7, Cavity Quantum ElectroDynamics (Cavity-QED)

1. Cavity Modes
2. Purcell effect
3. Input-Output Formulation
4. Intra-cavity Atomic Systems
5. Squeezed state generation

Ref:

Ch. 7, 13 in *"Quantum Optics,"* by D. Wall and G. Milburn.

Ch. 13 in *"Elements of Quantum Optics,"* by P. Meystre and M. Sargent III.

Ch. 10 in *"Introductory Quantum Optics,"* by C. Gerry and P. Knight.

Ch. 16 in *"Quantum Optics,"* by M. Scully and M. Zubairy.

"Theoretical Problems in Cavity Nonlinear Optics," by P. Mandel.

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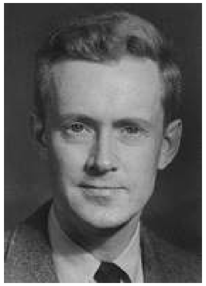
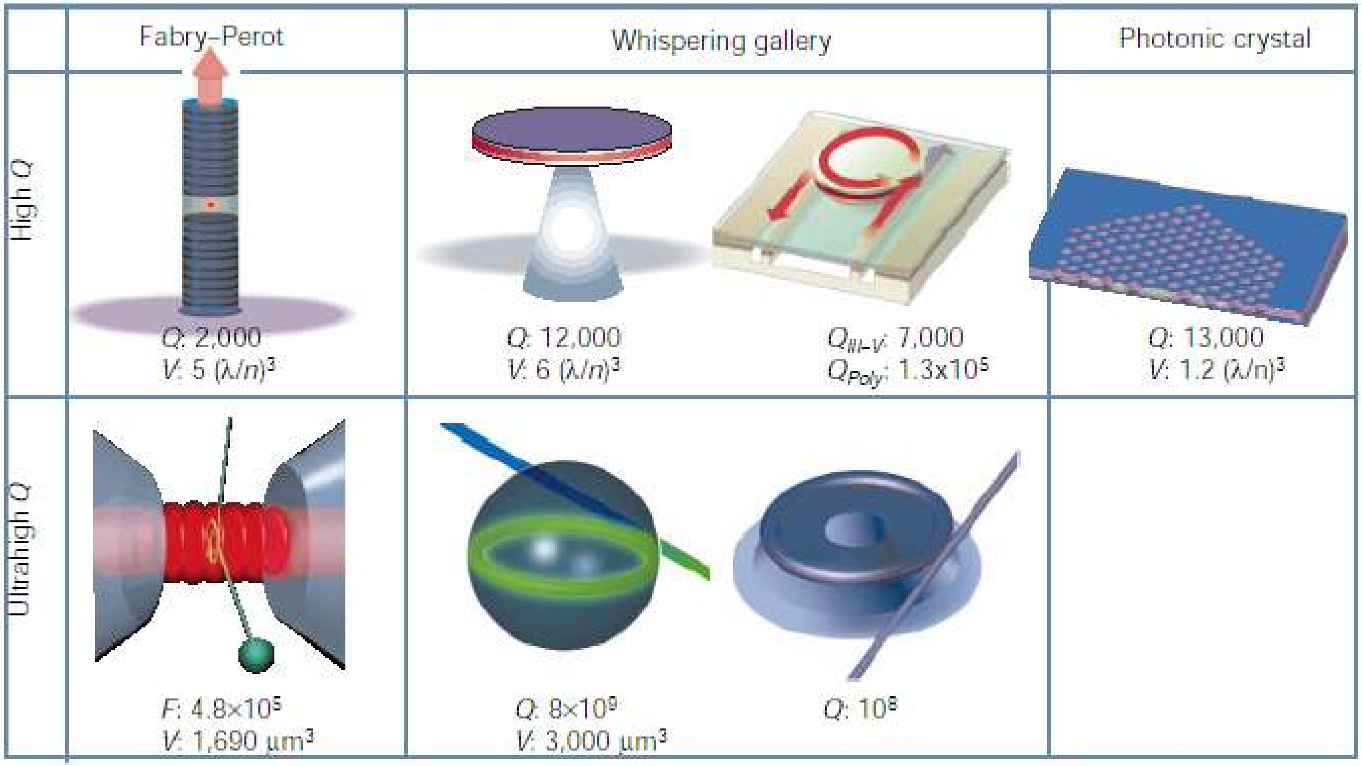
Purcell effect: Cavity-QED (Quantum ElectroDynamics)





E. M. Purcell, *Phys. Rev.* **69** (1946).

Nobel laureate Edward Mills Purcell (shared the prize with Felix Bloch) in 1952,

for their contribution to nuclear magnetic precision measurements.

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from: K. J. Vahala, *Nature* **424**, 839 (2003).

Field damping by field reservoirs

- consider a single-mode field in a cavity with a finite leakage rate,
- assume the reservoir density operator is a multimode thermal field,

$$\hat{\rho}_R = \prod_k \sum_n \frac{\exp(-\frac{\hbar\omega_k n}{k_B T})}{1-\exp(-\frac{\hbar\omega_k n}{k_B T})} |n\rangle_{kk}\langle n|,$$

- the equation of motion for the reduced density operator $\mathrm{Tr}_R[\hat{\rho}(t)] \equiv \hat{\rho}_f(t)$ is,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_f(t) &= (\frac{1}{i\hbar})^2 \int_0^t \mathrm{d}t' \mathrm{Tr}_R([\hat{H}_I(t), [\hat{H}_I(t'), \hat{\rho}_f(t) \otimes \hat{\rho}_R(0)]]), \\
&= -\int_{t_0}^t \mathrm{d}t' \sum_k g_k^2 \{n_{th}[\hat{a}\hat{a}^\dagger \hat{\rho}_f(t') - \hat{a}^\dagger \hat{\rho}_f(t')\hat{a}] e^{-i(\omega-\omega_k)(t-t')} \\
&\quad + (n_{th}+1)[\hat{a}^\dagger \hat{a}\hat{\rho}_f(t') - \hat{a}\hat{\rho}_f(t')\hat{a}^\dagger]) e^{i(\omega-\omega_k)(t-t')}\} + \text{H. C},
\end{aligned}$$

Field damping by field reservoirs

- the equation of motion for the reduced density operator $\mathrm{Tr}_R[\hat{\rho}(t)] \equiv \hat{\rho}_f(t)$ is,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_f(t) &= -\int_{t_0}^{t} \mathrm{d}t' \sum_k g_k^2 \{ n_{th}[\hat{a}\hat{a}^\dagger \hat{\rho}_f(t') - \hat{a}^\dagger \hat{\rho}_f(t')\hat{a}] e^{-i(\omega-\omega_k)(t-t')} \\
&\quad + (n_{th}+1)[\hat{a}^\dagger \hat{a}\hat{\rho}_f(t') - \hat{a}\hat{\rho}_f(t')\hat{a}^\dagger]) e^{i(\omega-\omega_k)(t-t')} \} + \mathrm{H.\ C},
\end{aligned}
$$

- again, by replacing $sum_k g_k^2$ with the integral $\int \mathrm{d}\omega_k D(\omega_k) g(\omega_k)^2$, and

$$
\begin{aligned}
\int_{t_0}^{t} \mathrm{d}t' \sum_k g_k^2 e^{\pm i(\omega-\omega_k)(t-t')} &= \int_{t_0}^{t} \mathrm{d}t' \int \mathrm{d}\omega_k D(\omega_k) g(\omega_k)^2 e^{\pm i(\omega-\omega_k)(t-t')}, \\
&\approx \int \mathrm{d}\omega_k D(\omega_k) g(\omega_k)^2 \pi\delta(\omega-\omega_k), \\
&\approx \pi D(\omega) g(\omega)^2 \equiv \frac{1}{2}(\frac{\omega}{Q_e}),
\end{aligned}
$$

where ω/Q_e is the cavity photon decay rate due to leakage (output coupling) via a partially reflecting mirror,

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Field damping by field reservoirs

- the equation of motion for the reduced density operator $\mathrm{Tr}_R[\hat{\rho}(t)] \equiv \hat{\rho}_f(t)$ is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_f(t) = -\frac{1}{2}(\frac{\omega}{Q_e})\{n_{th}[\hat{a}\hat{a}^{\dagger}\hat{\rho}_f(t') - \hat{a}^{\dagger}\hat{\rho}_f(t')\hat{a}] + (n_{th}+1)[\hat{a}^{\dagger}\hat{a}\hat{\rho}_f(t') - \hat{a}\hat{\rho}_f(t')\hat{a}^{\dagger}$$
$$+\text{H. C},$$

- compared to the case of atom damping by field reservoirs,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_a(t) = -\frac{1}{2}\Gamma\{n_{th}[\hat{\sigma}_-\hat{\sigma}_+\hat{\rho}_a - \hat{\sigma}_+\hat{\rho}_a\hat{\sigma}_-] + (n_{th}+1)[\hat{\sigma}_+\hat{\sigma}_-\hat{\rho}_a - \hat{\sigma}_-\hat{\rho}_a\hat{\sigma}_+])\} + \mathrm{H}$$

Input-output formulation of optical cavity

- in preceding chapters, we have used a *master* equation to calculate the photon statistics inside an optical cavity when the internal field is damped,
- in this approach, the field external to the cavity is treated as a *heat bath*, reservoir,
- the heat bath is simply a *passive* system with which the system gradually comes into equilibrium,
- now we would explicitly treat the heat bath as the external field, and determine the effect of the intra-cavity dynamics on the quantum statistics of the output field,

Cavity modes

- consider a single cavity mode interacting with an external field,
- the interaction Hamiltonian is

$$\hat{H}_I = i\hbar \int \mathrm{d}\omega g(\omega)[\hat{b}(\omega)\hat{a}^\dagger - \hat{a}\hat{b}^\dagger(\omega)],$$

- where $\hat{a}$ is the annihilation operator for the intra-cavity field, with the commutation relations,

$$[\hat{a}, \hat{a}^\dagger] = 1,$$

- where $\hat{b}(\omega)$ are the annihilation operators for the external field, with

$$[\hat{b}(\omega), \hat{b}^\dagger(\omega')] = \delta(\omega - \omega'),$$

- in actual fact the physical frequency limits are $(0, \infty)$,
- however, for high frequency optical systems we may shift the integration to a frequency ω_0, the cavity resonance frequency,
- the integration limits are $(-\omega_0, \infty)$, as ω_0 is large, then we approximate $\int_{-\infty}^{\infty}$,

Input-output theory

- the Heisenberg equation of motion for $\hat{b}(\omega)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{b}(\omega) = -i\omega\hat{b}(\omega) + g(\omega)\hat{a},$$

- with the initial condition at time $t_0 < t$, the *input*,

$$\hat{b}(\omega) = e^{-i\omega(t-t_0)}\hat{b}_0(\omega) + g(\omega)\int_{t_0}^{t} \mathrm{d}t' e^{-i\omega(t-t')}\hat{a}(t'),$$

where $t_0 < t$ and $\hat{b}_0(\omega)$ is the value of $\hat{b}(\omega)$ at $t = t_0$,

- or with the final condition at time $t_1 > t$, the *output*,

$$\hat{b}(\omega) = e^{-i\omega(t-t_1)}\hat{b}_1(\omega) - g(\omega)\int_{t}^{t_1} \mathrm{d}t' e^{-i\omega(t-t')}\hat{a}(t'),$$

where $t < t_1$ and $\hat{b}_1(\omega)$ is the value of $\hat{b}(\omega)$ at $t = t_1$,

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Input-output theory

- the system operator obeys the equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a} = -\frac{i}{\hbar}[\hat{a}, \hat{H}_S] - \int_{-\infty}^{\infty} \mathrm{d}\omega g(\omega)\hat{b}(\omega),$$

- in terms of the solutions with initial condition,

$$\hat{b}(\omega) = e^{-i\omega(t-t_0)}\hat{b}_0(\omega) + g(\omega)\int_{t_0}^{t} \mathrm{d}t' e^{-i\omega(t-t')}\hat{a}(t'),$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a} = -\frac{i}{\hbar}[\hat{a}, \hat{H}_S] - \int_{-\infty}^{\infty} \mathrm{d}\omega g(\omega)e^{-i\omega(t-t_0)}\hat{b}_0(\omega) - \int_{-\infty}^{\infty} \mathrm{d}\omega g^2(\omega)\int_{t_0}^{t} \mathrm{d}t' e^{-i\omega(t-t')}\hat{a}(t$$

- define an *input field*,

$$\hat{a}_{IN}(t) = \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{d}\omega e^{-i\omega(t-t_0)}\hat{b}_0(\omega),$$

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Input-output theory

- the system operator obeys the equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a} = -\frac{i}{\hbar}[\hat{a}, \hat{H}_S] - \int_{-\infty}^{\infty} \mathrm{d}\omega g(\omega) e^{-i\omega(t-t_0)}\hat{b}_0(\omega) - \int_{-\infty}^{\infty} \mathrm{d}\omega g^2(\omega) \int_{t_0}^{t} \mathrm{d}t' e^{-i\omega(t-t')}\hat{a}(t$$

- define an *input field*, $\hat{a}_I(t) = \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}\omega e^{-i\omega(t-t_0)}\hat{b}_0(\omega)$, which satisfy the commutation relation, $[\hat{a}_I(t), \hat{a}_I^\dagger(t')] = \delta(t-t')$,

- with Markovian approximation,

$$\begin{aligned} \int_{-\infty}^{\infty} \mathrm{d}\omega g^2(\omega) \int_{t_0}^{t} \mathrm{d}t' e^{-i\omega(t-t')}\hat{a}(t') &\approx g^2(\omega) \int_{-\infty}^{\infty} \mathrm{d}\omega \int_{t_0}^{t} \mathrm{d}t' e^{-i\omega(t-t')}\hat{a}(t'), \\ &= \frac{\gamma}{2\pi} 2\pi \int_{t_0}^{t} \mathrm{d}t' \delta(t-t')\hat{a}(t') = \frac{\gamma}{2} a(t), \end{aligned}$$

where we use following result,

$$\int_{t_0}^{t} \mathrm{d}t' \delta(t-t') f(t') = \int_{t}^{t_1} \mathrm{d}t' \delta(t-t') f(t') \frac{1}{2} f(t), \quad (t_0 < t < t_1),$$

Input-output theory

- the system operator obeys the equation,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{a} &= -\frac{i}{\hbar}[\hat{a},\hat{H}_S] - \int_{-\infty}^{\infty}\mathrm{d}\omega g(\omega)e^{-i\omega(t-t_0)}\hat{b}_0(\omega) - \int_{-\infty}^{\infty}\mathrm{d}\omega g^2(\omega)\int_{t_0}^{t}\mathrm{d}t' e^{-i\omega(t-t')}\hat{a}(\\
&= -\frac{i}{\hbar}[\hat{a},\hat{H}_S] - \frac{\gamma}{2}\hat{a}(t) + \sqrt{\gamma}\hat{a}_I(t),
\end{aligned}
$$

- this is a Langevin equation for the damped amplitude $\hat{a}(t)$ but with the noise term appears explicitly as the input field,
- the time reverse Langevin equation is

$$
\frac{\mathrm{d}}{\mathrm{d}t}\hat{a} = -\frac{i}{\hbar}[\hat{a},\hat{H}_S] + \frac{\gamma}{2}\hat{a}(t) - \sqrt{\gamma}\hat{a}_O(t),
$$

where

$$
\hat{a}_O(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}\omega e^{-i\omega(t-t_1)}\hat{b}_1(\omega),
$$

Input-output theory

- the system operator obeys the equation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a} = -\frac{i}{\hbar}[\hat{a}, \hat{H}_S] - \frac{\gamma}{2}\hat{a}(t) + \sqrt{\gamma}\hat{a}_I(t),$$
$$= -\frac{i}{\hbar}[\hat{a}, \hat{H}_S] + \frac{\gamma}{2}\hat{a}(t) - \sqrt{\gamma}\hat{a}_O(t),$$

- the relation between the external field and the intra-cavity field may be obtained,

$$\hat{a}_O(t) + \hat{a}_I(t) = \sqrt{\gamma}\hat{a}(t),$$

which is a boundary condition relating each of the far-field amplitudes outside the cavity to the internal cavity field,

- it is easy to see that interference between the input and the cavity field may contribute to the observed output field,

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Linear system

- for a linear system

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}(t) = \mathbf{A}\mathbf{a} - \frac{\gamma}{2}\hat{a}(t) + \sqrt{\gamma}\mathbf{a}_I(t),$$

where

$$\mathbf{a}(t) = \begin{pmatrix} \hat{a}(t) \\ \hat{a}^\dagger(t) \end{pmatrix}, \quad \text{and} \quad \mathbf{a}_I(t) = \begin{pmatrix} \hat{a}_I(t) \\ \hat{a}_I^\dagger(t) \end{pmatrix},$$

- define the Fourier components of the intra-cavity field,

$$\mathbf{a}(\omega) = \begin{pmatrix} \hat{a}(\omega) \\ \hat{a}^\dagger(\omega) \end{pmatrix}, \quad \text{where} \quad \hat{a}(t) = \frac{1}{\sqrt{2\pi}}\int \mathrm{d}\omega e^{-i\omega(t-t_0)}\hat{a}(\omega),$$

- then the equation of motion in frequency domain becomes,

$$[\mathbf{A} + (i\omega - \frac{\gamma}{2})\mathbf{1}]\mathbf{a}(\omega) = -\sqrt{\gamma}\mathbf{a}_I(\omega),$$

- by the same way,

$$[\mathbf{A} + (i\omega + \frac{\gamma}{2})\mathbf{1}]\mathbf{a}(\omega) = +\sqrt{\gamma}\mathbf{a}_O(\omega),$$

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Linear system

- for a linear system

$$[\mathbf{A} + (i\omega - \frac{\gamma}{2})\mathbf{1}]\mathbf{a}(\omega) = -\sqrt{\gamma}\mathbf{a}_I(\omega),$$
$$[\mathbf{A} + (i\omega + \frac{\gamma}{2})\mathbf{1}]\mathbf{a}(\omega) = +\sqrt{\gamma}\mathbf{a}_O(\omega),$$

  or

$$\mathbf{a}_O(\omega) = -[\mathbf{A} + (i\omega + \frac{\gamma}{2})\mathbf{1}][\mathbf{A} + (i\omega - \frac{\gamma}{2})\mathbf{1}]^{-1}\mathbf{a}_I(\omega),$$

- for example, consider an empty one-sided cavity,
- in this case the only source of loss in the cavity is through the mirror which couples the input and output fields,
- the system Hamiltonian is $\hat{H}_S = \hbar\omega_0\hat{a}^\dagger\hat{a}$, and

$$\mathbf{A} = \begin{pmatrix} -i\omega_0 & 0 \\ 0 & i\omega_0 \end{pmatrix},$$

Linear system

- for example, consider an empty one-sided cavity,
- in this case the only source of loss in the cavity is through the mirror which couples the input and output fields,
- the system Hamiltonian is $\hat{H}_S = \hbar\omega_0 \hat{a}^\dagger \hat{a}$, and

$$\mathbf{A} = \begin{pmatrix} -i\omega_0 & 0 \\ 0 & i\omega_0 \end{pmatrix},$$

then

$$\mathbf{a}_O(\omega) = \frac{\gamma/2 + i(\omega - \omega_0)}{\gamma/2 - i(\omega - \omega_0)} \mathbf{a}_I(\omega),$$

- there is a frequency dependent phase shift between the output and input,
- the relationship between the input and the internal field is,

$$\mathbf{a}(\omega) = \frac{\sqrt{\gamma}}{\gamma/2 - i(\omega - \omega_0)} \mathbf{a}_I(\omega),$$

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which leads to a *Lorentzian* of width $\gamma/2$ for the intensity transmission function,

Two-sided cavity

- a two-sided cavity has two partially transparent mirrors with associated loss coefficients γ_1 and γ_2,
- in this case there are two input ports and two output ports,
- the equation of motion for the internal field is

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}(t) = -i\omega_0\hat{a}(t) - \frac{1}{2}(\gamma_1 + \gamma_2)\hat{a}(t) + \sqrt{\gamma_1}\hat{a}_I(t) + \sqrt{\gamma_2}\hat{b}_I(t),$$

- the relationship between the internal and input field frequency components for an empty cavity is then

$$\mathbf{a}(\omega) = \frac{\sqrt{\gamma_1}\mathbf{a}_I(\omega) + \sqrt{\gamma_2}\mathbf{b}_I(\omega)}{\frac{\gamma_1+\gamma_2}{2} - i(\omega - \omega_0)},$$

Two-time correlation function

- two boundary conditions of the reservoir,

$$\hat{b}(\omega) = e^{-i\omega(t-t_0)}\hat{b}_0(\omega) + g(\omega)\int_{t_0}^{t} dt' e^{-i\omega(t-t')}\hat{a}(t'), \quad \text{at time } t_0 < t\text{, the } \textit{input},$$

$$\hat{b}(\omega) = e^{-i\omega(t-t_1)}\hat{b}_1(\omega) - g(\omega)\int_{t}^{t_1} dt' e^{-i\omega(t-t')}\hat{a}(t'), \quad \text{at time } t_1 > t\text{, the } \textit{output},$$

- the input and output fields,

$$\hat{a}_{IN}(t) = \frac{-1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t_0)}\hat{b}_0(\omega),$$

$$\hat{a}_{OUT}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t_1)}\hat{b}_1(\omega),$$

or

$$\hat{a}_{IN}(t) = \frac{\sqrt{\gamma}}{2}\hat{a}(t) - \frac{1}{\sqrt{2\pi}}\int d\omega \hat{b}(\omega, t),$$

$$\hat{a}_{OUT}(t) = \frac{\sqrt{\gamma}}{2}\hat{a}(t) + \frac{1}{\sqrt{2\pi}}\int d\omega \hat{b}(\omega, t),$$

Two-time correlation function

- the input and output fields,

$$\hat{a}_{IN}(t) = \frac{\sqrt{\gamma}}{2}\hat{a}(t) - \frac{1}{\sqrt{2\pi}}\int \mathrm{d}\omega \hat{b}(\omega, t),$$

$$\hat{a}_{OUT}(t) = \frac{\sqrt{\gamma}}{2}\hat{a}(t) + \frac{1}{\sqrt{2\pi}}\int \mathrm{d}\omega \hat{b}(\omega, t),$$

- let $\hat{c}(t)$ be any system operator, then

$$\begin{aligned}
[\hat{c}(t), \sqrt{\gamma}\hat{a}_{IN}(t')] &= \frac{\gamma}{2}[\hat{c}(t), \hat{a}(t')], \quad \text{for} \quad t = t', \\
[\hat{c}(t), \sqrt{\gamma}\hat{a}_{IN}(t')] &= 0, \quad \text{for} \quad t' > t, \\
[\hat{c}(t), \sqrt{\gamma}\hat{a}_{OUT}(t')] &= 0, \quad \text{for} \quad t' < t, \\
[\hat{c}(t), \sqrt{\gamma}\hat{a}_{IN}(t')] &= \gamma[\hat{c}(t), \hat{a}(t')], \quad \text{for} \quad t' < t,
\end{aligned}$$

with

$$\hat{a}_O(t) + \hat{a}_I(t) = \sqrt{\gamma}\hat{a}(t),$$

Two-time correlation function

- let $\hat{c}(t)$ be any system operator, then

$$\begin{aligned}
[\hat{c}(t), \sqrt{\gamma}\hat{a}_{IN}(t')] &= \frac{\gamma}{2}[\hat{c}(t), \hat{a}(t')], \quad \text{for} \quad t = t', \\
[\hat{c}(t), \sqrt{\gamma}\hat{a}_{IN}(t')] &= 0, \quad \text{for} \quad t' > t, \\
[\hat{c}(t), \sqrt{\gamma}\hat{a}_{IN}(t')] &= \gamma[\hat{c}(t), \hat{a}(t')], \quad \text{for} \quad t' < t,
\end{aligned}$$

with

$$\hat{a}_O(t) + \hat{a}_I(t) = \sqrt{\gamma}\hat{a}(t),$$

- the commutator for the output field is

$$[\hat{a}_O(t), \hat{a}_O^\dagger(t')] = [\hat{a}_I(t), \hat{a}_I^\dagger(t')],$$

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Spectrum of squeezing for the parametric oscillator

- below the threshold, the Hamiltonian for a parametric oscillator is

$$\hat{H}_S = \hbar\omega_0 \hat{a}^\dagger \hat{a} + \frac{i\hbar}{2}(\epsilon \hat{a}^{\dagger 2} - \epsilon^* \hat{a}^2),$$

then

$$[\mathbf{A} + (i\omega - \frac{\gamma}{2})\mathbf{1}]\mathbf{a}(\omega) = -\sqrt{\gamma}\mathbf{a}_I(\omega),$$
$$[\mathbf{A} + (i\omega + \frac{\gamma}{2})\mathbf{1}]\mathbf{a}(\omega) = +\sqrt{\gamma}\mathbf{a}_O(\omega),$$

where

$$\mathbf{A} = \begin{pmatrix} -i\omega_0 & \epsilon \\ \epsilon^* & i\omega_0 \end{pmatrix},$$

- the Fourier components for the output field is

$$\hat{a}_O(\omega) = \frac{1}{(\frac{\gamma}{2} - i(\omega - \omega_0)^2 - |\epsilon|^2}\{[(\frac{\gamma}{2})^2 + (\omega - \omega_0)^2 + |\epsilon|^2]\hat{a}_I(\omega) + \epsilon\gamma \hat{a}_I^\dagger(-\omega)\},$$

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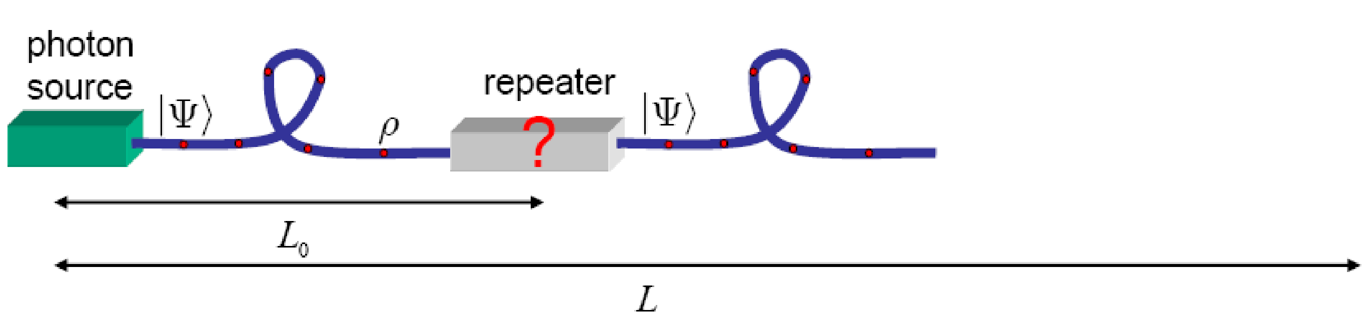
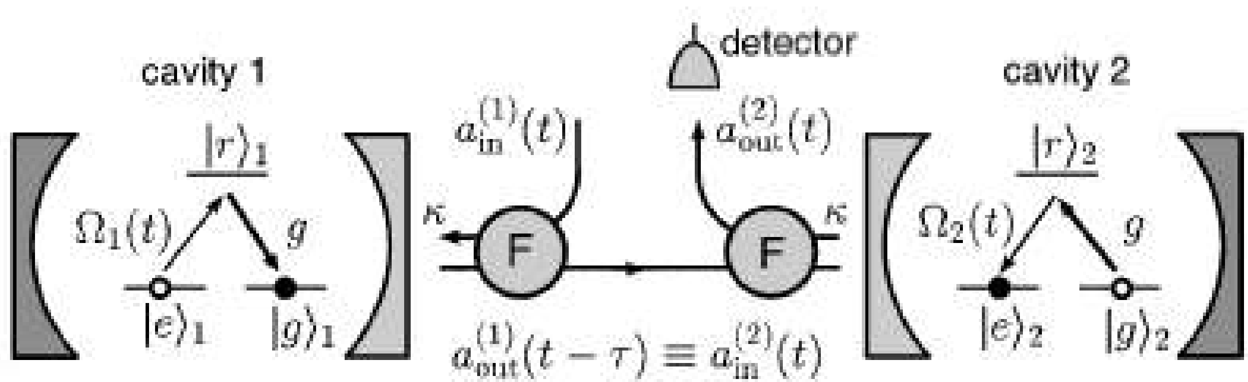
Quantum State Transfer as a Quantum Repeater





J. I. Cirac, P. Zoller, H. J. Kimble, and H. Mabuchi, *Phys. Rev. Lett.* **78**, 3221 (1997).

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Quantum State Transfer as a Quantum Repeater

- the Hamiltonian describing the interaction of each atom with the corresponding cavity mode is $(\hbar = 1)$,

$$\hat{H}_i = \omega \hat{a}_i^\dagger \hat{a}_i + \omega_0 |r\rangle_{ii}\langle r| + g(|r\rangle_{ii}\langle g|\hat{a}_i + \text{h.c}) + \frac{1}{2}\Omega_i(t)[e^{-i(\omega_L t+\phi_i)}|r\rangle_{ii}\langle e| + \text{h. c.}], \quad (i =$$

- in a quantum stochastic description employing the input¡Voutput formalism the cavity mode operators obey the quantum Langevin equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}_i = -\frac{i}{\hbar}[\hat{a}_i, \hat{H}_i] - \kappa \hat{a}_i(t) - \sqrt{2\kappa}\hat{a}_I^{(i)}(t), \quad (i = 1, 2),$$

- the output of each cavity is given by the equation,

$$\hat{a}_O^{(i)}(t) = \hat{a}_I^{(i)}(t) + \sqrt{2\kappa}\hat{a}_i(t),$$

Quantum State Transfer as a Quantum Repeater

- The output field of the first cavity constitutes the input for the second cavity with an appropriate time delay, i.e., $\hat{a}_I^{(2)}(t) = \hat{a}_O^{(1)}(t-\tau)$,
- The output field of the second cavity is

$$\hat{a}_O^{(2)}(t) = \hat{a}_I^{(1)}(t-\tau) + \sqrt{2\kappa}[\hat{a}_1(t-\tau) + \hat{a}_2(t)],$$

then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}_1 &= -\frac{i}{\hbar}[\hat{a}_1, \hat{H}_1] - \kappa\hat{a}_1(t) - \sqrt{2\kappa}\hat{a}_I^{(1)}(t), \\
\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}_2 &= -\frac{i}{\hbar}[\hat{a}_2, \hat{H}_2] - \kappa\hat{a}_2(t) - 2\kappa\hat{a}_1(t-\tau) - \sqrt{2\kappa}\hat{a}_I^{(1)}(t-\tau),
\end{aligned}$$