-representation may be related to the P-representation as,

$$Q(\alpha,\alpha^*) = \frac{1}{\pi}\int d^2\beta P(\beta,\beta^*)e^{-|\alpha-\beta|^2}.$$

For a number state $|n\rangle$, its Q-representation is,

$$Q(\alpha,\alpha^*) = \frac{1}{\pi}|\langle n|\alpha\rangle|^2 = \frac{e^{-|\alpha|^2}|\alpha|^{2n}}{\pi n!},$$

For a squeezed state $|\beta,\xi\rangle$, its Q-representation is,

$$Q(\alpha,\alpha^*) = \frac{1}{\pi}|\langle\alpha|\beta,\xi\rangle|^2, \tag{56}$$

$$= \frac{\mathrm{sech}r}{\pi}\exp\{-(|\alpha|^2+|\beta|^2)+(\alpha^*\beta+\beta^*\alpha)\mathrm{sech}r \tag{57}$$

$$- \frac{1}{2}[e^{i\theta}(\alpha^{*2}-\beta^{*2}+e^{-i\theta}(\alpha^2-\beta^2)]\mathrm{tanh}r\}, \tag{58}$$

In the quarature phase-space, $X_1 = (\alpha+\alpha^*)/2$ and $X_1 = (\alpha-\alpha^*)/2i$,

$$Q(\alpha,\alpha^*) = \frac{\mathrm{sech}r}{\pi}\exp\{-(|\alpha|^2+|\beta|^2)+(\alpha^*\beta+\beta^*\alpha)\mathrm{sech}r \tag{59}$$

$$- \frac{1}{2}[e^{i\theta}(\alpha^{*2}-\beta^{*2}+e^{-i\theta}(\alpha^2-\beta^2)]\mathrm{tanh}r\}, \tag{60}$$

V. W-REPRESENTATION, SYMMETRIC ORDERING

The quantum analog of the classical phase space distribution function in the s-ordering is,

$$f^{(s)}(\alpha,\alpha^*) = \frac{1}{\pi^2}\int \mathrm{d}^2\xi \exp[i(\alpha\xi+\alpha^*\xi^*)]\exp(-\frac{s}{2}|\xi|^2)\mathrm{Tr}\{\exp[-i(\xi\hat{a}+\xi^*\hat{a}^\dagger)]\hat{\rho}\},$$

For $s=0$,

$$f^{(0)}(\alpha,\alpha^*) = W(\alpha,\alpha^*) = \frac{1}{\pi^2}\int \mathrm{d}^2\xi \exp[i(\alpha\xi+\alpha^*\xi^*)]\mathrm{Tr}\{\exp[-i(\xi\hat{a}+\xi^*\hat{a}^\dagger)]\hat{\rho}\},$$

here the Wigner-Weyl distibution function $W(\alpha,\alpha^*)$ is associated with symmetric ordering. For example

$$\frac{1}{2}\langle\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}\rangle = \int \mathrm{d}^2\alpha W(\alpha,\alpha^*)\alpha\alpha^*,$$

the Wigner function can be measured experimentally, including its negative values. In terms of $\hat{q}$ and $\hat{p}$,

$$W(p,q) = \frac{1}{(2\pi)^2}\int \mathrm{d}\sigma \int \mathrm{d}\tau \exp[i(\tau p+\sigma q)]\mathrm{Tr}\{\exp[-i(\tau\hat{p}+\sigma\hat{q})]\hat{\rho}\}, \tag{61}$$

$$= \frac{1}{(2\pi)^2}\int \mathrm{d}\sigma \int \mathrm{d}\tau e^{[i(\tau p+\sigma q)]}\mathrm{Tr}\{e^{(-i\tau\hat{p}/2)}e^{(-i\sigma\hat{q}/2)}\hat{\rho}e^{(-i\tau\hat{p}/2)}\}e^{(-i\sigma\hat{q}/2)}, \tag{62}$$

$$= \frac{1}{(2\pi)^2}\int \mathrm{d}\sigma \int \mathrm{d}\tau e^{[i(\tau p+\sigma q)]}\int \mathrm{d}q'\langle q'|e^{(-i\tau\hat{p}/2)}e^{(-i\sigma\hat{q}/2)}\hat{\rho}e^{(-i\tau\hat{p}/2)}|q'\rangle e^{(-i\sigma\hat{q}/2)}. \tag{63}$$

As

$$\exp(-i\tau\hat{p}/2)|q'\rangle = |q'-\hbar\tau/2\rangle,$$

we have

$$W(p,q) = \frac{1}{(2\pi)^2}\int \mathrm{d}\sigma \int \mathrm{d}\tau e^{i\sigma(q-q')}\int \mathrm{d}q'\langle q'+\hbar\tau/2|\hat{\rho}|q'-\hbar\tau/2\rangle e^{i\tau p}, \tag{64}$$

$$= \frac{1}{\pi\hbar}\int \mathrm{d}y e^{(-2yp/\hbar)}\langle q'-y|\hat{\rho}|q'+y\rangle, \tag{65}$$

where $y=-\hbar\tau/2$.