Note for *Quantum Optics*: Squeezed states

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Reference:
Ch. 2, 4, 16, in *"Quantum Optics,"* by M. Scully and M. Zubairy.
Ch. 3, 4, in *"Mesoscopic Quantum Optics,"* by Y. Yamamoto and A. Imamoglu.
Ch. 6, in *"The Quantum Theory of Light,"* by R. Loudon.
Ch. 5, 7, in *"Introductory Quantum Optics,"* by C. Gerry fand P. Knight.
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I. COHERENT AND SQUEEZED STATES

From the *uncertainty Principle*:

$$\Delta\hat{X}_1\Delta\hat{X}_2 \geq 1,$$

then we have

1. Coherent states: $\Delta\hat{X}_1 = \Delta\hat{X}_2 = 1$,
2. Amplitude squeezed states: $\Delta\hat{X}_1 < 1$,
3. Phase squeezed states: $\Delta\hat{X}_2 < 1$,
4. Quadrature squeezed states.

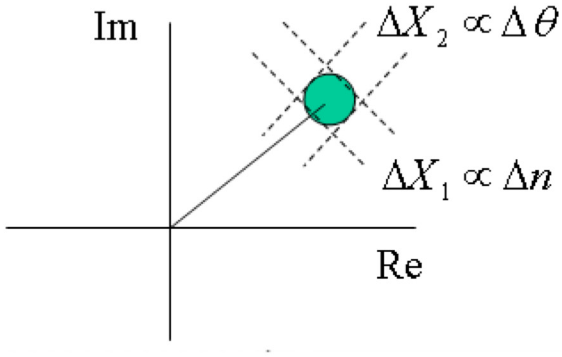

FIG. 1: Noise fluctuations in the phase space.

II. SQUEEZED STATES AND SHO

Suppose we can apply a dc field to the simple harmonic oscillator (SHO), but with a *wall* which limits the SHO to a finite region. In such a case, it would be expected that the wave packet would be deformed or *squeezed* when it is pushed against the barrier. Similarly the quadratic displacement potential would be expected to produce a squeezed wave packet:

$$\hat{H} = \frac{p^2}{2m} + \frac{1}{2}kx^2 - eE_0(ax - bx^2), \tag{1}$$

where the ax term will displace the oscillator and the bx^2 is added in order to give us a barrier. Then the corresponding Hamiltonian becomes

$$\hat{H} = \frac{p^2}{2m} + \frac{1}{2}(k + 2ebE_0)x^2 - eaE_0 x. \tag{2}$$

We again have a displaced ground state, but with the larger effective spring constant $k' = k + 2ebE_0$.

To generate squeezed state, we need quadratic terms in x, *i.e.*, terms of the form

$$(\hat{a} + \hat{a}^\dagger)^2. \tag{3}$$

For the degenerate parametric process, *i.e.*, two-photon process, its Hamiltonian is

$$\hat{H} = i\hbar(g\hat{a}^{\dagger 2} - g^*\hat{a}^2), \tag{4}$$

where g is a coupling constant. The state of fields generated by this Hamiltonian is

$$|\Psi(t)\rangle = \exp[(g\hat{a}^{\dagger 2} - g^*\hat{a}^2)t]|0\rangle. \tag{5}$$

Then, one can define the unitary squeeze operator

$$\hat{S}(\xi) = \exp[\frac{1}{2}\xi^*\hat{a}^2 - \frac{1}{2}\xi\hat{a}^{\dagger 2}], \tag{6}$$

where $\xi = r\exp(i\theta)$ is an arbitrary complex number.

III. PROPERTIES OF SQUEEZED OPERATOR

With the squeezed operator, we can define the squeezed state as,

$$|\Psi_s\rangle = \hat{S}(\xi)|\Psi\rangle. \tag{7}$$

For the unitary squeeze operator $\hat{S}(\xi) = \exp[\frac{1}{2}\xi^*\hat{a}^2 - \frac{1}{2}\xi\hat{a}^{\dagger 2}]$, it has following properties:

- Squeeze operator is unitary,

$$\hat{S}^\dagger(\xi) = \hat{S}^{-1}(\xi) = \hat{S}(-\xi), \tag{8}$$

  and the corresponding unitary transformation of the squeeze operator,

$$\hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi) = \hat{a}\cosh r - \hat{a}^\dagger e^{i\theta}\sinh r, \tag{9}$$

$$\hat{S}^\dagger(\xi)\hat{a}^\dagger\hat{S}(\xi) = \hat{a}^\dagger\cosh r - \hat{a}e^{-i\theta}\sinh r, \tag{10}$$

  with the formula $e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!}[\hat{A}, [\hat{A}, \hat{B}]], \ldots$

- A squeezed coherent state $|\alpha, \xi\rangle$ is obtained by first acting with the displacement operator $\hat{D}(\alpha)$ on the vacuum followed by the squeezed operator $\hat{S}(\xi)$, *i.e.*,

$$|\alpha, \xi\rangle = \hat{S}(\xi)\hat{D}(\alpha)|0\rangle, \tag{11}$$

  with $\alpha = |\alpha|\exp(i\psi)$.

- If $|\Psi\rangle$ is the vacuum state $|0\rangle$, then $|\Psi_s\rangle$ state is the *squeezed vacuum*,

$$|\xi\rangle = \hat{S}(\xi)|0\rangle. \tag{12}$$

- The variances for squeezed vacuum are

$$\Delta\hat{a}_1^2 = \frac{1}{4}[\cosh^2 r + \sinh^2 r - 2\sinh r\cosh r\cos\theta], \tag{13}$$

$$\Delta\hat{a}_2^2 = \frac{1}{4}[\cosh^2 r + \sinh^2 r + 2\sinh r\cosh r\cos\theta], \tag{14}$$

- For $\theta = 0$, we have

$$\Delta\hat{a}_1^2 = \frac{1}{4}e^{-2r}, \quad \text{and} \quad \Delta\hat{a}_2^2 = \frac{1}{4}e^{+2r}, \tag{15}$$

  and squeezing exists in the $\hat{a}_1$ quadrature.

- For $\theta = \pi$, the squeezing will appear in the $\hat{a}_2$ quadrature.

A. Quadrature Operators

One can define a rotated complex amplitude at an angle $\theta/2$

$$\hat{Y}_1 + i\hat{Y}_2 = (\hat{a}_1 + i\hat{a}_2)e^{-i\theta/2} = \hat{a}e^{-i\theta/2}, \tag{16}$$

where

$$\begin{pmatrix} \hat{Y}_1 \\ \hat{Y}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \cos\theta/2 \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}, \tag{17}$$

then

$$\hat{S}^\dagger(\xi)(\hat{Y}_1 + i\hat{Y}_2)\hat{S}(\xi) = \hat{Y}_1 e^{-r} + i\hat{Y}_2 e^{r}. \tag{18}$$

The corresponding quadrature variance are

$$\Delta\hat{Y}_1^2 = \frac{1}{4}e^{-2r}, \quad \Delta\hat{Y}_2^2 = \frac{1}{4}e^{+2r}, \quad \text{and} \quad \Delta\hat{Y}_1\Delta\hat{Y}_2 = \frac{1}{4}. \tag{19}$$

In the complex amplitude plane the coherent state error circle is squeezed into an *error ellipse* of the same area. The degree of squeezing is determined by $r = |\xi|$ which is called the squeezed parameter.

B. Squeezed Coherent State

A squeezed coherent state $|\alpha, \xi\rangle$ is obtained by first acting with the displacement operator $\hat{D}(\alpha)$ on the vacuum followed by the squeezed operator $\hat{S}(\xi)$, *i.e.*

$$|\alpha, \xi\rangle = \hat{D}(\alpha)\hat{S}(\xi)|0\rangle, \tag{20}$$

where $\hat{S}(\xi) = \exp[\frac{1}{2}\xi^*\hat{a}^2 - \frac{1}{2}\xi\hat{a}^{\dagger 2}]$. For $\xi = 0$, we obtain just a coherent state. The corresponding expectation value for a squeezed coherent state is,

$$\langle\alpha, \xi|\hat{a}|\alpha, \xi\rangle = \alpha, \tag{21}$$
$$\langle\hat{a}^2\rangle = \alpha^2 - e^{i\theta}\sinh r \cosh r, \tag{22}$$
$$\langle\hat{a}^\dagger\hat{a}\rangle = |\alpha|^2 + \sinh^2 r, \tag{23}$$

with helps of

$$\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a} + \alpha, \tag{24}$$
$$\hat{D}^\dagger(\alpha)\hat{a}^\dagger\hat{D}(\alpha) = \hat{a}^\dagger + \alpha^*. \tag{25}$$

Again, when $r \to 0$ we have coherent state, and $\alpha \to 0$ we have squeezed vacuum. Furthermore, the variance for a squeezed coherent state are

$$\langle\alpha, \xi|\hat{Y}_1 + i\hat{Y}_2|\alpha, \xi\rangle = \alpha e^{-i\theta/2}, \tag{26}$$
$$\langle\Delta\hat{Y}_1^2\rangle = \frac{1}{4}e^{-2r}, \tag{27}$$
$$\langle\Delta\hat{Y}_2^2\rangle = \frac{1}{4}e^{+2r}. \tag{28}$$

IV. SQUEEZED STATE AS A EIGEN-STATE

Since the vacuum state $\hat{a}|0\rangle = 0$, we have

$$\hat{S}(\xi)\hat{a}\hat{S}^\dagger(\xi)\hat{S}(\xi)|0\rangle = 0, \quad \text{or} \quad \hat{S}(\xi)\hat{a}\hat{S}^\dagger(\xi)|\xi\rangle = 0. \tag{29}$$

From the transformation,

$$\hat{S}(\xi)\hat{a}\hat{S}^\dagger(\xi) = \hat{a}\cosh r + \hat{a}^\dagger e^{i\theta}\sinh r \equiv \mu\hat{a} + \nu\hat{a}^\dagger, \tag{30}$$

we have,

$$(\mu\hat{a} + \nu\hat{a}^\dagger)|\xi\rangle = 0, \tag{31}$$

the squeezed vacuum state is an eigenstate of the operator $\mu\hat{a} + \nu\hat{a}^\dagger$ with eigenvalue zero. Similarly,

$$\hat{D}(\alpha)\hat{S}(\xi)\hat{a}\hat{S}^\dagger(\xi)\hat{D}^\dagger(\alpha)\hat{D}(\alpha)|\xi\rangle = 0, \tag{32}$$

with the relation $\hat{D}(\alpha)\hat{a}\hat{D}^\dagger(\alpha) = \hat{a} - \alpha$, we have

$$(\mu\hat{a} + \nu\hat{a}^\dagger)|\alpha,\xi\rangle = (\alpha\cosh r + \alpha^*\sinh r)|\alpha,\xi\rangle \equiv \gamma|\alpha,\xi\rangle. \tag{33}$$

A. Squeezed State and Minimum Uncertainty State

With the eigenvalue problem for the squeezed state

$$(\mu\hat{a} + \nu\hat{a}^\dagger)|\alpha,\xi\rangle = (\alpha\cosh r + \alpha^*\sinh r)|\alpha,\xi\rangle \equiv \gamma|\alpha,\xi\rangle, \tag{34}$$

in terms of in terms of $\hat{a} = (\hat{Y}_1 + i\hat{Y}_2)e^{i\theta/2}$, we have

$$(\hat{Y}_1 + ie^{-2r}\hat{Y}_2)|\alpha,\xi\rangle = \beta_1|\alpha,\xi\rangle, \tag{35}$$

where

$$\beta_1 = \gamma e^{-r}e^{-i\theta/2} = \langle\hat{Y}_1\rangle + i\langle\hat{Y}_2\rangle e^{-2r}. \tag{36}$$

In the other way, in terms of $\hat{a}_1$ and $\hat{a_2}$, we have

$$(\hat{a}_1 + i\lambda\hat{a}_2^\dagger)|\alpha,\xi\rangle = \beta_2|\alpha,\xi\rangle, \tag{37}$$

where

$$\lambda = \frac{\mu - \nu}{\mu + \nu}, \quad \text{and} \quad \beta_2 = \frac{\gamma}{\mu + \nu}. \tag{38}$$

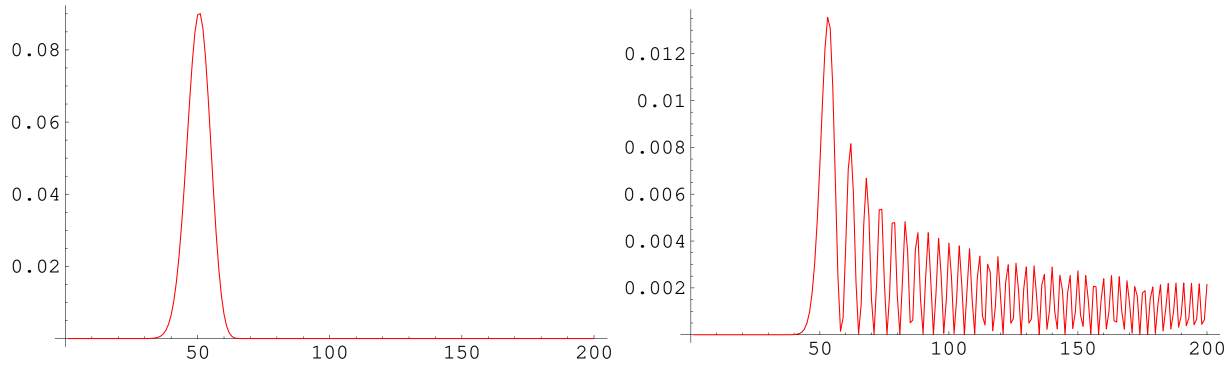

FIG. 2: Photon number distribution for squeezed coherent states. Left: $|\alpha|^2 = 50, \theta = 0, r = 0.5$; Right: $|\alpha|^2 = 50, \theta = 0, r = 4.0$.

V. SQUEEZED STATE IN THE BASIS OF NUMBER STATES

Consider squeezed vacuum state first,

$$|\xi\rangle = \sum_{n=0}^{\infty} C_n |n\rangle, \tag{39}$$

with the operator of $(\mu\hat{a} + \nu\hat{a}^\dagger)|\xi\rangle = 0$, we have

$$C_{n+1} = -\frac{\nu}{\mu}(\frac{n}{n+1})^{1/2} C_{n-1}. \tag{40}$$

It can be seen clearly that only the even photon states have the solutions,

$$C_{2m} = (-1)^m (e^{i\theta}\tanh r)^m [\frac{(2m-1)!!}{(2m)!!}]^{1/2} C_0, \tag{41}$$

where C_0 can be determined from the normalization, *i.e.*, $C_0 = \sqrt{\cosh r}$. In the basis of number state, the squeezed vacuum state is

$$|\xi\rangle = \frac{1}{\sqrt{\cosh r}} \sum_{m=0}^{\infty} (-1)^m \frac{\sqrt{(2m)!}}{2^m m!} e^{im\theta} \tanh^m r |2m\rangle. \tag{42}$$

The probability of detecting $2m$ photons in the field is

$$P_{2m} = |\langle 2m|\xi\rangle|^2 = \frac{(2m)!}{2^{2m}(m!)^2} \frac{\tanh^{2m} r}{\cosh r}. \tag{43}$$

Nevertheless, for detecting $2m + 1$ states, the probability is zero, $P_{2m+1} = 0$. Moreover, the photon probability distribution for a squeezed vacuum state is *oscillatory*, vanishing for all odd photon numbers. The shape of the squeezed vacuum state resembles that of thermal radiation.

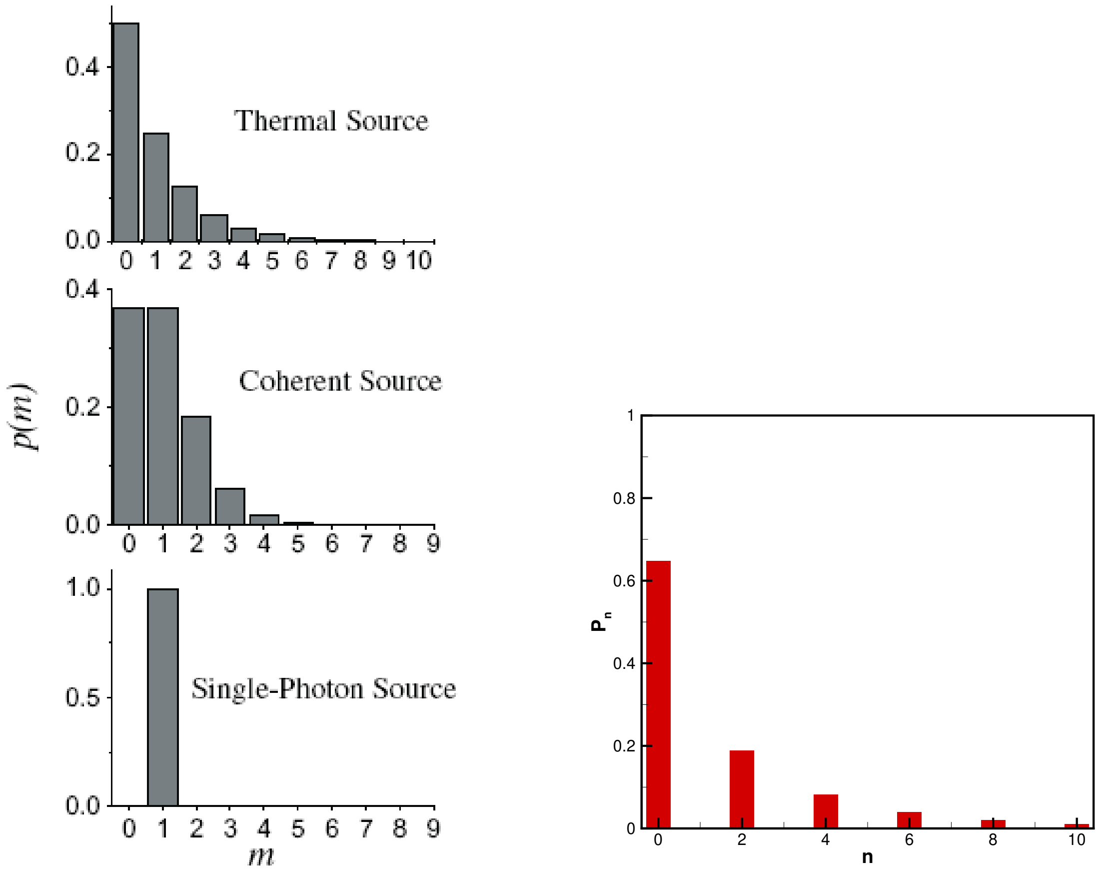

FIG. 3: Left: Photon number distributions for thermal state, coherent state, and single-photon state (from top to down). Right: Photon number distribution for a squeezed state.

VI. GENERATIONS OF SQUEEZED STATES

Generation of quadrature squeezed light are based on some sort of *parametric process* utilizing various types of nonlinear optical devices. For degenerate parametric down-conversion, the nonlinear medium is pumped by a field of frequency ω_p and that field are converted into pairs of identical photons, of frequency $\omega = \omega_p/2$ each,

$$\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\omega_p\hat{b}^\dagger\hat{b} + i\hbar\chi^{(2)}(\hat{a}^2\hat{b}^\dagger - \hat{a}^{\dagger 2}\hat{b}), \tag{44}$$

where b is the pump mode and a is the signal mode. Assume that the field is in a coherent state $|\beta e^{-i\omega_p t}\rangle$ and approximate the operators $\hat{b}$ and $\hat{b}^\dagger$ by classical amplitude $\beta e^{-i\omega_p t}$ and $\beta^* e^{i\omega_p t}$, respectively, we have the interaction Hamiltonian for *degenerate parametric down-conversion*,

$$\hat{H}_I = i\hbar(\eta^*\hat{a}^2 - \eta\hat{a}^{\dagger 2}), \tag{45}$$

where $\eta = \chi^{(2)}\beta$. The associated evolution operator is,

$$\hat{U}_I(t) = \exp[-i\hat{H}_I t/\bar{]}] = \exp[(\eta^*\hat{a}^2 - \eta\hat{a}^{\dagger 2})t] \equiv \hat{S}(\xi), \tag{46}$$

with $\xi = 2\eta t$. Similarly , for degenerate four-wave mixing, in which two pump photons are converted into two signal photons of the same frequency,

$$\hat{H} = \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\omega\hat{b}^\dagger\hat{b} + i\hbar\chi^{(3)}(\hat{a}^2\hat{b}^{\dagger 2} - \hat{a}^{\dagger 2}\hat{b}^2), \tag{47}$$

the associated evolution operator is,

$$\hat{U}_I(t) = \exp[(\eta^*\hat{a}^2 - \eta\hat{a}^{\dagger 2})t] \equiv \hat{S}(\xi), \tag{48}$$

with $\xi = 2\chi^{(3)}\beta^2 t$.

VII. DETECTION OF SQUEEZED STATES

In the homodyne detection scheme, the detectors measure the intensities $I_c = \langle \hat{c}^\dagger \hat{c} \rangle$ and $I_d = \langle \hat{d}^\dagger \hat{d} \rangle$, and the difference in these intensities is,

$$I_c - I_d = \langle \hat{n}_{cd} \rangle = \langle \hat{c}^\dagger \hat{c} - \hat{d}^\dagger \hat{d} \rangle = i \langle \hat{a}^\dagger \hat{b} - \hat{a} \hat{b}^\dagger \rangle. \qquad (49)$$

Assume that the b mode to be in the coherent state $|\beta e^{-i\omega t}\rangle$, where $\beta = |\beta| e^{-i\psi}$, we have

$$\langle \hat{n}_{cd} \rangle = |\beta| \{ \hat{a} e^{i\omega t} e^{-i\theta} + \hat{a}^\dagger e^{-i\omega t} e^{i\theta} \}, \qquad (50)$$

where $\theta = \psi + \pi/2$. Assume that a mode light is also of frequency ω (in practice both the a and b modes derive from the same laser), *i.e.*, $\hat{a} = \hat{a}_0 e^{-i\omega t}$, we have

$$\langle \hat{n}_{cd} \rangle = 2|\beta| \langle \hat{X}(\theta) \rangle, \qquad (51)$$

where $\hat{X}(\theta) = \frac{1}{2}(\hat{a}_0 e^{-i\theta} + \hat{a}_0^\dagger e^{i\theta})$ is the field quadrature operator at the angle θ. By changing the phase ψ of the local oscillator, we can measure an arbitrary quadrature of the signal field.

In other words, for homodyne detection, mode a contains the single field that is possibly squeezed; while mode b contains a strong coherent classical field, *local oscillator*, which may be taken as coherent state of amplitude β. For a balanced homodyne detection, through a $50:50$ beam splitter, the relation between input $(\hat{a}, \hat{b})$ and output $(\hat{c}, \hat{d})$ is,

$$\hat{c} = \frac{1}{\sqrt{2}}(\hat{a} + i\hat{b}), \qquad \hat{d} = \frac{1}{\sqrt{2}}(\hat{b} + i\hat{a}). \qquad (52)$$

The detectors measure the intensities $I_c = \langle \hat{c}^\dagger \hat{c} \rangle$ and $I_d = \langle \hat{d}^\dagger \hat{d} \rangle$, and the difference in these intensities is,

$$I_c - I_d = \langle \hat{n}_{cd} \rangle = \langle \hat{c}^\dagger \hat{c} - \hat{d}^\dagger \hat{d} \rangle = i \langle \hat{a}^\dagger \hat{b} - \hat{a} \hat{b}^\dagger \rangle. \qquad (53)$$