Note for *Quantum Optics*: Uncertainty Relation

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Reference:
Ch. 1-5 in P. Dirac *"The Principles of Quantum Mechanics,"* Oxford University Press (1958).
Chapter 2, in J. J. Sakurai, *"Modern Quantum Mechanics,"* Addison Wesley (1994).

I. UNCERTAINTY RELATION

1. Non-commuting observable do not admit common eigenvectors.
2. Non-commuting observables can not have definite values simultaneously.
3. Simultaneous measurement of non-commuting observables to an arbitrary degree of accuracy is thus *incompatible*.
4. Variance: one can define

$$\Delta\hat{A}^2 = \langle\Psi|(\hat{A} - \langle\hat{A}\rangle)^2|\Psi\rangle = \langle\Psi|\hat{A}^2|\Psi\rangle - \langle\Psi|\hat{A}|\Psi\rangle^2. \tag{1}$$

5. For any two non-commuting observables,

$$[\hat{A}, \hat{B}] = i\hat{C},$$

   we have the *uncertainty relation*:

$$\Delta A^2 \Delta B^2 \geq \frac{1}{4}[\langle\hat{F}\rangle^2 + \langle\hat{C}\rangle^2], \tag{2}$$

   where

$$\hat{F} = \hat{A}\hat{B} + \hat{B}\hat{A} - 2\langle\hat{A}\rangle\langle\hat{B}\rangle, \tag{3}$$

   where the operator $\hat{F}$ is a measure of correlations between $\hat{A}$ and$\hat{B}$.

For example, take the operators $\hat{A} = \hat{q}$ (position) and $\hat{B} = \hat{p}$ (momentum) for a free particle, one have

$$[\hat{q}, \hat{p}] = i\hbar \rightarrow \langle\Delta\hat{q}^2\rangle\langle\Delta\hat{p}^2\rangle \geq \frac{\hbar^2}{4}. \tag{4}$$

A. Proof of the Uncertainty Relation

Start from the *Schwarz inequality*

$$\langle\phi|\phi\rangle\langle\psi|\psi\rangle \geq \langle\phi|\psi\rangle\langle\psi|\phi\rangle, \tag{5}$$

where the equality holds *if and only if* the two states are *linear dependent*, $|\psi\rangle = \lambda|\phi\rangle$, where λ is a complex number. Define two states,

$$|\psi_1\rangle = [\hat{A} - \langle\hat{A}\rangle]|\psi\rangle, \tag{6}$$

$$|\psi_2\rangle = [\hat{B} - \langle\hat{B}\rangle]|\psi\rangle. \tag{7}$$

To have a minimum value of the uncertainty product, we have $|\psi_1\rangle = -i\lambda|\psi_2\rangle$, or

$$[\hat{A} + i\lambda\hat{B}]|\psi\rangle = [\langle\hat{A}\rangle + i\lambda\langle\hat{B}\rangle]|\psi\rangle = z|\psi\rangle, \tag{8}$$

where z is a complex number. And the state $|\psi\rangle$ is called a *minimum uncertainty state*. There are difference values for the coefficient λ,

1. If $Re(\lambda) = 0$, then $\hat{A} + i\lambda\hat{B}$ is a normal operator, which have orthonormal eigenstates, with the

$$\Delta\hat{A}^2 = -\frac{i\lambda}{2}[\langle\hat{F}\rangle + i\langle\hat{C}\rangle], \tag{9}$$

$$\Delta\hat{B}^2 = -\frac{i}{2\lambda}[\langle\hat{F}\rangle - i\langle\hat{C}\rangle]. \tag{10}$$

2. If we set $\lambda = \lambda_r + i\lambda_i$, then

$$\Delta\hat{A}^2 = \frac{1}{2}[\lambda_i\langle\hat{F}\rangle + \lambda_r\langle\hat{C}\rangle], \tag{11}$$

$$\Delta\hat{B}^2 = \frac{1}{|\lambda|^2}\Delta\hat{A}^2, \tag{12}$$

along with the condition that

$$\lambda_i\langle\hat{C}\rangle - \lambda_r\langle\hat{F}\rangle = 0. \tag{13}$$

- If $|\lambda| = 1$, we have

$$\Delta\hat{A}^2 = \Delta\hat{B}^2, \tag{14}$$

which are *equal variance minimum uncertainty states*.

- If $|\lambda| = 1$ along with $\lambda_i = 0$, we have

$$\Delta\hat{A}^2 = \Delta\hat{B}^2 \qquad \text{and} \qquad \langle\hat{F}\rangle = 0, \tag{15}$$

which are *uncorrelated equal variance minimum uncertainty states*.

- If $\lambda_r \neq 0$, we have

$$\langle\hat{F}\rangle = \frac{\lambda_i}{\lambda_r}\langle\hat{C}\rangle, \tag{16}$$

$$\Delta\hat{A}^2 = \frac{|\lambda|^2}{2\lambda_r}\langle\hat{C}\rangle, \tag{17}$$

$$\Delta\hat{B}^2 = \frac{1}{2\lambda_r}\langle\hat{C}\rangle. \tag{18}$$

If $\hat{C}$ is a positive operator then the minimum uncertainty states exist only if $\lambda_r > 0$.

II. UNCERTAINTY RELATION FOR $\hat{q}$ AND $\hat{p}$

Take the operators $\hat{A} = \hat{q}$ (position) and $\hat{B} = \hat{p}$ (momentum) for a free particle, then we have

$$[\hat{q}, \hat{p}] = i\hbar \rightarrow \langle\Delta\hat{q}^2\rangle\langle\Delta\hat{p}^2\rangle \geq \frac{\hbar^2}{4}. \tag{19}$$

If we define two states,

$$|\psi_1\rangle = [\hat{A} - \langle\hat{A}\rangle]|\psi\rangle \equiv \hat{\alpha}|\psi\rangle, \tag{20}$$

$$|\psi_2\rangle = [\hat{B} - \langle\hat{B}\rangle]|\psi\rangle \equiv \hat{\beta}|\psi\rangle. \tag{21}$$

For *uncorrelated minimum uncertainty states*, one has

$$\hat{\alpha}|\psi\rangle = -i\lambda\hat{\beta}|\psi\rangle, \qquad \langle\psi|\hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\alpha}|\psi\rangle = 0, \tag{22}$$

where λ is a real number. If $\hat{A} = \hat{q}$ and $\hat{B} = \hat{p}$, we have

$$(\hat{q} - \langle\hat{q}\rangle)|\psi\rangle = -i\lambda(\hat{p} - \langle\hat{p}\rangle)|\psi\rangle. \tag{23}$$

By defining the complex number

$$\lambda = e^{-2r}, \tag{24}$$

then

$$(e^r \hat{q} + ie^{-r}\hat{p})|\psi\rangle = (e^r \langle \hat{q}\rangle + ie^{-r}\langle \hat{p}\rangle)|\psi\rangle. \tag{25}$$

To have the minimum uncertainty state, we define it as an *eigenstate* of a non-Hermitian operator:

$$e^r \hat{q} + ie^{-r}\hat{p}, \tag{26}$$

with a c-number eigenvalue $e^r\langle \hat{q}\rangle + ie^{-r}\langle \hat{p}\rangle$, and the corresponding variances of $\hat{q}$ and $\hat{p}$ are

$$\langle \Delta\hat{q}^2\rangle = \frac{\hbar}{2}e^{-2r}, \tag{27}$$

$$\langle \Delta\hat{p}^2\rangle = \frac{\hbar}{2}e^{2r}, \tag{28}$$

here r is referred as the *squeezing parameter.*

III. GAUSSIAN WAVE PACKETS

In the x-space, we have a Gaussian wave packet with the form,

$$\Psi(x) = \langle x|\Psi\rangle = [\frac{1}{\pi^{1/4}\sqrt{d}}]\exp[ikx - \frac{x^2}{2d^2}], \tag{29}$$

which is a plane wave with wave number k and width d. The expectation value of $\hat{X}$ is zero due to the symmetry, *i.e.*,

$$\langle \hat{X}\rangle = \int_{-\infty}^{\infty} \mathrm{d}x \langle\Psi|x\rangle\hat{X}\langle x|\Psi\rangle = 0,$$

where the variation of $\hat{X}$ is

$$\langle \Delta\hat{X}^2\rangle = \frac{d^2}{2}. \tag{30}$$

In the p-space, the expectation value of $\hat{P}$ is $\langle\hat{P}\rangle = \hbar k$, *i.e.*,

$$\langle x|\hat{P}|\Psi\rangle = -i\hbar\frac{\partial}{\partial x}\langle x|\Psi\rangle, \tag{31}$$

while the variation of $\hat{P}$ is

$$\langle \Delta\hat{P}^2\rangle = \frac{\hbar^2}{2d^2}. \tag{32}$$

The Heisenberg uncertainty product for a Gaussian wave packet is

$$\langle \Delta\hat{X}^2\rangle\langle \Delta\hat{P}^2\rangle = \frac{\hbar^2}{4}. \tag{33}$$

A Gaussian wave packet is called a *minimum uncertainty wave packet.*

IV. TIME EVOLUTION OF A MINIMUM UNCERTAINTY STATE

For a free particle, the corresponding Hamiltonian is

$$\hat{H} = \frac{\hat{p}^2}{2m}, \tag{34}$$

with the unitary operator

$$\hat{U} = \exp(-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}t). \tag{35}$$

In the Schrödinger picture, the wave function evolves accordingly

$$\Psi(q,t) = \langle q|\hat{U}|\Psi(0)\rangle \ = \ \int_{-\infty}^{\infty} \mathrm{d}p\langle|p\rangle\Psi(p,0)\exp(-\frac{i}{\hbar}\frac{p^2}{2m}t), \tag{36}$$

$$= \ \frac{1}{(2\pi)^{1/4}(\Delta q + i\hbar t/2m\Delta q)^{1/2}}\exp[-\frac{q^2}{4(\Delta q)^2 + 2i\hbar t/m}], \tag{37}$$

where the variance in $x(q)0$-space is defined as

$$\Delta q \ = \ \hbar/2\langle\hat{p}^2\rangle^{1/2}. \tag{38}$$

It can be shown that even though the momentum uncertainty $\langle\Delta\hat{p}^2\rangle$ is preserved, but the position uncertainty increases as time develops,

$$\langle\Delta\hat{q}^2(t)\rangle = (\Delta\hat{q})^2 + \frac{\hbar^2 t^2}{4m^2(\Delta q)^2}. \tag{39}$$

This is known as the *free particle expansion.*

V. GAUSSIAN OPTICS

In free space, the vector potential, A, is defined as $A(r,t) = \vec{n}\psi(x,y,z)e^{j\omega t}$, which obeys the vector wave equation,

$$\nabla^2\psi + k^2\psi = 0. \tag{40}$$

With the paraxial wave approximation, $\psi(x,y,z) = u(x,y,z)e^{-jkz}$, one obtains

$$\nabla_T^2 u - 2jk\frac{\partial u}{\partial z} = 0, \tag{41}$$

where $\nabla_T \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}$. The solution of the scalar paraxial wave equation is,

$$u_{00}(x,y,z) = \frac{\sqrt{2}}{\sqrt{\pi}w}exp(j\phi)exp(-\frac{x^2+y^2}{w^2})exp[-\frac{jk}{2R}(x^2+y^2], \tag{42}$$

which is also a *Gaussian wave packet.*