3, Atom-field interaction, semi-classical and quantum theories

1. Semiclassical theory
2. Jaynes-Cummings Hamiltonian
3. Multi-mode squeezing
4. Rabi Oscillation
5. Superradiance

Ref:

Ch. 5, 6 in *"Quantum Optics,"* by M. Scully and M. Zubairy.

Ch. 5, 6 in *"Mesoscopic Quantum Optics,"* by Y. Yamamoto and A. Imamoglu.

Ch. 5 in *"The Quantum Theory of Light,"* by R. Loudon.

Ch. 10 in *"Quantum Optics,"* by D. Wall and G. Milburn.

Ch. 13 in *"Elements of Quantum Optics,"* by P. Meystre and M. Sargent III.

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Einstein on Radiation

Zur Quantentheorie der Strahlung.

Von A. Einstein[1]).

Die formale Ähnlichkeit der Kurve der chromatischen Verteilung der Temperaturstrahlung mit Maxwellschen Geschwindigkeits-Verteilungsgesetz ist zu frappant, als daß sie lange hätte verborgen bleiben können. In der Tat wurde bereits W. Wien in der wichtigen theoretischen Arbeit, in welcher er sein Verschiebungsgesetz

$$\rho = \nu^3 f\left(\frac{\nu}{T}\right) \qquad (1)$$

ableitete, durch diese Ähnlichkeit auf eine weitergehende Bestimmung der Strahlungsformel geführt. Er fand hierbei bekanntlich die Formel

$$\rho = \alpha \nu^3 e^{-\frac{h\nu}{kT}} \qquad (2)$$

welche als Grenzgesetz für große Werte von

"On the Quantum Theory of Radiation"

$$D(\omega) = \frac{A/B}{e^{\hbar\omega/k_B T} - 1}$$

$$\frac{A}{B} = \frac{\hbar\omega^3}{\pi^2 c^3}$$

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A. Einstein, *Phys. Z.* **18**, 121 (1917).

D. Kleppner, "Rereading Einstein on Radiation," *Physics Today* **58**, 30 (Feb. 2005).

Quantization of the Electromagnetic Field

- Like simple harmonic oscillator, $\hat{H} = \frac{p^2}{2m} + \frac{1}{2}kx^2$, where $[\hat{x}, \hat{p}] = i\hbar$,
- For EM field, $\hat{H} = \frac{1}{2}\sum_j [m_j \omega_m^2 q_j^2 + \frac{p_j^2}{m_j}]$, , where $[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$,
- the Hamiltonian for EM fields becomes: $\hat{H} = \sum_j \hbar\omega_j (\hat{a}_j^\dagger \hat{a}_j + \frac{1}{2})$,
- the electric and magnetic fields become,

$$
\begin{aligned}
\hat{E}_x(z,t) &= \sum_j (\frac{\hbar\omega_j}{\epsilon_0 V})^{1/2} [\hat{a}_j e^{-i\omega_j t} + \hat{a}_j^\dagger e^{i\omega_j t}] \sin(k_j z), \\
\hat{H}_y(z,t) &= -i\epsilon_0 c \sum_j (\frac{\hbar\omega_j}{\epsilon_0 V})^{1/2} [\hat{a}_j e^{-i\omega_j t} - \hat{a}_j^\dagger e^{i\omega_j t}] \cos(k_j z),
\end{aligned}
$$

- energy level for quantized field, $E_n = (n + \frac{1}{2})\hbar\omega$.

Planck's Law

- In the thermal equilibrium at temperature *T*, the probability P_n that the mode oscillator is thermally excited to the n-th excited state is given by the *Boltzman factor*,

$$P_n = \frac{\exp[-E_n/k_B T]}{\sum_n \exp[-E_n/k_B t]},$$

- the mean number $\bar{n}$ of photons is,

$$\bar{n} = \sum_n n P_n = \frac{U}{1-U} = \frac{1}{\exp(\hbar\omega/k_B T) - 1},$$

where $U \equiv \exp(-\hbar\omega/k_B T)$ and $\sum_{n=0}^{\infty} U^n = 1/(1-U)$.

- energy density of the radiation:

$$\begin{aligned} D(\omega)\mathrm{d}\omega &= \bar{n}\hbar\omega \mathrm{d}\omega = \bar{n}\hbar\omega\rho_\omega \mathrm{d}\omega, \\ &= \bar{n}\hbar\omega^3 \mathrm{d}\omega/\pi^2 c^3 = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{\mathrm{d}\omega}{\exp[\hbar\omega/k_B T] - 1}. \end{aligned}$$

- total electromagnetic energy density: $\int_0^\infty D(\omega)\mathrm{d}\omega = 1/2V \int_{\text{cavity}} \epsilon_0 |\mathbf{E}(r,t)|^2 \mathrm{d}V$.

Fluctuations in Photon Number

- the ergodic theorem of statistical mechanics: time averages are equivalent to averages taken over a large number of exactly similar systems, each maintained in a fixed state (ensemble).

- the probability of finding $\bar{n}$ photons,

$$P_n = \frac{\exp[-E_n/k_B T]}{\sum_n \exp[-E_n/k_B t]} = (1-U)U^n = \frac{\bar{n}^n}{(1+\bar{n})^{1+n}},$$

  which is a thermal distribution or the geometric distribution.

- the root-mean-square deviation:

$$\Delta n^2 = \sum_n (n - \bar{n}^2) P_n = \bar{n}^2 + \bar{n},$$

  then

$$\Delta n \approx \bar{n} + \frac{1}{2}, \quad \text{for} \quad \bar{n} \gg 1.$$

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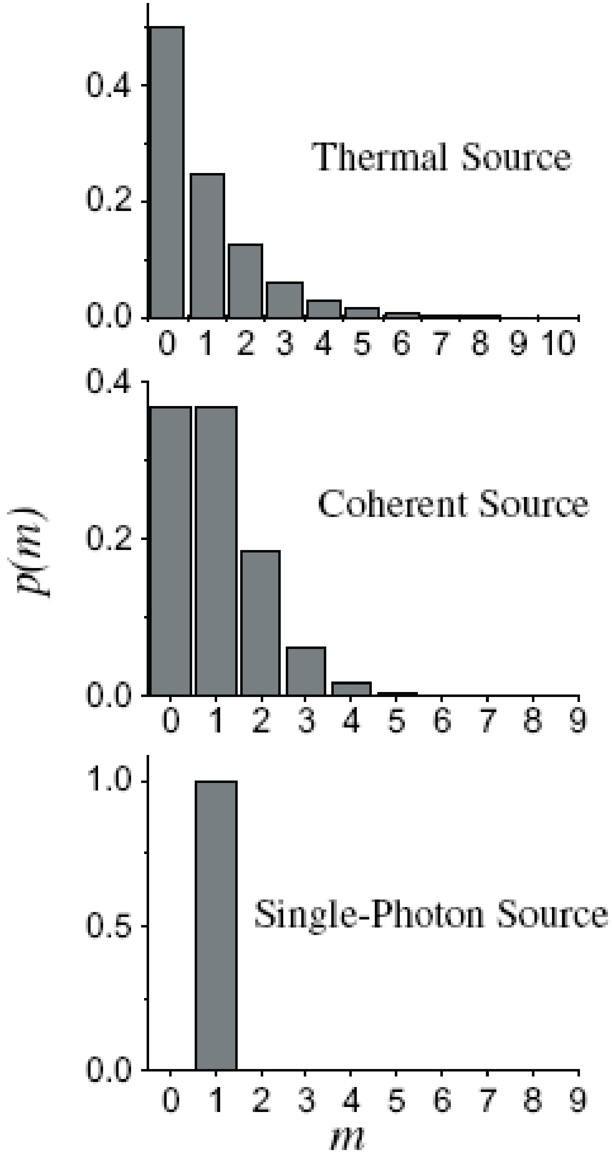
Probability distribution for $\bar{n} = 1$



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Einstein's A and B coefficients

- For a two-level atom, the rates of changes of N_1 and N_2 are,

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = -\frac{\mathrm{d}N_2}{\mathrm{d}t} = N_2 A_{21} - N_1 B_{12} D(\omega) + N_2 B_{21} D(\omega),$$

- A_{21} is the probability of photon in state 2 spontaneously fall into the lower state 1, i.e. spontaneous emission;
- B_{12} is the probability of photon absorption in state 1 into state 2, i.e. absorption;
- B_{21} is the probability of photon emission from state 2 into state 1, i.e. stimulated emission;
- in thermal equilibrium, $\frac{\mathrm{d}N_1}{\mathrm{d}t} = -\frac{\mathrm{d}N_2}{\mathrm{d}t} = 0$,

$$D(\omega) = \frac{A_{21}}{(N_1/N_2)B_{12} - B_{21}},$$

where the populations N_1 and N_2 are related by Boltzmann's law,

$$N_1/N_2 = (g_1/g_2)\mathrm{exp}[\hbar\omega/k_B T],$$

Einstein's A and B coefficients

- the density distribution of EM fields in a two-level atom,

$$D(\omega) = \frac{A_{21}}{(g_1/g_2)\text{exp}[\hbar\omega/k_B T]B_{12} - B21},$$

  where g_1 and g_2 are the level degenerate parameters.

- compare it in free space,

$$D(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3}\frac{1}{\text{exp}[\hbar\omega/k_B T] - 1},$$

- at all temperatures T, we have

$$\begin{aligned}(g_1/g_2)B_{12} &= B_{21},\\ (\hbar\omega^3/\pi^2 c^3)B_{21} &= A_{21},\end{aligned}$$

- the consistency between the Einstein theory and Planck's law could not have been achieved without the introduction of the *stimulated emission* process.

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Einstein's A and B coefficients

- for nondegenerate two-level atom, $g_1 = g_2 = 1$ and $N_1 + N_2 = N$,

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = -\frac{\mathrm{d}N_2}{\mathrm{d}t} = N_2 A + (N_2 - N_1) B D(\omega),$$

- the solution for N_1 is,

$$N_1 = [N_1^0 - \frac{N(A + BD(\omega))}{A + 2BD(\omega)}]\exp[-(A + 2BD(\omega))t] + \frac{N[A + BD(\omega)]}{A + 2BD(\omega)}$$

where N_1^0 is the initial value of N_1 at $t = 0$,

- if $N_2^0 = 0$, all atoms are in the ground state at $t = 0$,

$$N_2 = \frac{NBD(\omega)}{A + 2BD(\omega)}[1 - \exp[-(A + 2BD(\omega))t],$$

- in the steady-state,

$$N_2 = \frac{NBD(\omega)}{A + 2BD(\omega)} \approx 0.5, \quad \text{if} \quad BD(\omega) \gg A,$$

Macroscopic theory of Absorption

- for the excited state,

$$\frac{\mathrm{d}N_2}{\mathrm{d}t} = -N_2 A,$$

  with the solution $N_2 = N_2^0 \exp[-At]$, where $A \equiv 1/\tau_R$ the radiative lifetime of the excited states.

- in macroscopic, the polarization **P** by an applied electric field **E** is related with $\mathbf{P} = \epsilon_0 \chi \mathbf{E}$, where the susceptibility $\chi = \chi_1 + i\chi_2$,

- the relation between frequency and the wavevector, $kc/\omega = 1 + \chi = n^2 = (\eta + i\kappa)^2$, where $\eta^2 - \kappa^2 = 1 + \chi_1$ and $2\eta\kappa = \chi_2$,

- the traveling-wave solution propagated in the $z-$direction becomes,

$$\exp[i(kz - \omega t)] = \exp[i\omega(\frac{\eta z}{c} - t) - \frac{\omega \kappa z}{c}],$$

- the averaged Poynting vector, $\bar{I} = \langle \mathbf{E} \times \mathbf{B}/\mu_0 \rangle = \frac{1}{2}\epsilon_0 c \eta |\mathbf{E}(r,t)|^2$, where

$$\bar{I}(z) = \bar{I}_0 \exp[-2\omega\kappa z/c],$$

  where $2\omega\kappa/c$ is called the absorption coefficient.

Microscopic theory of Absorption

- total electromagnetic energy density: $\int_0^\infty D(\omega)\mathrm{d}\omega = 1/2V\int_{\text{cavity}} \epsilon_0 |\mathbf{E}(r,t)|^2 \mathrm{d}V$.
- for a lossy dielectric medium, $\int_0^\infty D(\omega)\mathrm{d}\omega = 1/2V\int_{\text{cavity}} \epsilon_0 \eta^2 |\mathbf{E}(r,t)|^2 \mathrm{d}V$.
- in steady-state condition, $-\frac{\mathrm{d}N_2}{\mathrm{d}t} = N_2 A + (N_2 - N_1)BD(\omega)/\eta^2 = 0$, with an additional factor η^2 for the energy density,
- the attenuation energy within a small section of $\mathrm{d}z$, cross-section A is,

$$\frac{\partial}{\partial t} D(\omega)\mathrm{d}\omega A\mathrm{d}z = -(N_1 - N_2)F(\omega)\mathrm{d}\omega BD(\omega)/\eta^2\hbar\omega(A\mathrm{d}z/V),$$

- for the absorption, $-\frac{\partial}{\partial t} D(\omega)\mathrm{d}\omega A\mathrm{d}z = -\frac{\partial}{\partial z}\bar{I}\mathrm{d}\omega A\mathrm{d}z$, or $\frac{\partial}{\partial t}D(\omega) = \frac{\partial}{\partial z}\bar{I}$,
- for $\bar{I} = \frac{1}{2}\epsilon_0 c\eta |\mathbf{E}(r,t)|^2$, we have $cD(\omega) = \eta\bar{I}$, then,

$$\frac{\partial}{\partial z}\bar{I} = -(N_1 - N_2)F(\omega)(B\hbar\omega/Vc\eta)\hbar I,$$

where $F(\omega)$ is the distribution of atomic transition frequencies.

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Microscopic theory of Absorption

- if $N_2^0 = 0$, all atoms are in the ground state at $t = 0$,

$$N_2 = \frac{NBD(\omega)}{A + 2BD(\omega)/\eta^2}[1 - \exp[-(A + 2BD(\omega))t] \approx \frac{NBD(\omega)}{A + 2BD(\omega)/\eta^2},$$

and we have,

$$N_1 - N_2 = \frac{NA}{A + 2BD(\omega)/\eta^2} = \frac{NA}{A + 2B\bar{I}/c\eta},$$

- the equation for the average beam intensity becomes,

$$\frac{1}{\bar{I}}(1 + \frac{2B\bar{I}}{Ac\eta})\frac{\partial}{\partial z}\bar{I} = -\frac{NB\hbar\omega F(\omega)}{Vc\eta}$$

- for all ordinary light beams, $\frac{2B\bar{I}}{Ac\eta} \ll 1$, then we have,

$$\begin{aligned} \bar{I}(z) &= \bar{I}_0 \exp[-NB\hbar\omega F(\omega)z/Vc\eta], \\ &= \bar{I}_0 \exp[-Kz], \end{aligned}$$

where the absorption coefficient, $K = 2\omega\kappa/c$.

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Microscopic theory of Absorption

- A dielectric with one single resonance may be modeled as a distribution of "$+$" and "$-$" charges, the $+$ charges immobile and the $-$ charges tied to the $+$ charges by a spring constant k,

$$m(\frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2)\mathbf{d} = -\frac{e}{m}\mathbf{E},$$

- for the incident field $\mathbf{E} = E_0 \text{exp}[-i(\omega t - kz)]$ and the dipole $\mathbf{d} = a \text{exp}[-i(\omega t - kz)]$, we have

$$a = \frac{-(e/m)E_0}{\omega^2 - \omega_0^2 + 2i\beta\omega},$$

- the polarization $\mathbf{P} = Np = N\sum_j e\mathbf{d}_j = N\alpha(\omega)E_0 e^{-i(\omega t - kz)}$, where $\alpha(\omega) = \frac{-e^2/m}{\omega^2 - \omega_0^2 + 2i\beta\omega}$.

Ch. 2, 3, 7, 8 in *"Lasers,"* by P. Milonni and J. Eberly.

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Microscopic theory of Absorption

- the dispersion relation,

$$k^2 = \frac{\omega^2}{c^2}[1 + \frac{N\alpha(\omega)}{\epsilon_0}] = \frac{\omega^2}{c^2} n^2(\omega^2),$$

- the real index of refraction,

$$n_R(\omega) = 1 + \frac{Ne^2}{m\epsilon_0} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2},$$

- the absorption coefficient or extinction coefficient,

$$a(\omega) = 2n_I(\omega)\omega/c = \frac{2Ne^2}{m\epsilon_0 c} \frac{\beta\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2},$$

which has the lineshape of the *Lorentzian* function,

$$a(\omega) = \frac{Ne^2}{2m\epsilon_0 c} \frac{\delta\omega_0}{(\omega_0 - \omega)^2 + \delta\omega_0^2},$$

where $\delta\omega = \beta$

Population Inversion: the Laser

- for a three level atom, $N_1 + N_2 + N_3 = N$, the rate equations are:

$$\begin{aligned}
\frac{\mathrm{d}N_2}{\mathrm{d}t} &= -N_2A_{21} - N_2A_{23} + D_pB_{23}(N_3 - N_2) - D(\omega)B_{21}(N_2 - N_1), \\
\frac{\mathrm{d}N_1}{\mathrm{d}t} &= N_2A_{21} - N_1A_{13} + D(\omega)B_{21}(N_2 - N_1), \\
\frac{\mathrm{d}N_3}{\mathrm{d}t} &= -N_2A_{23} + N_1A_{13} - D_pB_{23}(N_3 - N_2),
\end{aligned}$$

- the pumping rate $\gamma = D_pB_{23}(N_3 - N_2)/N$,
- in steady-state,

$$\begin{aligned}
N_2[A_{21} + B_{21}D(\omega)] &= N_1[A_{12} + B_{21}D(\omega)], \\
N_2A_{23} + N_1A_{13} &= N\gamma,
\end{aligned}$$

- for $A_{21} < A_{13}$, we have $N_2 > N_1$.

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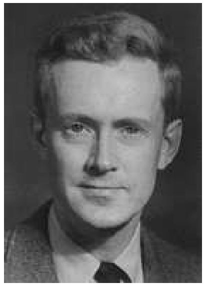
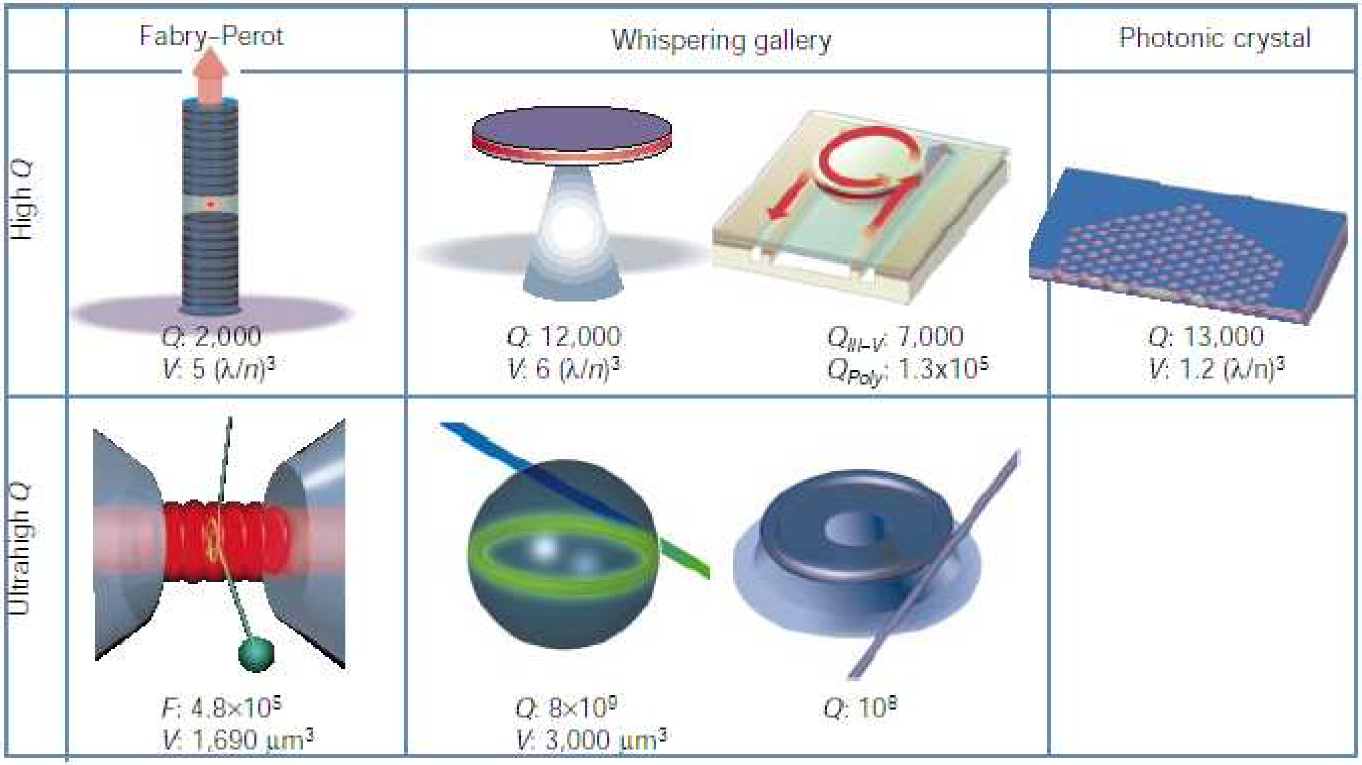
Purcell effect: Cavity-QED (Quantum ElectroDynamics)





E. M. Purcell, *Phys. Rev.* **69** (1946).

Nobel laureate Edward Mills Purcell (shared the prize with Felix Bloch) in 1952,

for their contribution to nuclear magnetic precision measurements.

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from: K. J. Vahala, *Nature* **424**, 839 (2003).

The motion of a free electron

- the motion of a free electron is described by the Schr odinger equation,

$$\frac{-\hbar^2}{2m}\nabla^2\Psi = i\hbar\frac{\partial\Psi}{\partial t},$$

- the probability density of finding an electron at position r and time t is

$$P(r,t) = |\Psi(r,t)|^2,$$

- is $\Psi(r,t)$ is a solution os the Schrödinger equation so is

$$\Psi_1(r,t) = \Psi(r,t)\text{exp}[i\chi],$$

  where χ is an arbitrary constant phase,

- the probability density $P(r,t)$ would remain unaffected by an arbitrary choice of χ,
- the choice of the phase of the wave function $\Psi(r,t)$ is completely arbitrary,
- two functions differing only by a constant phase factor represent the same physical state,

Local gauge (phase) invariance

- the motion of a free electron is described by the Schr odinger equation,

$$\frac{-\hbar^2}{2m}\nabla^2\Psi = i\hbar\frac{\partial\Psi}{\partial t},$$

- if the phase of the wave function is allowed to vary *locally*, i.e.

$$\Psi_1(r,t) \rightarrow \Psi(r,t)\text{exp}[i\chi(r,t)],$$

- the probability $P(r,t)$ remains unaffected but the Schrödinger equation is no longer satisfied,
- to satisfy *local gauge (phase) invariance*, then the Schrödinger equation must be modified by adding new terms,

$$\{\frac{-\hbar^2}{2m}[\nabla - i\frac{e}{\hbar}\mathbf{A}(r,t)]^2 + e\mathbf{U}(r,t)\}\Psi = i\hbar\frac{\partial\Psi}{\partial t},$$

where $\mathbf{A}(r,t)$ and $\mathbf{U}(r,t)$ are the vector and scalar potentials of the external field, respectively,

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Minimal-coupling Hamiltonian

- to satisfy *local gauge (phase) invariance*, then the Schrödinger equation must be modified by adding new terms,

$$\{\frac{-\hbar^2}{2m}[\nabla - i\frac{e}{\hbar}\mathbf{A}(r,t)]^2 + e\mathbf{U}(r,t)\}\Psi = i\hbar\frac{\partial\Psi}{\partial t},$$

  and

$$\begin{aligned} \mathbf{A}(r,t) &\rightarrow \mathbf{A}(r,t) + \frac{\hbar}{e}\nabla\chi(r,t), \\ \mathbf{U}(r,t) &\rightarrow \mathbf{U}(r,t) - \frac{\hbar}{e}\frac{\partial\chi(r,t)}{\partial t}, \end{aligned}$$

  where $A(r,t)$ and $U(r,t)$ are the vector and scalar potentials of the external field, respectively,
- $A(r,t)$ and $U(r,t)$ are the gauge-dependent potentials,
- the gauge-independent quantities are the electric and magnetic fields,

$$\begin{aligned} \mathbf{E} &= -\nabla\mathbf{U} - \frac{\partial\mathbf{A}}{\partial t}, \\ \mathbf{B} &= \nabla\mathbf{A}, \end{aligned}$$

Minimal-coupling Hamiltonian

- an electron of charge e and mass m interacting with an external EM field is described by the *minimal-coupling* Hamiltonian,

$$\hat{H} = \frac{1}{2m}[\mathbf{p} - e\mathbf{A}(r,t)]^2 + e\mathbf{U}(r,t),$$

  where $\mathbf{p} = -i\hbar\nabla$ is the canonical momentum operator, $\mathbf{A}(r,t)$ and $\mathbf{U}(r,t)$ are the vector and scalar potentials of the external field, respectively,

- the electrons are described by the wave function $\Psi(r,t)$,
- the field is described by the vector and scalar potentials **A** and **U**,
- in this way, the *photon* has been 'derived' from the Schrödinger equation plus the local gauge invariance arguments,
- the gauge field theory leads to the unification of the weak and the electromagnetic interactions,

Dipole approximation and r · E Hamiltonian

- if the entire atom is immersed in a plane EM wave,

$$\mathbf{A}(r_0 + r, t) = \mathbf{A}(t)\text{exp}[ik \cdot (r_0 + r)] \approx A(t)\text{exp}(ik \cdot r_0),$$

  where r_0 is the location of the electron,

- in this way, the dipole approximation, $\mathbf{A}(r,t) \approx \mathbf{A}(r_0, t)$,
- and the minimal-coupling Hamiltonian becomes,

$$\hat{H} = \frac{1}{2m}[\mathbf{p} - e\mathbf{A}(r_0, t)]^2 + e\mathbf{U}(r,t) + V(r),$$

  where $V(r)$ is the atomic binding potential,

- in the *radiation gauge*, R-gauge,

$$\mathbf{U}(r,t) = 0, \quad \text{and} \quad \nabla \cdot \mathbf{A}(r,t) = 0,$$

- the minimal-coupling Hamiltonian becomes,

$$\hat{H} = \frac{\mathbf{p}^2}{2m} + V(r) + e\mathbf{r} \cdot \frac{\partial \mathbf{A}(r_0, t)}{\partial t},$$

Dipole approximation and r · E Hamiltonian

- in the dipole approximation the minimal-coupling Hamiltonian becomes,

$$\hat{H} = \frac{1}{2m}[\mathbf{p} - e\mathbf{A}(r_0, t)]^2 + e\mathbf{U}(r, t) + V(r),$$

- the wave function with a local phase,

$$\Psi(r, t) = \Phi(r, t)\text{exp}[\frac{ie}{\hbar}\mathbf{A}(r_0, t) \cdot r],$$

then

$$i\hbar[\frac{ie}{\hbar}\mathbf{r} \cdot \frac{\partial \mathbf{A}(r_0, t)}{\partial t}\psi(r, t) + \frac{\partial \psi(r, t)}{\partial t}]\text{exp}[\frac{ie}{\hbar}\mathbf{A} \cdot r] = [\frac{\mathbf{p}^2}{2m} + V(r)]\text{exp}[\frac{ie}{\hbar}\mathbf{A} \cdot r],$$

- in terms of the gauge-independent field **E**, the Hamiltonian for $\Psi(r, t)$ is,

$$\begin{aligned} \hat{H} &= \frac{\mathbf{p}^2}{2m} + V(r) + e\mathbf{r} \cdot \frac{\partial \mathbf{A}(r_0, t)}{\partial t}, \\ &= \frac{\mathbf{p}^2}{2m} + V(r) - e\mathbf{r} \cdot \mathbf{E}(r_0, t) = \hat{H}_0 + \hat{H}_1, \end{aligned}$$

Dipole approximation and r · E Hamiltonian

- in the dipole approximation the minimal-coupling Hamiltonian becomes,

$$\begin{aligned}\hat{H} &= \frac{\mathbf{p}^2}{2m} + V(r) + e\mathbf{r}\cdot\frac{\partial \mathbf{A}(r_0,t)}{\partial t},\\ &= \frac{\mathbf{p}^2}{2m} + V(r) - e\mathbf{r}\cdot\mathbf{E}(r_0,t) = \hat{H}_0 + \hat{H}_1,\end{aligned}$$

  in terms of the gauge-independent field **E** and where

$$\begin{aligned}\hat{H}_0 &= \frac{\mathbf{p}^2}{2m} + V(r),\\ \hat{H}_1 &= -e\mathbf{r}\cdot\mathbf{E}(r_0,t),\end{aligned}$$

- this Hamiltonian is for the atom-field interaction,

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$\mathbf{p} \cdot \mathbf{A}$ Hamiltonian

- in the *radiation gauge*, R-gauge,

$$\mathbf{U}(r,t) = 0, \quad \text{and} \quad \nabla \cdot \mathbf{A}(r,t) = 0,$$

  the latter one implies $[\mathbf{p}, \mathrm{A}] = 0$, then

- and the minimal-coupling Hamiltonian becomes,

$$\hat{H} = \frac{1}{2m}[\mathbf{p} - e\mathbf{A}(r_0,t)]^2 + V(r) = \hat{H}_0 + \hat{H}_2,$$

  where

$$\begin{aligned} \hat{H}_0 &= \frac{\mathbf{p}^2}{2m} + V(r), \\ \hat{H}_2 &= -\frac{e}{m}\mathbf{p} \cdot \mathbf{A}(r_0,t) + \frac{e^2}{2m}\mathbf{A}^2(r_0,t) \approx -\frac{e}{m}\mathbf{p} \cdot \mathbf{A}(r_0,t), \end{aligned}$$

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Differences in $\mathbf{r} \cdot \mathbf{E}$ and $\mathbf{p} \cdot \mathbf{A}$ Hamiltonian

- in $\mathbf{r} \cdot \mathbf{E}$

$$\hat{H}_1 = -e\mathbf{r} \cdot \mathbf{E}(r_0, t),$$

- in $\mathbf{p} \cdot \mathbf{A}$ Hamiltonian

$$\hat{H}_2 = -\frac{e}{m}\mathbf{p} \cdot \mathbf{A}(r_0, t),$$

- these two different Hamiltonian $\hat{H}_1$ and $\hat{H}_2$ give different physical results,
- for example, consider a linearly polarized monochromatic plane-wave field,

$$\mathbf{E}(r_0 = 0, t) = E_0 \cos \omega t, \quad \text{and} \quad \mathbf{A}(r_0 = 0, t) = -\frac{1}{\omega} E_0 \sin \omega t,$$

- the ratio of the matrix elements for the Hamiltonian $\hat{H}_1$ and $\hat{H}_2$ is

$$\left|\frac{\langle f|\hat{H}_2|i\rangle}{\langle f|\hat{H}_1|i\rangle}\right| = \left| -\frac{(e/m\omega)\langle f|\mathbf{p}|i\rangle \cdot E_0}{e\langle f|\mathbf{r}|i\rangle \cdot E_0}\right| = \frac{\omega_{fi}}{\omega},$$

- As was first pointed out by Lamb, this makes a difference in measurable quantities like transition rates,

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Interaction of a single two-level atom with a single-mode field

- consider the interaction of a single-mode radiation field of frequency ν,
- and a two-level atom with upper and lower level states $|a\rangle$ and $|b\rangle$,
- the unperturbed part of the Hamiltonian $\hat{H}_0$ has the eigenvalues $\hbar\omega_a$ and $\hbar\omega_b$ for the atom,
- the wave function of a two-level atom can be written in the form,

$$|\Psi t\rangle = C_a(t)|a\rangle + C_b(t)|b\rangle,$$

the corresponding Schrödinger equation is

$$i\hbar\frac{\partial\Psi(t)}{\partial t} = (\hat{H}_0 + \hat{H}_1)\Psi(t),$$

where

$$\begin{aligned}
\hat{H}_0 &= |a\rangle\langle a| + |b\rangle\langle b|)\hat{H}_0|a\rangle\langle a| + |b\rangle\langle b|) = \hbar\omega_a|a\rangle\langle a| + \hbar\omega_b|b\rangle\langle b|, \\
\hat{H}_1 &= -e\mathbf{r}\cdot\mathbf{E}(t) = -e(|a\rangle\langle a| + |b\rangle\langle b|)\mathbf{r}(|a\rangle\langle a| + |b\rangle\langle b|)\mathbf{E}, \\
&= -(\mathbf{p}_{ab}|a\rangle\langle b| + \mathbf{p}_{ba}|a\rangle\langle b|)\mathbf{E}(t),
\end{aligned}$$

Probability amplitude method

- in the dipole approximation,

$$\begin{aligned} \hat{H}_0 &= \hbar\omega_a|a\rangle\langle a| + \hbar\omega_b|b\rangle\langle b|, \\ \hat{H}_1 &= -(\mathbf{p}_{ab}|a\rangle\langle b| + \mathbf{p}_{ba}|a\rangle\langle b|)\mathbf{E}(t), \end{aligned}$$

  where $\mathbf{p}_{ab} = \mathbf{p}^*_{ba} = e\langle a|\mathbf{r}|b\rangle$,

- for a single-mode field,

$$\mathbf{E}(t) = E_0 \cos\nu t,$$

- the equation of motion for the probability amplitude are

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}C_a &= -i\omega_a C_a + i\Omega_R\cos(\nu t)e^{-i\phi}C_b, \\ \frac{\mathrm{d}}{\mathrm{d}t}C_b &= -i\omega_b C_b + i\Omega_R\cos(\nu t)e^{+i\phi}C_a, \end{aligned}$$

- where $\Omega_R = \frac{|\mathbf{p}_{ab}|E_0}{\hbar}$ is the Rabi frequency which is proportional to the amplitude of the classical field,
- and ϕ is the phase of the dipole matrix element $\mathbf{p}_{ab} = |\mathbf{p}_{ab}|\exp(i\phi)$,

Probability amplitude method

- the equation of motion for the probability amplitude are

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}C_a &= -i\omega_a C_a + i\Omega_R\cos(\nu t)e^{-i\phi}C_b,\\ \frac{\mathrm{d}}{\mathrm{d}t}C_b &= -i\omega_b C_b + i\Omega_R\cos(\nu t)e^{+i\phi}C_a,\end{aligned}$$

- define the slowly varying amplitudes,

$$c_a = C_a e^{i\omega_a t}, \quad \text{and} \quad c_b = C_b e^{i\omega_b t},$$

then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}c_a &= i\frac{\Omega_R}{2}e^{-i\phi}[e^{i(\omega-\nu)t}+e^{i(\omega+\nu)t}]c_b \approx i\frac{\Omega_R}{2}e^{-i\phi}e^{i(\omega-\nu)t}c_b,\\ \frac{\mathrm{d}}{\mathrm{d}t}c_b &= i\frac{\Omega_R}{2}e^{i\phi}[e^{-i(\omega-\nu)t}+e^{-i(\omega+\nu)t}]c_a \approx i\frac{\Omega_R}{2}e^{i\phi}e^{-i(\omega-\nu)t}c_a,\end{aligned}$$

where $\omega = \omega_a - \omega_b$ is the atomic transition frequency,

- we also apply the *rotating-wave approximation* by neglecting terms with $\exp[\pm i(\omega+\nu)t]$,

Probability amplitude method

- the equation of motion for the probability amplitude are

$$\frac{\mathrm{d}}{\mathrm{d}t} c_a = i\frac{\Omega_R}{2} e^{-i\phi} e^{i(\omega-\nu)t} c_b,$$
$$\frac{\mathrm{d}}{\mathrm{d}t} c_b = i\frac{\Omega_R}{2} e^{i\phi} e^{-i(\omega-\nu)t} c_a,$$

- the solutions are

$$c_a(t) = \{[\cos(\frac{\Omega t}{2}) - i\frac{\Delta}{\Omega}\sin(\frac{\Omega t}{2})]c_a(0) + i\frac{\Omega_R}{\Omega}\sin(\frac{\Omega t}{2})e^{-i\phi}c_b(0)\}e^{i\Delta t/2},$$
$$c_b(t) = \{[\cos(\frac{\Omega t}{2}) + i\frac{\Delta}{\Omega}\sin(\frac{\Omega t}{2})]c_b(0) + i\frac{\Omega_R}{\Omega}\sin(\frac{\Omega t}{2})e^{i\phi}c_b(0)\}e^{-i\Delta t/2},$$

where

$$\Delta = \omega - \nu, \quad \text{frequency detuning},$$
$$\Omega = \sqrt{\Omega_R^2 + \Delta^2},$$

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Rabi oscillation

- it is easy to verify that

$$|c_a(t)|^2 + |c_b(t)|^2 = 1$$

- assume that the atom is initially in the excited state $|a\rangle$, i.e $c_a(0) = 1$ and $c_b(0) = 0$, then the population inversion is

$$W(t) = |c_a(t)|^2 - |c_b(t)|^2 = \frac{\Delta^2 - \Omega_2^R}{\Omega^2} \sin^2(\frac{\Omega}{2}t) + \cos^2(\frac{\Omega}{2}t)$$

- the population oscillates with the frequency $\Omega = \sqrt{\Omega_R^2 + \Delta^2}$,
- when the atom is at resonance with the incident field $\Delta = 0$, we get $\Omega = \Omega_R$, and

$$W(t) = \cos(\Omega_R t),$$

the inversion oscillates between -1 and $+1$ at a frequency Ω_R,

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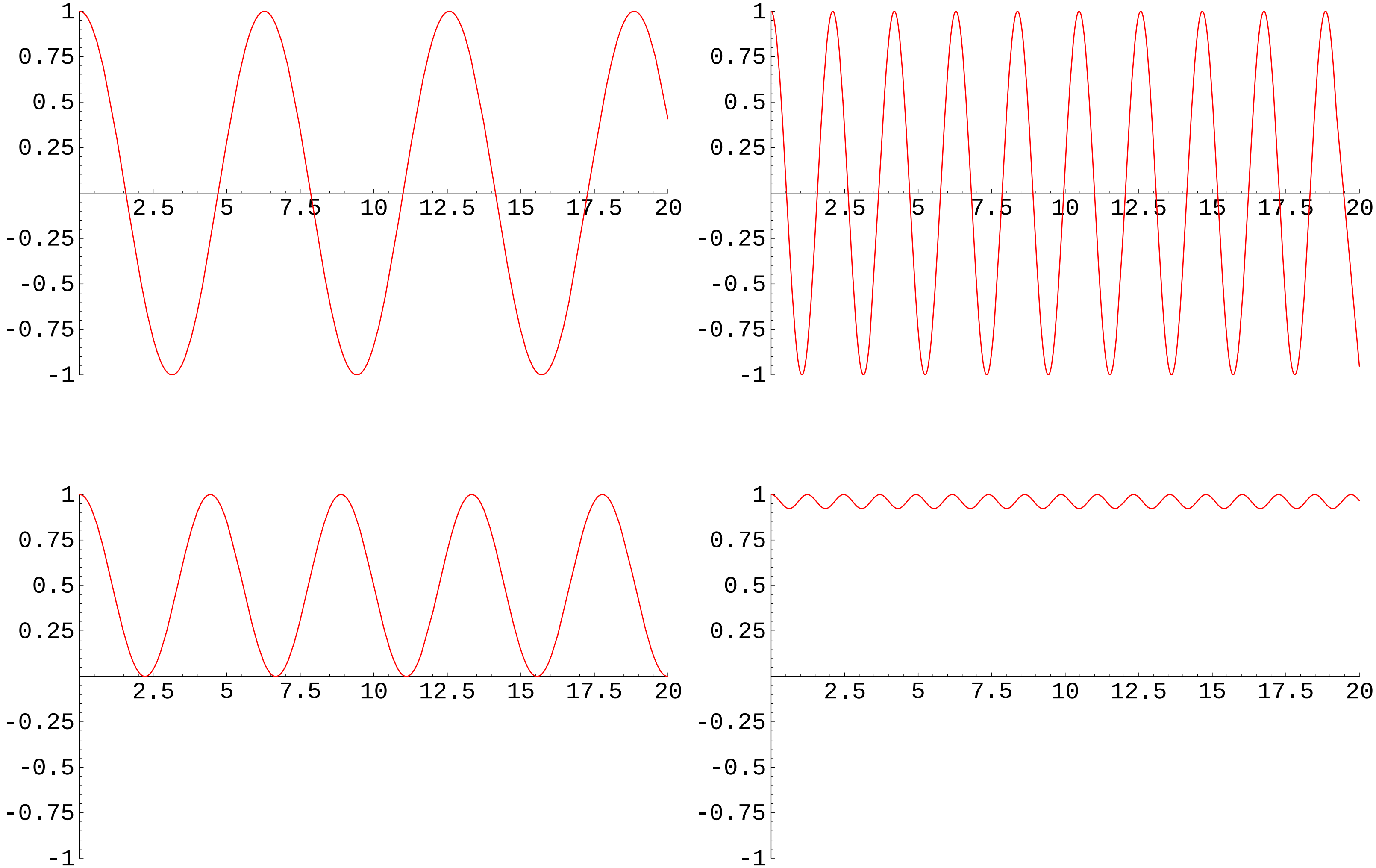
Rabi oscillation

$\Omega_R = 1.0,\ \Delta = 0.0$ $\Omega_R = 3.0,\ \Delta = 0.0$



$\Omega_R = 1.0,\ \Delta = 1.0$ $\Omega_R = 3.0,\ \Delta = 5.0$

Interaction picture

- Consider a system described by $|\Psi(t)\rangle$ evolving under the action of a hamiltonian $\hat{H}(t)$ decomposable as,
$$\hat{H}(t) = \hat{H}_0 + \hat{H}_1(t),$$
where $\hat{H}_0$ is time-independent.

- Define
$$|\Psi_I(t)\rangle = \text{exp}(i\hat{H}_0 t/\hbar)|\Psi(t)\rangle,$$
then $|\Psi_I(t)\rangle$ evolves accords to
$$i\hbar\frac{\text{d}}{\text{d}t}|\Psi_I(t)\rangle = \hat{H}_I(t)|\Psi_I(t)\rangle,$$
where
$$\hat{H}_I(t) = \text{exp}(i\hat{H}_0 t/\hbar)\hat{H}_1(t)\text{exp}(-i\hat{H}_0 t/\hbar).$$

- The evolution is in the **interaction picture** generated by $\hat{H}_0$.

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Interaction picture

- in the dipole approximation,

$$\hat{H}_0 = \hbar\omega_a|a\rangle\langle a| + \hbar\omega_b|b\rangle\langle b|,$$
$$\hat{H}_1 = -(\mathbf{p}_{ab}|a\rangle\langle b| + \mathbf{p}_{ba}|a\rangle\langle b|)\mathbf{E}(t) = -\hbar\Omega_R(e^{-i\phi}|a\rangle\langle b| + e^{i\phi}|a\rangle\langle b|)\cos\nu t,$$

where $\mathbf{p}_{ab} = \mathbf{p}_{ba}^* = e\langle a|\mathbf{r}|b\rangle$ and $\Omega_R = \frac{|\mathbf{p}_{ab}|E_0}{\hbar}$,

- the interaction picture Hamiltonian is

$$\begin{aligned}\hat{H}_I(t) &= \mathrm{exp}(i\hat{H}_0t/\hbar)\hat{H}_1(t)\mathrm{exp}(-i\hat{H}_0t/\hbar),\\ &= -\frac{\hbar}{2}\Omega_R[e^{-i\phi}|a\rangle\langle b|e^{i(\omega-\nu)t} + e^{i\phi}|b\rangle\langle a|e^{-i(\omega-\nu)t}\\ &+ e^{-i\phi}|a\rangle\langle b|e^{i(\omega+\nu)t} + e^{i\phi}|b\rangle\langle a|e^{-i(\omega+\nu)t}],\end{aligned}$$

- in the *rotating-wave approximation*,

$$\hat{H}_I(t) = -\frac{\hbar}{2}\Omega_R[e^{-i\phi}|a\rangle\langle b|e^{i(\omega-\nu)t} + e^{i\phi}|b\rangle\langle a|e^{-i(\omega-\nu)t}],$$

Interaction picture

- on resonance $\omega - \nu = 0$,

$$\hat{H}_I(t) = -\frac{\hbar}{2}\Omega_R[e^{-i\phi}|a\rangle\langle b| + e^{i\phi}|b\rangle\langle a|],$$

- the time-evolution operator in the interaction picture $\hat{U}_I(t)$ is

$$\begin{aligned}
\hat{U}_I(t) &= \overleftarrow{T}\,\text{exp}[-\frac{i}{\hbar}\int_{t_0}^{t}\text{d}\tau\hat{H}_I(\tau)],\\
&= \cos(\frac{\Omega_R t}{2})(|a\rangle\langle a| + |b\rangle\langle b|) + i\sin(\frac{\Omega_R t}{2})(e^{-i\phi}|a\rangle\langle b| + e^{i\phi}|b\rangle\langle a|),
\end{aligned}$$

- if the atom is initially in the excited state $|\Psi(t=0)\rangle = |a\rangle$, then

$$\begin{aligned}
|\Psi(t)\rangle &= \hat{U}_I(t)|a\rangle,\\
&= \cos(\frac{\Omega_R t}{2})|a\rangle + i\sin(\frac{\Omega_R t}{2})e^{i\phi}|b\rangle,
\end{aligned}$$

Density Operator

- for the quantum mechanical description, if we know that the system is in state $|\psi\rangle$, then an operator $\hat{O}$ has the expectation value,

$$\langle \hat{O} \rangle_{\text{qm}} = \langle \psi | \hat{O} | \psi \rangle,$$

- but we typically do not know that we are in state $|\psi\rangle$, then an ensemble average must be performed,

$$\langle \langle \hat{O} \rangle_{\text{qm}} \rangle_{\text{ensemble}} = \sum_{\psi} P_{\psi} \langle \psi | \hat{O} | \psi \rangle,$$

  where the P_{ψ} is the probability of being in the state $|\psi\rangle$ and we introduce a density operator,

$$\hat{\rho} = \sum_{\psi} P_{\psi} |\psi\rangle \langle \psi|,$$

- the expectation value of any operator $\hat{O}$ is given by,

$$\langle \hat{O}) \rangle_{\text{qm}} = \text{Tr}[\hat{\rho} \hat{O}],$$

  where Tr stands for trace.

Equation of motion for the density matrix

- density operator is defined as,

$$\hat{\rho} = \sum_{\psi} P_{\psi} |\psi\rangle\langle\psi|,$$

- in the Schrödiner picture,

$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = \hat{H} |\Psi\rangle,$$

then we have

$$i\hbar \frac{\partial}{\partial t} \hat{\rho} = \hat{H}\hat{\rho} - \hat{\rho}\hat{H} = [\hat{H}, \hat{\rho}],$$

which is called the Liouville or Von Neumann equation of motion for the density matrix,

- using density operator instead of a specific state vector can give statistical as well as quantum mechanical information,
- compared to the Heisenberg equation, $i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \hat{A}(t) = [\hat{A}, \hat{H}(t)]$

Decay processes in the density matrix

- equation of motion for the density matrix,

$$i\hbar\frac{\partial}{\partial t}\hat{\rho} = [\hat{H}, \hat{\rho}],$$

- the excited atomic levels can also decay due to spontaneous emission or collisions and other phenomena,
- the decay rates can be incorporated by a relaxation matrix Γ,

$$\langle n|\Gamma|m\rangle = \gamma_n \delta_{nm},$$

  then the density matrix equation of motion becomes,

$$\frac{\partial}{\partial t}\hat{\rho} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] - \frac{1}{2}\{\Gamma, \hat{\rho}\},$$

  where $\{\Gamma, \hat{\rho}\} = \Gamma\hat{\rho} + \hat{\rho}\Gamma$,
- the ijth matrix element is,

$$\frac{\partial}{\partial t}\rho_{ij} = -\frac{i}{\hbar}\sum_k (H_{ik}\rho_{kj} - \rho_{ik}H_{kj}) - \frac{1}{2}\sum_k (\Gamma_{ik}\rho_{kj} + \rho_{ik}\Gamma_{kj}),$$

Two-level atom

- a two-level atom with upper and lower level states $|a\rangle$ and $|b\rangle$,

$$|\Psi t\rangle = C_a(t)|a\rangle + C_b(t)|b\rangle,$$

- the density matrix operator is

$$\begin{aligned}\hat{\rho} &= |\Psi\rangle\langle\Psi| = |C_a|^2|a\rangle\langle a| + C_aC_b^*|a\rangle\langle b| + C_bC_a^*|b\rangle\langle a| + |C_b|^2|b\rangle\langle b|,\\ &= \rho_{aa}|a\rangle\langle a| + \rho_{ab}|a\rangle\langle b| + \rho_{ba}|b\rangle\langle a| + \rho_{bb}|b\rangle\langle b|,\end{aligned}$$

- diagonal elements, ρ_{aa} and ρ_{bb}, are the probabilities in the upper and lower states,
- off-diagonal elements, ρ_{ab} and ρ_{ba}, are the atomic polarizations,
- from the equation of motion for the two-level atom $\frac{\partial}{\partial t}\hat{\rho} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] - \frac{1}{2}\{\Gamma,\hat{\rho}\}$, we have

$$\begin{aligned}\frac{\partial}{\partial t}\rho_{aa} &= \frac{i}{\hbar}[\mathbf{p}_{ab}\mathbf{E}\rho_{ba} - \text{c.c}] - \gamma_a\rho_{aa},\\ \frac{\partial}{\partial t}\rho_{bb} &= -\frac{i}{\hbar}[\mathbf{p}_{ab}\mathbf{E}\rho_{ba} - \text{c.c}] - \gamma_b\rho_{bb},\\ \frac{\partial}{\partial t}\rho_{ab} &= -\frac{i}{\hbar}\mathbf{p}_{ab}\mathbf{E}(\rho_{aa} - \rho_{bb}) - (i\omega + \frac{\gamma_a + \gamma_b}{2})\rho_{ab},\end{aligned}$$

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Inclusion of elastic collisions between atoms

- the physical interpretation of the elements of the density matrix allows us to include terms associated wither certain processes,
- for example, one can have *elastic* collision between atoms in a gas,
- during an atom-atom collision the energy levels experience random Stark shifts,

$$\frac{\partial}{\partial t}\rho_{ab} = -i[i\omega + i\delta\omega(t) + \gamma_{ab}]\rho_{ab},$$

after integration,

$$\rho_{ab} = \text{exp}[-(i\omega + \gamma_{ab})t - i\int_0^t \text{d}t'\delta\omega(t')]\rho_{ab}(0),$$

- for a zero-mean random process, $\langle\delta\omega(t)\rangle = 0$,
- the variations in $\delta\omega(t)$ are usually rapid compared to other changes which occur in times like γ_{ph},

$$\langle\delta\omega(t)\delta\omega(t')\rangle = 2\gamma_{\text{ph}}\delta(t-t'),$$

Inclusion of elastic collisions between atoms

- assume that $\delta\omega(t)$ is described by a Gaussian random process, then

$$\langle \exp[-i\int_0^t \mathrm{d}t'\delta\omega(t')]\rangle = \exp[-\gamma_{\mathrm{ph}}t],$$

which gives for the average of ρ_{ab},

$$\rho_{ab} = \exp[-(i\omega + \gamma_{ab} - \gamma_{\mathrm{ph}})t]\rho_{ab}(0),$$

- for the process of atom-atom collisions,

$$\frac{\partial}{\partial t}\rho_{ab} = -i[i\omega + \gamma]\rho_{ab} - \frac{i}{\hbar}\mathbf{p}_{ab}\mathbf{E}(\rho_{aa} - \rho_{bb}),$$

where $\gamma = \gamma_{ab} + \gamma_{\mathrm{ph}}$ is the new decay rate,

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Population matrix

- for a single two-level atom, its density operator at time t and position z is

$$\hat{\rho}(z,t,t_0) = \sum_{\alpha,\beta} \rho_{\alpha\beta}(z,t,t_0)|\alpha\rangle\langle\beta|,$$

where $\alpha, \beta = a, b$ and the atom starts interacting with the field at an initial time t_0,

- for a medium consists of two-level homogeneously broadened atoms,
- the effect of all atoms which are pumped at the rate $r_a(z,t_0)$ atoms per second per unit volume is the population matrix,

$$\hat{\rho}(z,t) = \int_{-\infty}^{t} \mathrm{d}t_0 r_a(z,t_0)\hat{\rho}(z,t,t_0) = \sum_{\alpha,\beta} \int_{-\infty}^{t} \mathrm{d}t_0 r_a(z,t_0 \rho_{\alpha\beta}(z,t,t_0)|\alpha\rangle\langle\beta|,$$

where the excitation $r_a(z,t_0)$ generally varies slowly and can be taken to be a constant, i.e.

$$\hat{\rho}(z,t) = \sum_{\alpha,\beta} \rho_{\alpha\beta}(z,t)|\alpha\rangle\langle\beta|,$$

Population matrix

- the macroscopic polarization of the medium, $P(z,t)$ is the ensemble of atoms that arrive at z at time t, regardless of their time of excitation,

$$\mathbf{P}(z,t) = \mathrm{Tr}[\hat{\mathbf{p}} \cdot \hat{\rho}(z,t)] = \sum_{\alpha,\beta} \rho_{\alpha\beta}(z,t)\mathbf{p}_{\beta\alpha},$$

- for a two-level atom, $\mathbf{p}_{ab} = \mathbf{p}_{ba} = \mathbf{p}$,

$$\mathbf{P}(z,t) = \mathbf{p}[\rho_{ab}(z,t) + \rho_{ba}(z,t)] = \mathbf{p}[\rho_{ab}(z,t) + \mathrm{c.c}],$$

- the *off-diagonal* elements of the population matrix determine the macroscopic polarization,

$$\begin{aligned}
\frac{\partial}{\partial t}\rho_{aa} &= \frac{i}{\hbar}[\mathbf{p}_{ab}\mathbf{E}\rho_{ba} - \mathrm{c.c}] - \gamma_a\rho_{aa}, \\
\frac{\partial}{\partial t}\rho_{bb} &= -\frac{i}{\hbar}[\mathbf{p}_{ab}\mathbf{E}\rho_{ba} - \mathrm{c.c}] - \gamma_b\rho_{bb}, \\
\frac{\partial}{\partial t}\rho_{ab} &= -\frac{i}{\hbar}\mathbf{p}_{ab}\mathbf{E}(\rho_{aa} - \rho_{bb}) - (i\omega + \frac{\gamma_a + \gamma_b}{2})\rho_{ab},
\end{aligned}$$

Maxwell-Schrödinger equations

- the equations for the two-level atomic medium coupled to the field **E** are

$$\begin{aligned}
\frac{\partial}{\partial t}\rho_{aa} &= \frac{i}{\hbar}[\mathbf{p}_{ab}\mathbf{E}\rho_{ba} - \text{c.c}] - \gamma_a\rho_{aa}, \\
\frac{\partial}{\partial t}\rho_{bb} &= -\frac{i}{\hbar}[\mathbf{p}_{ab}\mathbf{E}\rho_{ba} - \text{c.c}] - \gamma_b\rho_{bb}, \\
\frac{\partial}{\partial t}\rho_{ab} &= -\frac{i}{\hbar}\mathbf{p}_{ab}\mathbf{E}(\rho_{aa} - \rho_{bb}) - (i\omega + \frac{\gamma_a + \gamma_b}{2})\rho_{ab},
\end{aligned}$$

- the condition of *self-consistency* requires that the equation of motion for the field **E** is driven by the atomic population matrix elements,
-
- the field is described by the Maxwell's equation,

$$\nabla \cdot \mathbf{D} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = J + \frac{\partial \mathbf{D}}{\partial t},$$

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Maxwell-Schrödinger equations

- the field is described by the Maxwell's equation,

$$\nabla \times (\nabla \times \mathbf{E}) + \mu_0 \sigma \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2},$$

- for a running wave polarized along x-direction,

$$\mathbf{E}(r,t) = \hat{x}\frac{1}{2}E(z,t)\text{exp}[-i(\nu t - kz + \phi)] + \text{c.c},$$

- the response of the medium is assumed

$$\mathbf{P}(r,t) = \hat{x}\frac{1}{2}P(z,t)\text{exp}[-i(\nu t - kz + \phi)] + \text{c.c},$$

where $E(z,t)$, $\phi(z,t)$, and $P(z,t)$ are all slowly varying function of position and time, i.e.

$$\frac{\partial E}{\partial t} \ll \nu E, \frac{\partial E}{\partial z} \ll kE, \frac{\partial}{\partial t} \ll \nu, \frac{\partial}{\partial z} \ll k,$$

$$\frac{\partial P}{\partial t} \ll \nu P, \frac{\partial P}{\partial z} \ll kP,$$

Maxwell-Schrödinger equations

- the response of the medium is assumed

$$\mathbf{P}(r,t) = \hat{x}\frac{1}{2}P(z,t)\text{exp}[-i(\nu t - kz + \phi)] + \text{c.c},$$

in terms of the population matrix,

$$P(z,t) = 2\mathbf{P}\rho_{ab}\text{exp}[i(\nu t - kz + \phi)],$$

- the Maxwell's equation for the slowly varying envelope function is,

$$(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t})(-\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t})E = -\mu_0\sigma\frac{\partial E}{\partial t} - \mu_0\frac{\partial^2 P}{\partial t^2},$$

- along with the equations of motion for the two-level atom,

$$\begin{aligned}
\frac{\partial}{\partial t}\rho_{aa} &= \frac{i}{\hbar}[\mathbf{p}_{ab}\mathbf{E}\rho_{ba} - \text{c.c}] - \gamma_a\rho_{aa}, \\
\frac{\partial}{\partial t}\rho_{bb} &= -\frac{i}{\hbar}[\mathbf{p}_{ab}\mathbf{E}\rho_{ba} - \text{c.c}] - \gamma_b\rho_{bb}, \\
\frac{\partial}{\partial t}\rho_{ab} &= -\frac{i}{\hbar}\mathbf{p}_{ab}\mathbf{E}(\rho_{aa} - \rho_{bb}) - (i\omega + \frac{\gamma_a + \gamma_b}{2})\rho_{ab},
\end{aligned}$$

Jaynes-Cummings Hamiltonian

- in the dipole approximation, the semi-classical Hamiltonian is

$$\begin{aligned}\hat{H}_0 &= \hbar\omega_a|a\rangle\langle a| + \hbar\omega_b|b\rangle\langle b|,\\ \hat{H}_1 &= -(\mathbf{p}_{ab}|a\rangle\langle b| + \mathbf{p}_{ba}|a\rangle\langle b|)\mathbf{E}(t),\end{aligned}$$

- to include the quantized field,

$$\begin{aligned}\hat{H} &= \hat{H}_A + \hat{H}_F - e\mathsf{r}\cdot\mathsf{E},\\ &= \sum_i \hbar\omega_i\hat{\sigma}_{ii} + \sum_k \hbar\nu_k(\hat{a}_k^\dagger\hat{a}_k + \frac{1}{2}) - \sum_{i,j}\mathbf{P}_{ij}\hat{\sigma}_{ij}\sum_k E_k(\hat{a}_k + \hat{a}_k^\dagger),\\ &= \hbar\omega_i\hat{\sigma}_{ii} + \sum_k \hbar\nu_k(\hat{a}_k^\dagger\hat{a}_k + \frac{1}{2}) + \hbar\sum_{i,j}\sum_k g_k^{ij}\hat{\sigma}_{ij}(\hat{a}_k + \hat{a}_k^\dagger),\end{aligned}$$

where

$$g_k^{ij} = -\frac{\mathbf{P}_{ij}\cdot E_k}{\hbar}$$

is the coupling constant,

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Jaynes-Cummings Hamiltonian

- to include the quantized field,

$$\hat{H} = \hbar\omega_i \hat{\sigma}_{ii} + \sum_k \hbar\nu_k (\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}) + \sum_{i,j}\sum_k g_k^{ij} \hat{\sigma}_{ij} (\hat{a}_k + \hat{a}_k^\dagger),$$

- for a two-level atom, $\mathbf{P}_{ab} = \mathbf{P}_{ba}$, we have $g_k = g_k^{ab} = g_k^{ba}$, then

$$\hat{H} = \hbar\omega_a \hat{\sigma}_{aa} + \hbar\omega_b \hat{\sigma}_{bb} + \sum_k \hbar\nu_k (\hat{a}_k^\dagger \hat{a}_k + \hbar\frac{1}{2}) + \hbar\sum_k g_k (\hat{\sigma}_{ab} + \hat{\sigma}_{ba})(\hat{a}_k + \hat{a}_k^\dagger),$$

- define new operators,

$$\begin{aligned} \hat{\sigma}_z &= \hat{\sigma}_{aa} - \hat{\sigma}_{bb} = |a\rangle\langle a| - |b\rangle\langle b|, \\ \hat{\sigma}_+ &= \hat{\sigma}_{ab} = |a\rangle\langle b|, \\ \hat{\sigma}_- &= \hat{\sigma}_{ba} = |b\rangle\langle a|, \end{aligned}$$

and the new energy level

$$\hbar\omega_a \hat{\sigma}_{aa} + \hbar\omega_b \hat{\sigma}_{bb} = \frac{1}{2}\hbar\omega\hat{\sigma}_z + \frac{1}{2}(\omega_a + \omega_b),$$

Jaynes-Cummings Hamiltonian

- the Hamiltonian for a two-level atom interaction with quantized fields becomes

$$\hat{H} = \frac{1}{2}\hbar\omega\hat{\sigma}_z + \sum_k \hbar\nu_k(\hat{a}_k^\dagger\hat{a}_k + \frac{1}{2}) + \hbar\sum_k g_k(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a}_k + \hat{a}_k^\dagger),$$

where the atomic operators satisfy the spin-$1/2$ algebra of the Pauli matrices, i.e.

$$[\hat{\sigma}_-, \hat{\sigma}_+] = -\hat{\sigma}_z, \quad \text{and} \quad [\hat{\sigma}_-, \hat{\sigma}_z] = 2\hat{\sigma}_-,$$

- in the *rotating-wave approximation*, we drop terms $\hat{a}_k\hat{\sigma}_-$ and $\hat{a}_k^\dagger\hat{\sigma}_+$, then we have **Jaynes-Cummings** Hamiltonian

$$\hat{H} = \frac{1}{2}\hbar\omega\hat{\sigma}_z + \sum_k \hbar\nu_k(\hat{a}_k^\dagger\hat{a}_k + \frac{1}{2}) + \hbar\sum_k g_k(\hat{\sigma}_+\hat{a}_k + \hat{a}_k^\dagger\hat{\sigma}_-),$$

Interaction of a single two-level atom with a single-mode field

- the **Jaynes-Cummings** Hamiltonian,

$$\hat{H} = \frac{1}{2}\hbar\omega\hat{\sigma}_z + \hbar\nu\hat{a}^\dagger\hat{a} + \hbar g(\hat{\sigma}_+\hat{a} + \hat{a}^\dagger\hat{\sigma}_-),$$

- the interaction Hamiltonian is,

$$\begin{aligned}\hat{V} &= \text{exp}[i\hat{H}_0 t/\hbar]\hat{H}_1\text{exp}[-i\hat{H}_0 t/\hbar], \\ &= \hbar g(\hat{\sigma}_+\hat{a}e^{i\Delta t} + \hat{a}^\dagger\hat{\sigma}_- e^{-i\Delta t}),\end{aligned}$$

  where $\Delta = \omega - \nu$,

- the equation of motion for the state $|\Psi$ is

$$i\hbar\frac{\partial}{\partial t}|\Psi\rangle = \hat{V}|\Psi\rangle,$$

  where the state $|\Psi\rangle$ is the superposition of

$$|\Psi(t)\rangle = \sum_n [c_{a,n}(t)|a,n\rangle + b_{a,n}(t)|b,n\rangle],$$

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Interaction of a single two-level atom with a single-mode field

- the interaction Hamiltonian is,

$$\hat{V} = \hbar g(\hat{\sigma}_+ \hat{a} e^{i\Delta t} + \hat{a}^\dagger \hat{\sigma}_- e^{-i\Delta t}),$$

which only cause transitions between the states $|a, n\rangle$ and $|b, n+1\rangle$, and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} c_{a,n} &= -ig\sqrt{n+1} e^{i\Delta t} c_{b,n+1}, \\ \frac{\mathrm{d}}{\mathrm{d}t} c_{b,n+1} &= -ig\sqrt{n+1} e^{-i\Delta t} c_{a,n}, \end{aligned}$$

- compared to the semi-classical equations,

$$\frac{\mathrm{d}}{\mathrm{d}t} c_a = i\frac{\Omega_R}{2} e^{-i\phi} e^{i(\omega-\nu)t} c_b,$$
$$\frac{\mathrm{d}}{\mathrm{d}t} c_b = i\frac{\Omega_R}{2} e^{i\phi} e^{-i(\omega-\nu)t} c_a,$$

Interaction of a single two-level atom with a single-mode field

- for the initially excited state, $c_{a,n}(0) = c_n(0)$ and $c_{b,n+1}(0) = 0$, and here $c_n(0)$ is the probability amplitude for the field along,
- the solutions are

$$\begin{aligned} c_{a,n}(t) &= c_n(0)[\cos(\frac{\Omega_n t}{2}) - \frac{i\Delta}{\Omega_n}\sin(\frac{\Omega_n t}{2})]e^{i\Delta t/2}, \\ c_{b,n+1}(t) &= -c_n(0)\frac{2ig\sqrt{n+1}}{\Omega_n}\sin(\frac{\Omega_n t}{2})e^{i\Delta t/2}, \end{aligned}$$

- the Rabi frequency is $\Omega_n = \Delta^2 + 4g^2(n+1)$, which is proportional to the photon number of the field,
- the probability $p(n)$ that there are n photons in the field at time t is,

$$\begin{aligned} p(n) &= |c_{a,n}(t)|^2 + |c_{b,n}(t)|^2, \\ &= |c_n(0)|^2[\cos^2(\frac{\Omega_n t}{2}) + (\frac{\Delta}{\Omega_n})^2\sin^2(\frac{\Omega_n t}{2})] + |c_{n-1}(0)|^2(\frac{4g^2 n}{\Omega_{n-1}^2})\sin^2(\frac{\Omega_{n-1} t}{2} \end{aligned}$$

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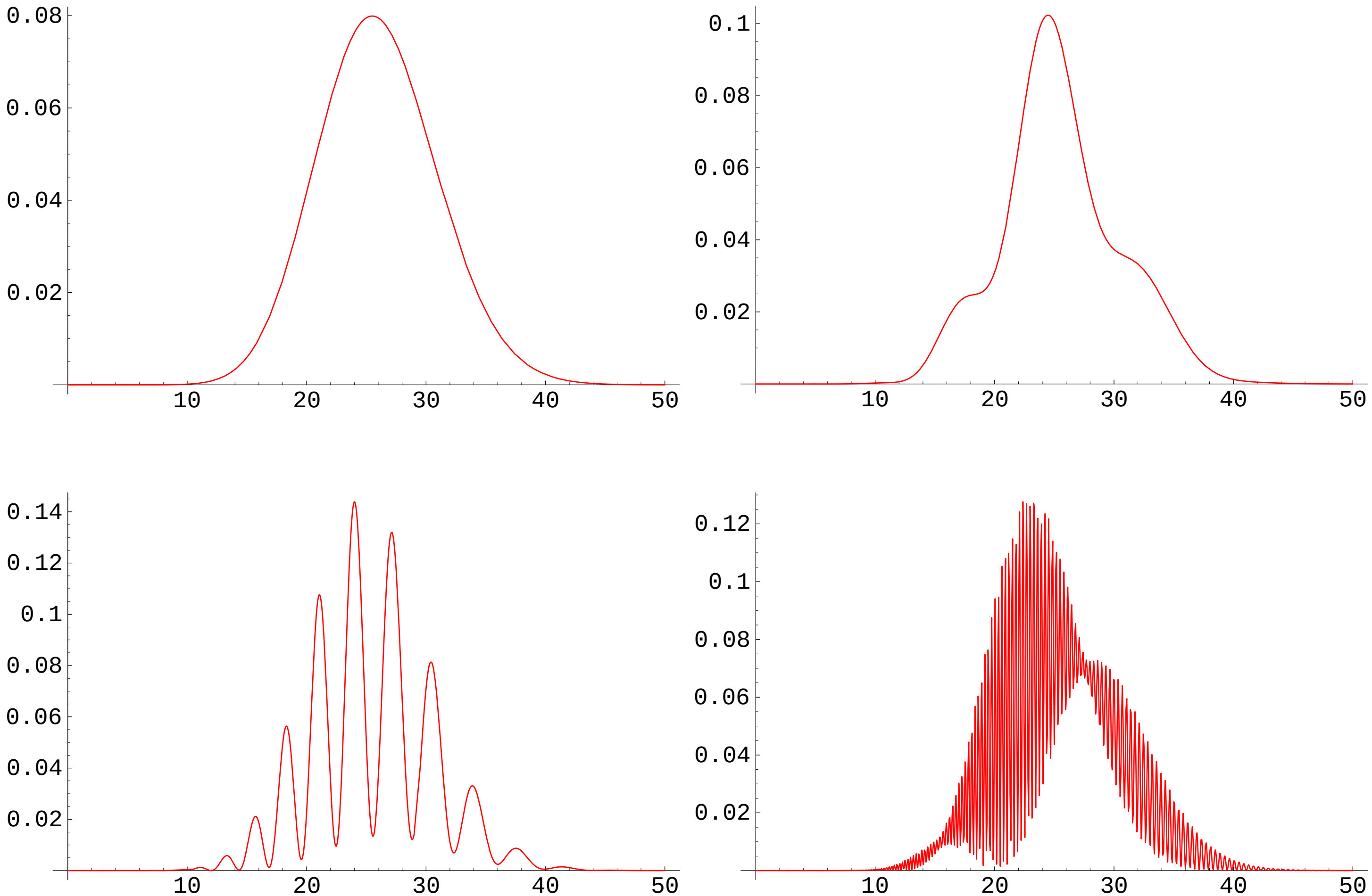
Interaction of a single two-level atom with a single-mode field

- for n photons in the field at time $t = 0$ with a coherent state, $|c_n(0)|^2 = \frac{\langle n \rangle^n e^{-\langle n \rangle}}{n!}$,

$\Delta = 0$, $\langle n \rangle = 25$, $gt = 0$ $\qquad$ $gt = 3.0$

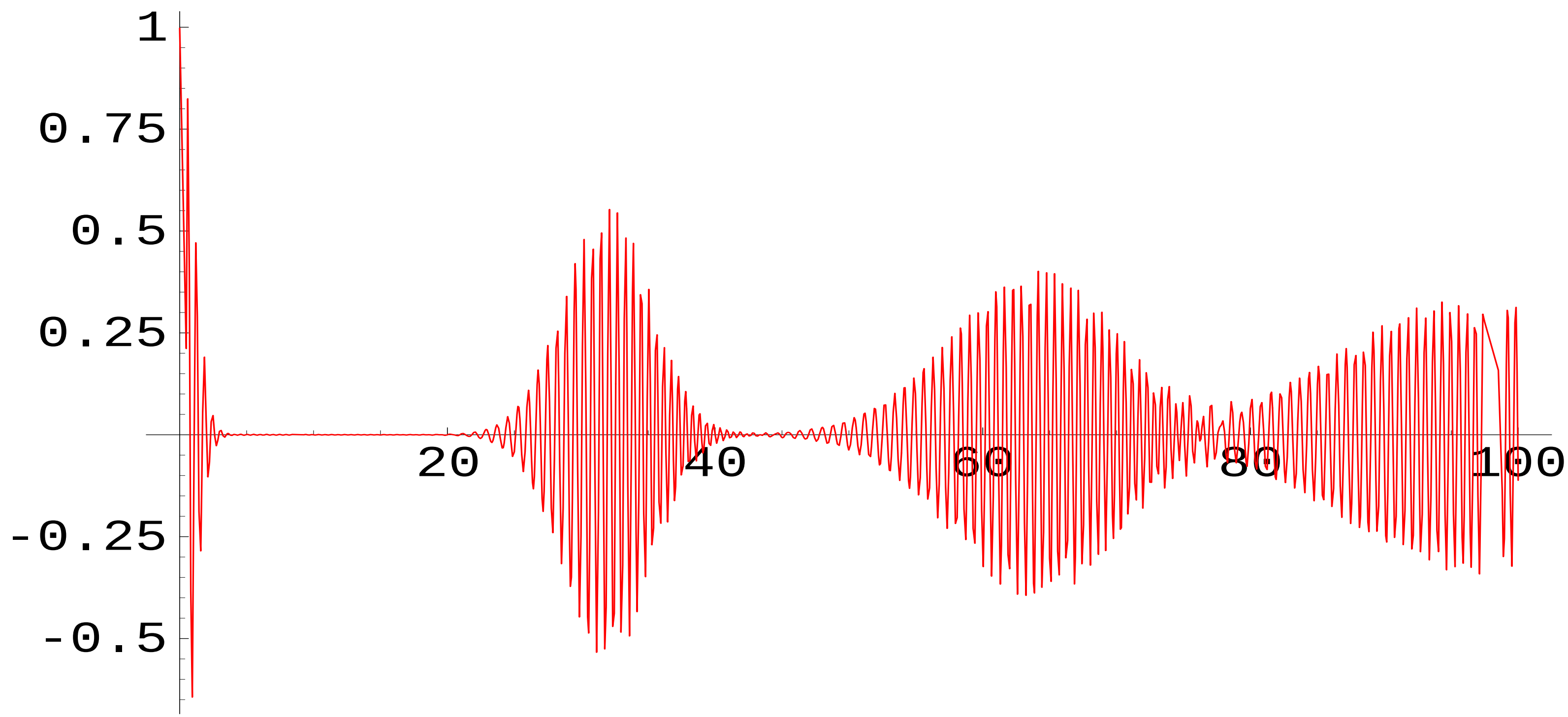


$\Delta = 0$, $\langle n \rangle = 25$, $gt = 10$ $\qquad$ $gt = 100$

Interaction of a single two-level atom with a single-mode field

- the population inversion,

$$W(t) = \sum_n |c_{a,n}(t)|^2 - |c_{b,n}(t)|^2 = \sum_0^\infty |c_n(0)|^2 [\frac{\Delta^2}{\Omega_n^2} + \frac{4g^2(n+1)}{\Omega_n^2} \cos(\Omega_n t)],$$



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$\Delta = 0, \langle n \rangle = 25,$

Revival and Collapse of the population inversion

- the population inversion,

$$W(t) = \sum_n |c_{a,n}(t)|^2 - |c_{b,n}(t)|^2 = \sum_0^\infty |c_n(0)|^2 [\frac{\Delta^2}{\Omega_n^2} + \frac{4g^2(n+1)}{\Omega_n^2} \cos(\Omega_n t)],$$

  each term in the summation represents Rabi oscillation for a definite value of n,

- at the initial time $t = 0$, the atom is prepared in a definite state and therefore all the terms in the summation are *correlated*,
- as times increases, the Rabi oscillations associated with different frequent excitations have different frequencies and there fore become *uncorrelated*, leading to a collapse of inversion,
- as time is further increased, the correlation is restored and revival occurs,
- in the semi-classical theory, the population inversion evolves with sinusoidal Rabi oscillations, and collapses to zero when on resonance,
- for the quantized fields, the collapse and revival of inversion is repeated with increasing time, but the amplitude of Rabi oscillations decreasing and the time duration in which revival takes place increasing,

Vacuum Rabi Oscillation

- the revivals occur only because of the quantized photon distribution,
- for a continuous photon distribution, like a classical random field, there is only a collapse but no revivals,
- compared to Fourier transform and Discrete Fourier transform,
- even for initial vacuum field, $|c_n(0)|^2 = \delta_{n0}$, the inversion is

$$W(t) = \frac{1}{\Delta^2 + 4g^2}[\Delta^2 + 4g^2 \cos(\sqrt{\Delta^2 + 4g^2}t)],$$

- the Rabi oscillation take place due to the vacuum state,
- the transition from the upper level to the lower level in the *vacuum* becomes possible due to spontaneous emission,

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Collective angular momentum operators

- for a two-level atom, one can use Pauli spin operator to describe,

$$\hat{s} = \frac{1}{2}\hbar\sigma,$$

where

$$\begin{aligned} \hat{\sigma}_z &= |a\rangle\langle a| - |b\rangle\langle b|, \quad \hat{\sigma}_+ = |a\rangle\langle b|, \quad \hat{\sigma}_- = |b\rangle\langle a|, \\ \hat{\sigma}_x &= |a\rangle\langle b| + |b\rangle\langle a|, \quad \text{and} \quad \hat{\sigma}_y = -i(|a\rangle\langle b| - |b\rangle\langle a|), \end{aligned}$$

- for an assembly of N two-level atoms, the corresponding Hilbert space is spanned by the set of 2^N product states,

$$|\Phi\rangle = \prod_{n=1}^{N} |\Psi_n\rangle,$$

- we can define the collective angular momentum operators,

$$\hat{J}_\mu = \frac{1}{2}\hat{\sigma}_{n\mu}, \quad (\mu = x, y, z),$$

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Analogs between $\hat{J}$ and $\hat{a}$, $\hat{a}^\dagger$

- the analogies between the free-field quantization, $\hat{a}$ and $\hat{a}^\dagger$, and the free atom quantization,

$$\begin{aligned}
[\hat{J}_x, \hat{J}_y] = i\hat{J}_z &\quad\leftrightarrow\quad [\hat{q},\hat{p}] = i\hbar, \\
\hat{J}_- = \hat{J}_x - i\hat{J}_y &\quad\leftrightarrow\quad \hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + i\hat{p}), \\
\hat{J}_+ = \hat{J}_x + i\hat{J}_y &\quad\leftrightarrow\quad \hat{a}^\dagger = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} - i\hat{p}), \\
\hat{J}_z = \frac{1}{2}(\hat{J}_+\hat{J}_- \hat{J}_-\hat{J}_+) &\quad\leftrightarrow\quad \hat{n} = \hat{a}^\dagger\hat{a},
\end{aligned}$$

- and the commutation relations,

$$\begin{aligned}
[\hat{J}_-, \hat{J}_+] = -2\hat{J}_z &\quad\leftrightarrow\quad [\hat{a},\hat{a}^\dagger] = 1, \\
[\hat{J}_-, \hat{J}_z] = \hat{J}_- &\quad\leftrightarrow\quad [\hat{a},\hat{n}] = \hat{a}, \\
[\hat{J}_+, \hat{J}_z] = -\hat{J}_+ &\quad\leftrightarrow\quad [\hat{a}^\dagger,\hat{n}] = -\hat{a}^\dagger,
\end{aligned}$$

- when all the atoms are in the ground state, the eigenvalue of $\hat{J}_z$ is $-J = -\frac{N}{2}$, the commutation relation is reduced to a *bosonlike* one, $[\hat{J}_-, \hat{J}_+] = N \leftrightarrow [\hat{a},\hat{a}^\dagger] = 1$,

Angular momentum eigenstates (Dicke states)

- the Dicke states are defined as the simultaneous eigenstates of the Hermitian operators $\hat{J}_z$ and $\hat{J}^2$, i.e.

$$\hat{J}_z|M,J\rangle = M|M,J\rangle, \quad \text{and} \quad , \hat{J}^2|M,J\rangle = J(J+1)|M,J\rangle,$$

where $(M = -J, -J+1, \ldots, J-1, J)$ and

$$\begin{aligned}
\hat{J}_+|M,J\rangle &= \sqrt{J(J+1)-M(M+1)}|M+1,J\rangle &\leftrightarrow\quad & \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \\
\hat{J}_-|M,J\rangle &= \sqrt{J(J+1)-M(M-1)}|M-1,J\rangle &\leftrightarrow\quad & \hat{a}|n\rangle = \sqrt{n}|n-1\rangle, \\
\hat{J}_-|-J,J\rangle &= 0 &\leftrightarrow\quad & \hat{a}|0\rangle = 0,
\end{aligned}$$

$$|M,J\rangle = \frac{1}{(M+J)!}\left(\begin{array}{c} 2J \\ M+J \end{array}\right)^{-1/2} \hat{J}_+^{(M+J)}|-J,J\rangle \quad \leftrightarrow \quad |n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle,$$

- the Dicke states is the counterpart of the Fock state, the state $|M,J\rangle$ denotes an atomic ensemble where exactly $J+M$ atoms are in the excited state out of $N = 2J$ atoms,

Interaction between N two-level atoms and a single-mode field

- the Dicke states $|-J, J\rangle$ corresponds to the case in which all the atoms are in the ground state, $J = N/2$,
- the Dicke states $|-J+1, J\rangle$ corresponds to the case in which only one atom is in the excited state,
- the Dicke states $|J, J\rangle$ corresponds to the case in which all the atom are in the excited state,
- the total Hamiltonian for N two-level atoms with a single-mode field is,

$$\hat{H} = \frac{1}{2}\hbar\omega\hat{J}_z + \hbar\nu(\hat{a}^\dagger\hat{a} + \frac{1}{2}) + \hbar g(\hat{J}_+\hat{a} + \hat{a}^\dagger\hat{J}_-),$$

- collective Rabi oscillation

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Spontaneous emission of a two-level atom

- the interaction hamiltonian, in the rotating-wave approximation, for a two-level atom is,

$$\hat{V} = \hbar \sum_k (g_k(r_0)^* \hat{\sigma}_+ \hat{a}_k e^{i(\omega-\nu_k)t} + g_k(r_0) \hat{a}_k^\dagger \hat{\sigma}_- e^{-i(\omega-\nu_k)t}),$$

  where $g_k(r_0) = g_k \mathrm{exp}(-ik \cdot r_0)$ is the spatial dependent coupling coefficient,

- assume at $t = 0$ the atom is in the excited state $|a\rangle$ and the field modes are in the vacuum state $|0\rangle$,

$$|\Psi(t)\rangle = c_a(t)|a, 0\rangle + \sum_k c_{b,k}|b, 1_k\rangle,$$

  with $c_a(0) = 1$ and $c_{b,k}(0) = 0$,

- in the interaction picture, $|\dot{\Psi}(t)\rangle = -\frac{i}{\hbar}|\Psi(t)\rangle$, we have

$$\begin{aligned}
\dot{c}_a(t) &= -i \sum_k g_k^*(r_0) e^{i(\omega-\nu_k)t} c_{b,k}(t), \\
\dot{c}_b(t) &= -i g_k(r_0) e^{-i(\omega-\nu_k)t} c_a(t),
\end{aligned}$$

Weisskopf-Wigner theory of spontaneous emission

- in the interaction picture, $|\dot{\Psi}(t)\rangle = -\frac{i}{\hbar}|\Psi(t)\rangle$, we have

$$\begin{aligned}
\dot{c}_a(t) &= -i\sum_k g_k^*(r_0)e^{i(\omega-\nu_k)t}c_{b,k}(t), \\
\dot{c}_b(t) &= -ig_k(r_0)e^{-i(\omega-\nu_k)t}c_a(t),
\end{aligned}$$

- the exact solutions are

$$\begin{aligned}
c_b(t) &= -ig_k(r_0)\int_0^t \mathrm{d}t' e^{-i(\omega-\nu_k)t'}c_a(t'), \\
\dot{c}_a(t) &= -\sum_k |g_k(r_0)|^2\int_0^t \mathrm{d}t' e^{i(\omega-\nu_k)(t-t')}c_a(t'),
\end{aligned}$$

- assuming that the filed modes are closely spaced in frequency,

$$\sum_k \to 2\frac{V}{(2\pi)^3}\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\theta\sin\theta\int_0^{\infty}\mathrm{d}k k^2,$$

where V is the quantization volume,

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Weisskopf-Wigner theory of spontaneous emission

- the exact solutions are

$$\dot{c}_a(t) = -\sum_k |g_k(r_0)|^2 \int_0^t \mathrm{d}t' e^{i(\omega-\nu_k)(t-t')} c_a(t'),$$

- the coupling coefficient,

$$|g_k(r_0)|^2 = |\frac{\mathbf{P}\cdot E_k}{\hbar}|^2 = \frac{\nu_k}{2\hbar\epsilon_0 V}\mathbf{P}_{ab}^2 \cos^2\theta,$$

where θ is the angle between the atomic dipole moment $\mathbf{P}_{ab}$ and the electric field polarization vector $\hat{\epsilon}_k$, i.e. $\hat{E}_k(r,t) = \hat{\epsilon}_k(\frac{\hbar\nu_k}{\epsilon_0 V})^{1/2}[\hat{a}_k + \hat{a}_k^\dagger]$,

- the equation for $c_a(t)$ becomes

$$\dot{c}_a(t) = -\frac{4\mathbf{P}_{ab}^2}{(2\pi)^2 6\hbar\epsilon_0 c^3}\int_0^\infty \mathrm{d}\nu_k \int_0^t \mathrm{d}t' \nu_k^3 e^{i(\omega-\nu_k)(t-t')} c_a(t'),$$

where we have use $k = \nu_k/c$,

Weisskopf-Wigner theory of spontaneous emission

- the equation for $c_a(t)$ becomes

$$\dot{c}_a(t) = -\frac{4\mathbf{P}_{ab}^2}{(2\pi)^2 6\hbar\epsilon_0 c^3}\int_0^\infty \mathrm{d}\nu_k \int_0^t \mathrm{d}t' \nu_k^3 e^{i(\omega-\nu_k)(t-t')} c_a(t'),$$

- for most of the optical problems, ν_k varies little around the atomic transition frequency ω,
- we can safely replace ν_k^3 by ω^3 and the lower limit in the ν_k integration by $-\infty$, then

$$\begin{aligned}
\dot{c}_a(t) &= -\frac{4\mathbf{P}_{ab}^2\omega^3}{(2\pi)^2 6\hbar\epsilon_0 c^3}\int_{-\infty}^\infty \mathrm{d}\nu_k \int_0^t \mathrm{d}t' e^{i(\omega-\nu_k)(t-t')} c_a(t'), \\
&= -\frac{4\mathbf{P}_{ab}^2\omega^3}{(2\pi)^2 6\hbar\epsilon_0 c^3}\int_0^t \mathrm{d}t' 2\pi\delta(t-t') c_a(t'), \\
&\equiv -\frac{\Gamma}{2} c_a(t),
\end{aligned}$$

where $\Gamma = \frac{4\mathbf{P}_{ab}^2\omega^3}{12\pi^2\hbar\epsilon_0 c^3}$ is the decay rate of the excited state,

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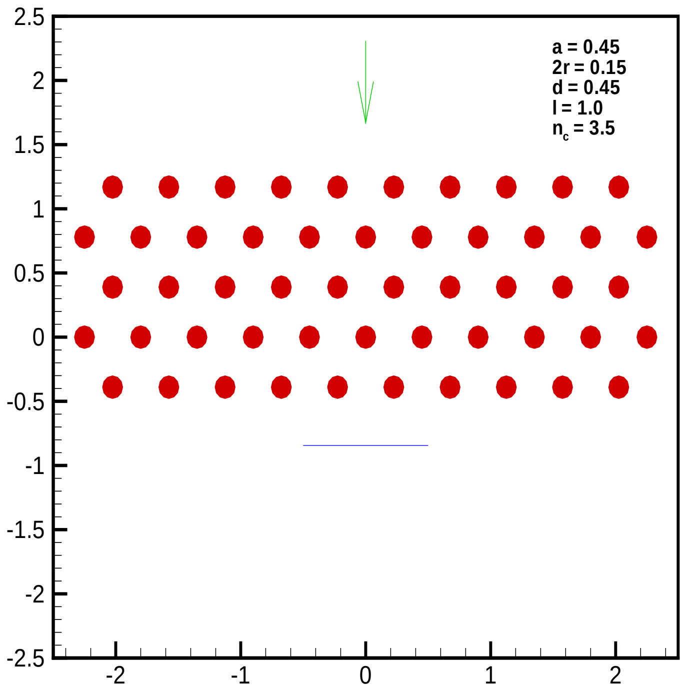
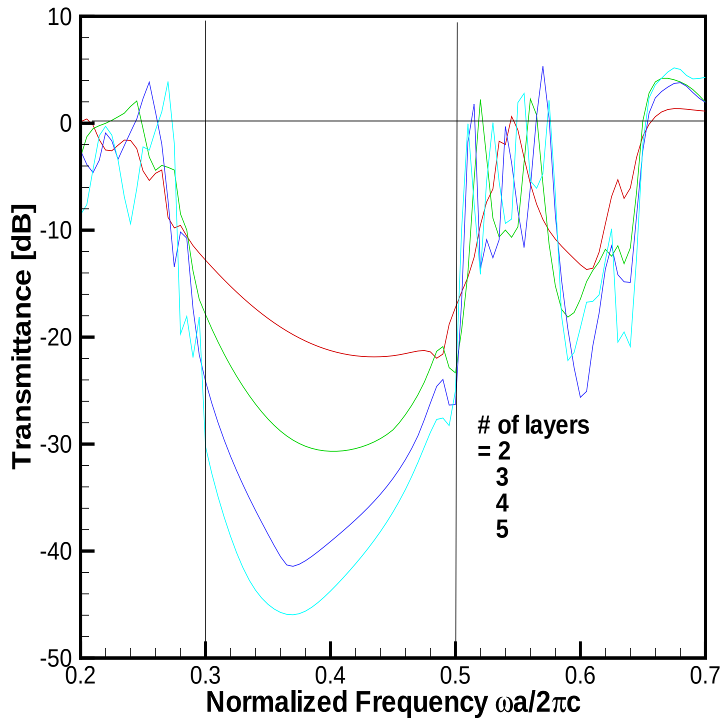
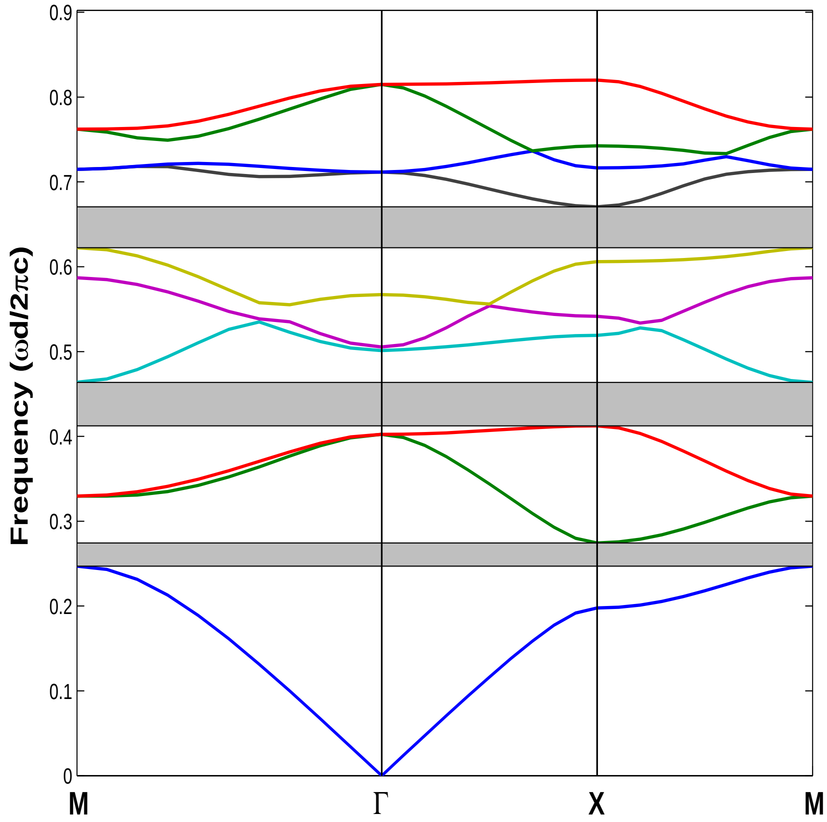
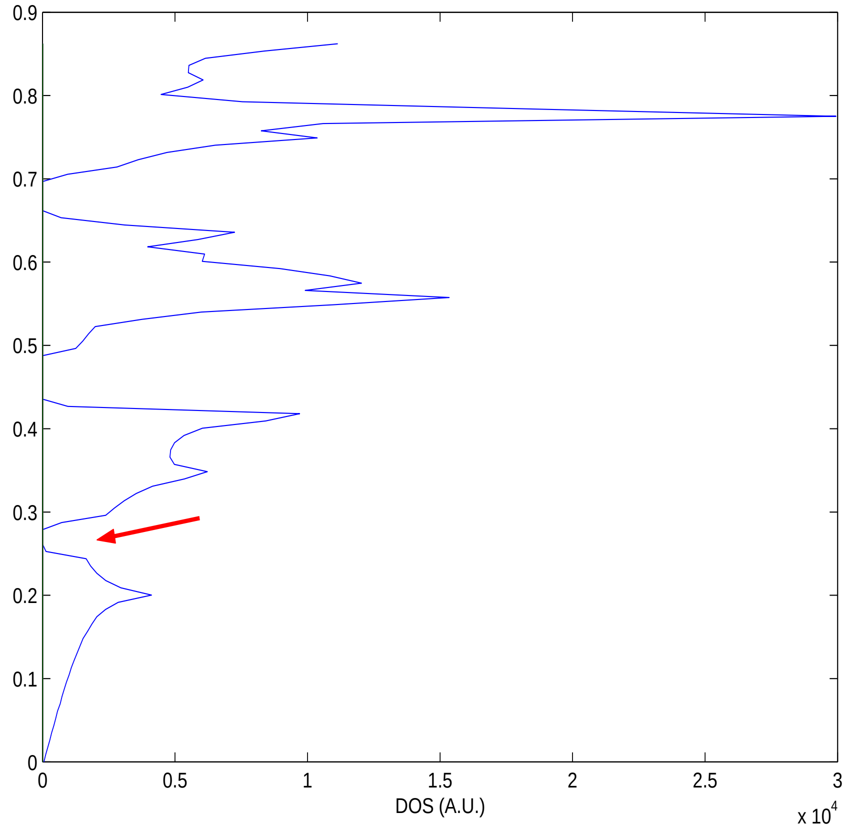
Photonic Bandgap Crystals: two(high)-dimension



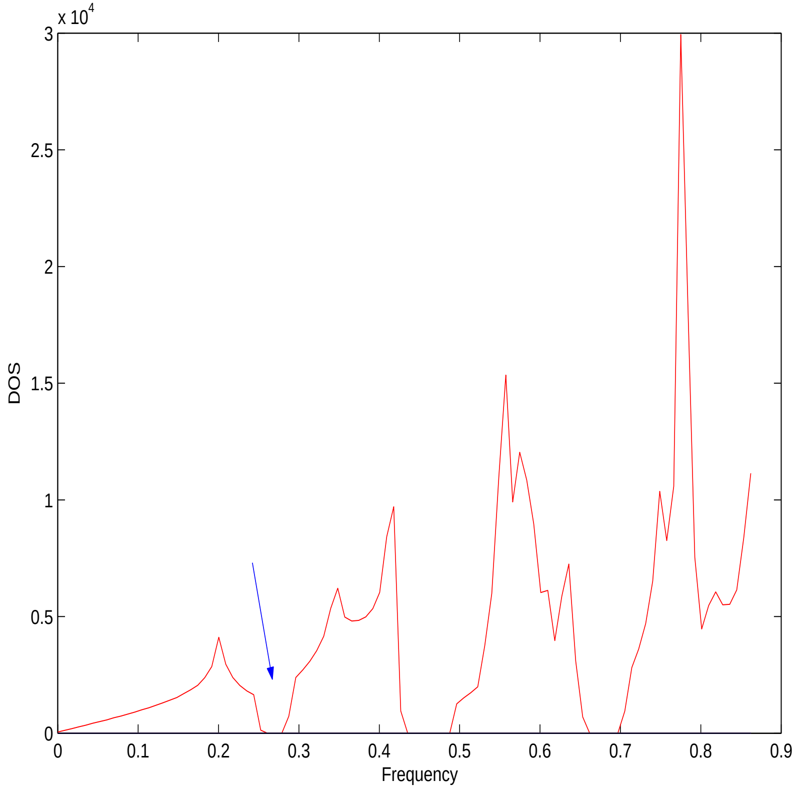


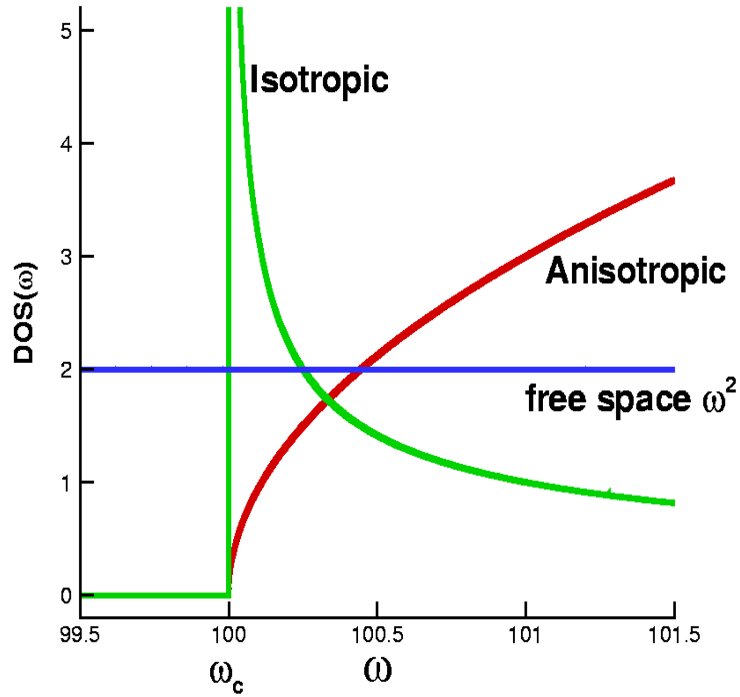
Band diagram and Density of States





Modeling DOS of PBCs





anisotropic model: $\omega_k = \omega_c + A|\mathbf{k} - \mathbf{k}_0^i|^2$

$$D(\omega) = \sqrt{\frac{\omega - \omega_c}{A^3}}\Theta(\omega - \omega_c)$$

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S. Y. Zhu, et al., *Phys. Rev. Lett.* **84**, 2136 (2000).

Remarks:

1. coupling constant:

$$g_k \equiv g_k(\hat{\mathbf{d}}, \overrightarrow{r}_0) = |d|\omega_a \sqrt{\frac{1}{2\hbar\epsilon_0\omega_k V}}\hat{\mathbf{d}} \cdot \mathbf{E}_k^*(\overrightarrow{r_0})$$

2. memory functions:

$$\begin{aligned} G(\tau) &\equiv \sum_k |g_k|^2 e^{i\Delta_k t}\Theta(\tau) \\ G_c(\tau) &\equiv \sum_k |g_k|^2 e^{-i\Delta_k t}\Theta(\tau) \end{aligned}$$

3. Markovian approximation:

$$G(t) = G_c(t) = \Gamma\delta(t)$$

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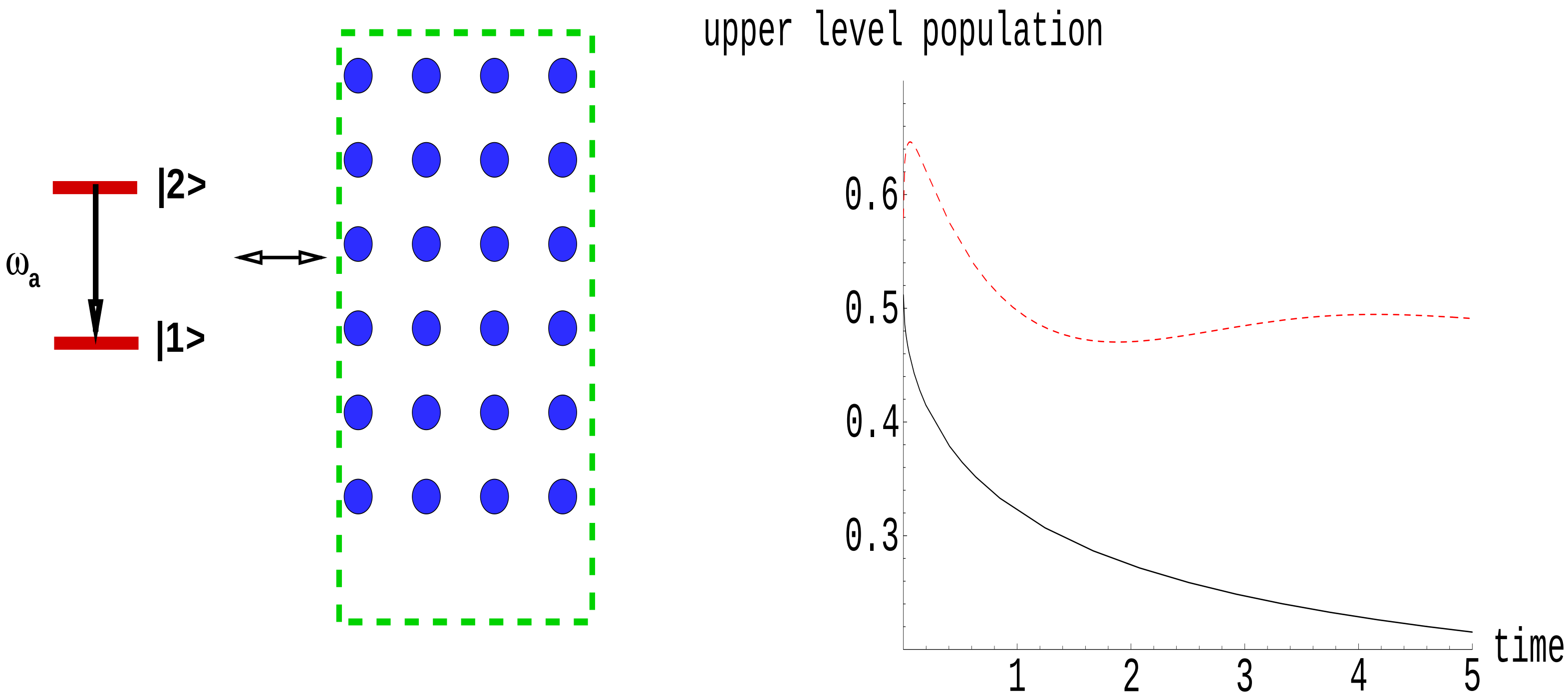
photon-atom bound state



S. John and H. Wang, *Phys. Rev. Lett.* **64**, 2418 (1990).

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Hamiltonian of our system: Jaynes-Cummings model

$$\begin{aligned} H &= \frac{\hbar}{2}\omega_a\sigma_z + \hbar\sum_k \omega_k a_k^\dagger a_k + \frac{\Omega}{2}\hbar(\sigma_- e^{i\omega_L t} + \sigma_+ e^{-i\omega_L t}) \\ &+ \hbar\sum_k (g_k\sigma_+ a_k + g_k^* a_k^\dagger \sigma_-) \end{aligned}$$

And we want to solve the generalized Bloch equations:

$$\begin{aligned} \dot{\sigma}_-(t) &= i\frac{\Omega}{2}\sigma_z(t)e^{-i\Delta t} + \int_{-\infty}^{t} dt' G(t-t')\sigma_z(t)\sigma_-(t') + n_-(t) \\ \dot{\sigma}_+(t) &= -i\frac{\Omega}{2}\sigma_z(t)e^{i\Delta t} + \int_{-\infty}^{t} dt' G_c(t-t')\sigma_+(t')\sigma_z(t) + n_+(t) \\ \dot{\sigma}_z(t) &= i\Omega(\sigma_-(t)e^{i\Delta t} - \sigma_+(t)e^{-i\Delta t}) + n_z(t) \\ &- 2\int_{-\infty}^{t} dt'[G(t-t')\sigma_+(t)\sigma_-(t') + G_c(t-t')\sigma_+(t')\sigma_-(t)] \end{aligned}$$

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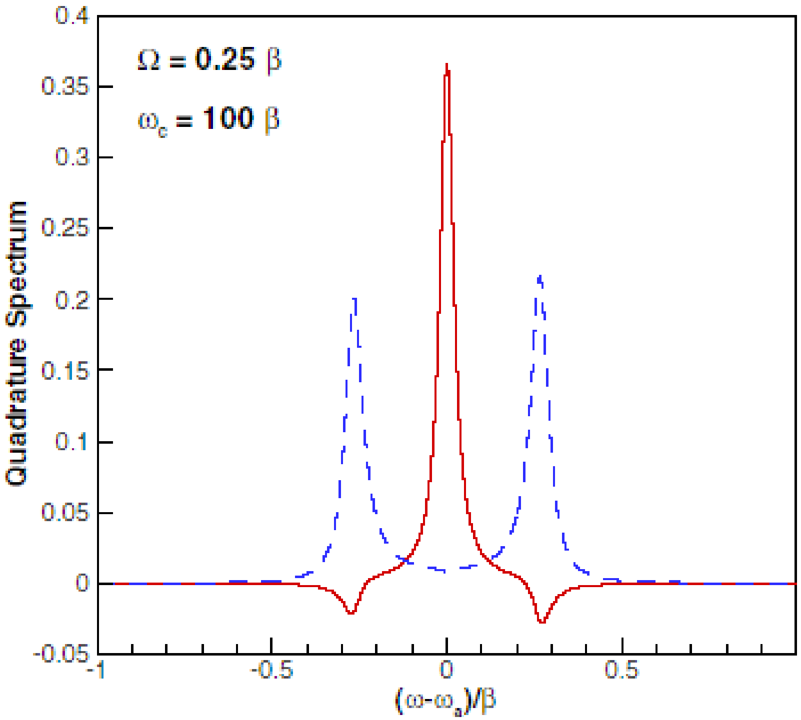
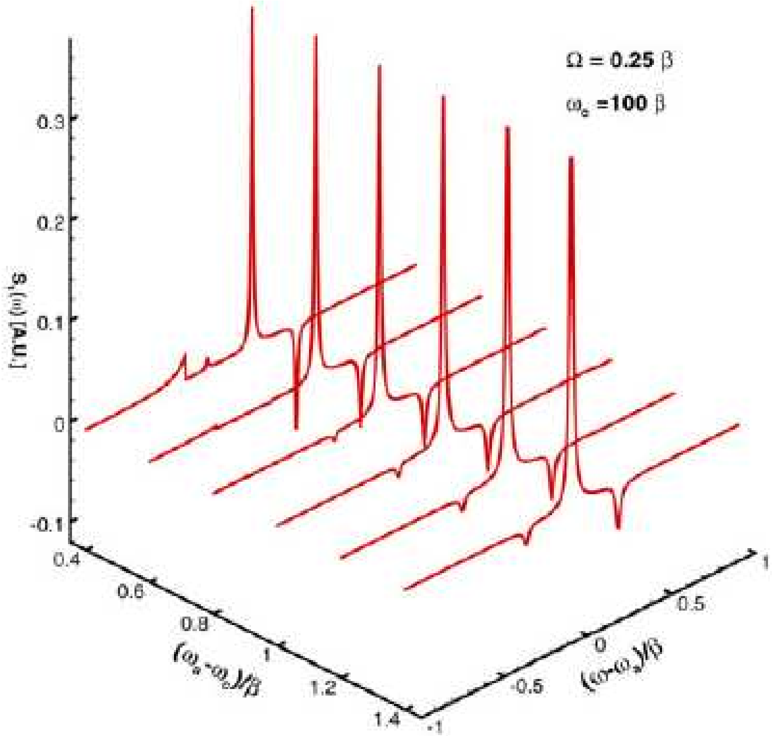
Fluorescence quadrature spectra near the band-edge





R.-K. Lee and Y. Lai, *J. Opt. B*, **6**, S715 (Special Issue 2004).

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