1, A brief review about Quantum Mechanics

1. Basic Quantum Theory
2. Time-Dependent Perturbation Theory
3. Simple Harmonic Oscillator
4. Quantization of the field
5. Canonical Quantization

Ref:

Ch. 2 in *"Introductory Quantum Optics,"* by C. Gerry and P. Knight.

Ch. 2 in *"Mesoscopic Quantum Optics,"* by Y. Yamamoto and A. Imamoglu.

Ch. 1 in *"Quantum Optics,"* by D. Wall and G. Milburn.

Ch. 4 in *"The Quantum Theory of Light,"* by R. Loudon.

Ch. 1, 2, 3, 6 in *"Mathematical Methods of Quantum Optics,"* by R. Puri

Ch. 3 in *"Elements of Quantum Optics,"* by P. Meystre and M. Sargent III.

Ch. 9 in *"Modern Foundations of Quantum Optics,"* by V. Vedral.

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Field Quantization

1. Simple Harmonic Oscillator
2. Quantization of a single-mode field
3. Basic Quantum Theory
4. Time-Dependent Perturbation Theory
5. Canonical Quantization
6. Quantum fluctuations of a single-mode field
7. Quadrature operators for a single-mode field
8. Multimode fields
9. Thermal fields
10. Vacuum fluctuations and the zero-point energy
11. Casimir force

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Role of Quantum Optics

- photons occupy an *electromagnetic mode*, we will always refer to modes in quantum optics, typically a plane wave;
- the energy in a mode is not continuous but discrete in quanta of $\hbar\omega$;
- the observables are just represented by probabilities as usual in quantum mechanics;
- there is a *zero point energy* inherent to each mode which is equivalent with fluctuations of the electromagnetic field in vacuum, due to uncertainty principle.

Vacuum

vacuum is not just nothing, it is full of energy.

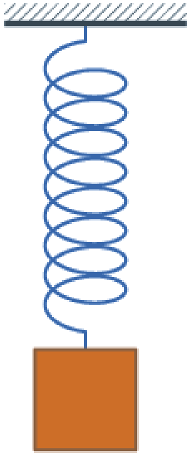
Vacuum

- **spontaneous emission** is actually stimulated by the vacuum fluctuation of the electromagnetic field,
- one can modify vacuum fluctuations by resonators and photonic crystals,
- **atomic stability**: the electron does not crash into the core due to vacuum fluctuation of the electromagnetic field,
- **gravity** is not a fundamental force but a side effect matter modifies the vacuum fluctuations, by Sakharov,
- **Casimir effect**: two charged metal plates repel each other until Casimir effect overcomes the repulsion,
- **Lamb shift**: the energy level difference between $2S_{1/2}$ and $2P_{1/2}$ in hydrogen.
- $\ldots$

Simple Harmonic Oscillator

- The simple harmonic oscillator has no driving force, and no friction (damping), so the net force is just:

$$F = -kx = ma = m\frac{\mathrm{d}^2x}{\mathrm{d}t^2},$$



- if define $\omega_0^2 = k/m$, then

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + {\omega_0}^2 x = 0$$

- the general solution $x = A\text{Cos}(\omega_0 t + \phi)$,
- The kinetic energy is $T = \frac{1}{2}m\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = \frac{1}{2}kA^2\sin^2(\omega_0 t + \phi)$,
- the potential energy is $U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi)$
- the total energy of the system has the constant value $E = \frac{1}{2}kA^2$.

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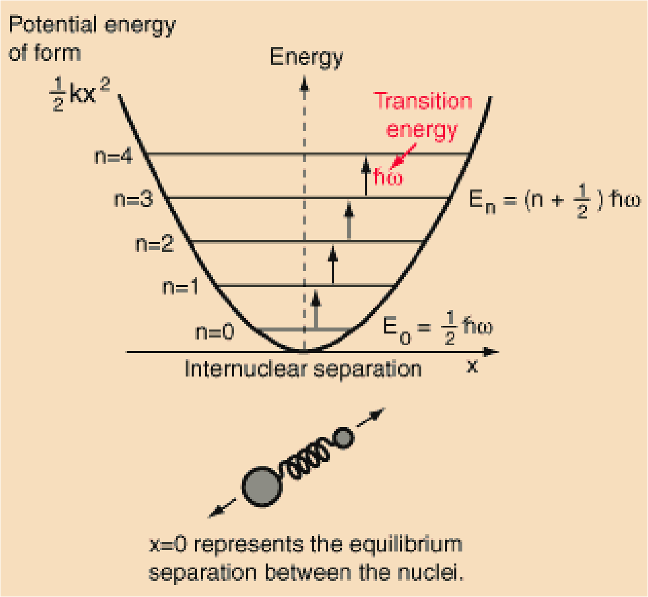
Quantum Harmonic Oscillator: 1D

- In the one-dimensional harmonic oscillator problem, a particle of mass m is subject to a potential $V(x) = \frac{1}{2}m\omega^2 x^2$.
- In classical mechanics, $m\omega^2 = k$ is called the spring stiffness coefficient or force constant, and ω the circular frequency.
- The Hamiltonian of the particle is: $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$ where x is the position operator, and p is the momentum operator $\left(p = -i\hbar\frac{d}{dx}\right)$. The first term represents the kinetic energy of the particle, and the second term represents the potential energy in which it resides.



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Maxwell's equations in Free space

- Faraday's law:

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B},$$

- Ampére's law:

$$\nabla \times \mathbf{H} = \frac{\partial}{\partial t}\mathbf{D},$$

- Gauss's law for the electric field:

$$\nabla \cdot \mathbf{D} = 0,$$

- Gauss's law for the magnetic field:

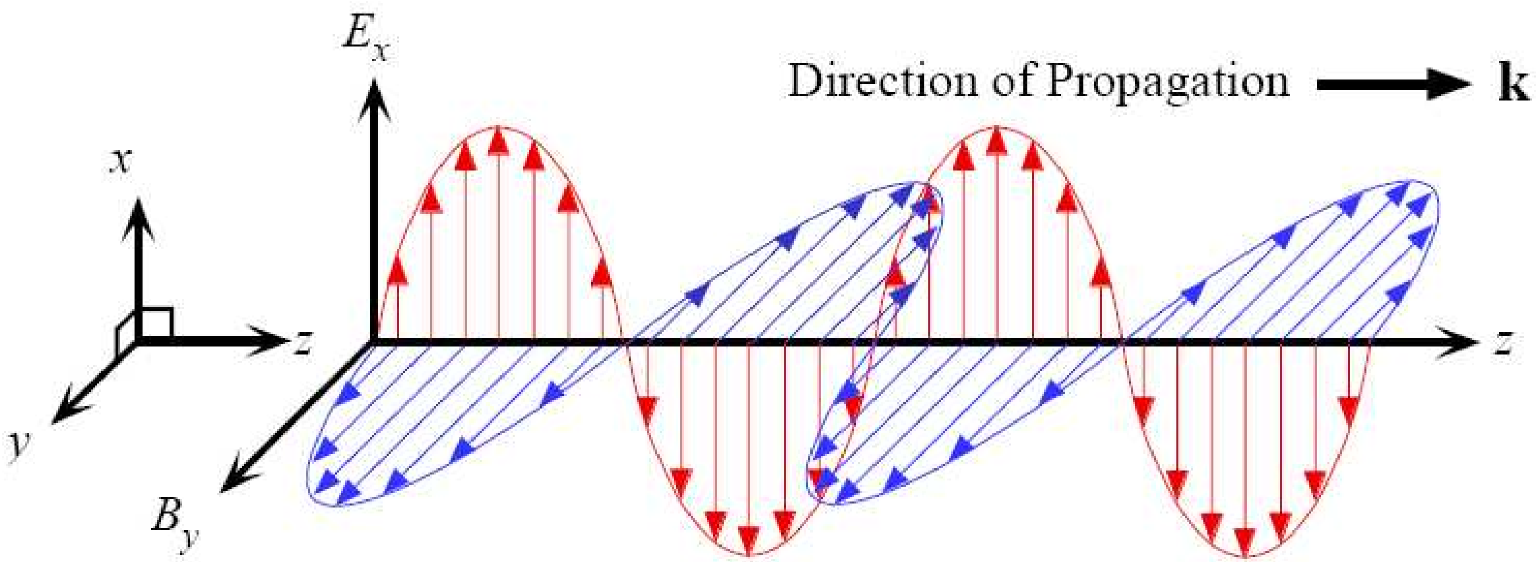
$$\nabla \cdot \mathbf{B} = 0,$$

- the constitutive relation: $\mathbf{B} = \mu_0\mathbf{H}$ and $\mathbf{D} = \epsilon_0\mathbf{E}$.

Plane electromagnetic waves

- Maxwell's equations in free space, there is vacuum, no free charges, no currents, $\mathbf{J} = \rho = 0$,
- both **E** and **B** satisfy wave equation, $\nabla^2 \mathbf{E} = \epsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$,
- we can use the solutions of *wave optics*,

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= E_0 \exp(i\omega t) \exp(-i\mathbf{k} \cdot \mathbf{r}), \\ \mathbf{B}(\mathbf{r}, t) &= B_0 \exp(i\omega t) \exp(-i\mathbf{k} \cdot \mathbf{r}), \end{aligned}$$



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Mode Expansion of the Field

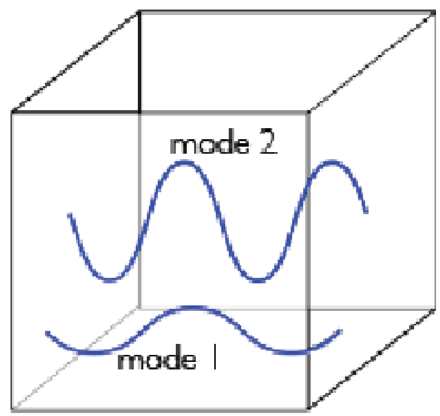
- A single-mode field, polarized along the x-direction, in the cavity:

$$\mathbf{E}(r,t) = \hat{x}E_x(z,t) = \sum_j \left(\frac{2m_j\omega_j^2}{V\epsilon_0}\right)^{1/2} q_j(t)\,\mathrm{Sin}(k_j z),$$

where $k=\omega/c$, $\omega_j = c(j\pi/L)$, $j=1,2,\ldots$, V is the effective volume of the cavity, and $q(t)$ is the normal mode amplitude with the dimension of a length (acts as a canonical position, and $p_j = m_j\dot{q}_j$ is the canonical momentum).

- the magnetic field in the cavity:

$$\mathbf{H}(r,t) = \hat{y}H_y(z,t) = \left(m_j\frac{2\omega_j^2}{V\epsilon_0}\right)^{1/2}\left(\frac{\dot{q}_j(t)\epsilon_0}{k_j}\right)\mathrm{Cos}(k_j z),$$



- the classical Hamiltonian for the field:

$$\begin{aligned} H &= \frac{1}{2}\int_V \mathrm{d}V[\epsilon_0 E_x^2 + \mu_0 H_y^2], \\ &= \frac{1}{2}\sum_j [m_j\omega_m^2 q_j^2 + m_j\dot{q}_j^2] = \frac{1}{2}\sum_j [m_j\omega_m^2 q_j^2 + \frac{p_j^2}{m_j}]. \end{aligned}$$

Quantization of the Electromagnetic Field

- Like simple harmonic oscillator, $\hat{H} = \frac{p^2}{2m} + \frac{1}{2}kx^2$, where $[\hat{x}, \hat{p}] = i\hbar$,
- For EM field, $\hat{H} = \frac{1}{2}\sum_j [m_j \omega_m^2 q_j^2 + \frac{p_j^2}{m_j}]$, where $[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$,
- *annihilation* and *creation* operators:

$$\hat{a}_j e^{-i\omega_j t} = \frac{1}{\sqrt{2m_j\hbar\omega_j}}(m_j\omega_j\hat{q}_j + i\hat{p}_j),$$
$$\hat{a}_j^\dagger e^{i\omega_j t} = \frac{1}{\sqrt{2m_j\hbar\omega_j}}(m_j\omega_j\hat{q}_j - i\hat{p}_j),$$

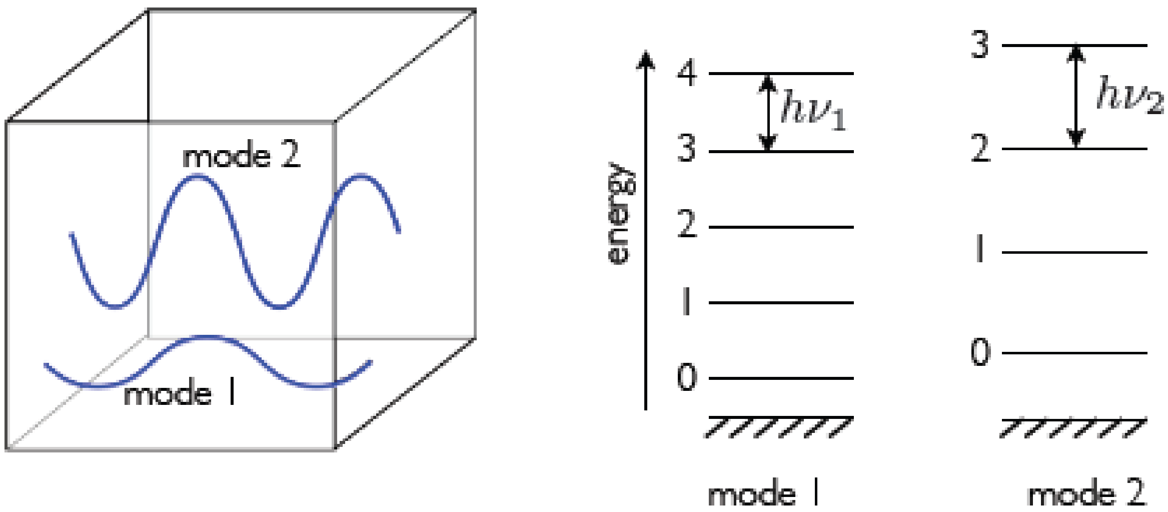
- the Hamiltonian for EM fields becomes: $\hat{H} = \sum_j \hbar\omega_j(\hat{a}_j^\dagger\hat{a}_j + \frac{1}{2})$,
- the electric and magnetic fields become,

$$\hat{E}_x(z,t) = \sum_j (\frac{\hbar\omega_j}{\epsilon_0 V})^{1/2}[\hat{a}_j e^{-i\omega_j t} + \hat{a}_j^\dagger e^{i\omega_j t}]\text{Sin}(k_j z),$$
$$\hat{H}_y(z,t) = -i\epsilon_0 c\sum_j (\frac{\hbar\omega_j}{\epsilon_0 V})^{1/2}[\hat{a}_j e^{-i\omega_j t} - \hat{a}_j^\dagger e^{i\omega_j t}]\text{Cos}(k_j z),$$

Quantization of EM fields

- the Hamiltonian for EM fields becomes: $\hat{H} = \sum_j \hbar\omega_j(\hat{a}_j^\dagger \hat{a}_j + \frac{1}{2})$,
- the electric and magnetic fields become,

$$\begin{aligned}\hat{E}_x(z,t) &= \sum_j \left(\frac{\hbar\omega_j}{\epsilon_0 V}\right)^{1/2}[\hat{a}_j e^{-i\omega_j t} + \hat{a}_j^\dagger e^{i\omega_j t}]\sin(k_j z),\\ &= \sum_j c_j[\hat{a}_{1j}\cos\omega_j t + \hat{a}_{2j}\sin\omega_j t]u_j(r),\end{aligned}$$



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Simple Harmonic Oscillator in Schrödinger picture

- one-dimensional harmonic oscillator, $\hat{H} = \frac{p^2}{2m} + \frac{1}{2}kx^2$,
- Schrödinger equation,

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\psi(x) + \frac{2m}{\hbar^2}[E - \frac{1}{2}kx^2]\psi(x) = 0,$$

with dimensionless coordinates $\xi = \sqrt{m\omega/\hbar}x$ and dimensionless quantity $\epsilon = 2E/\hbar\omega$, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\psi(x) + [\epsilon - \xi^2]\psi(x) = 0,$$

which has Hermite-Gaussian solutions,

$$\psi(\xi) = \mathrm{H}_n(\xi)e^{-\xi^2/2}, \qquad E = \frac{1}{2}\hbar\omega\epsilon = \hbar\omega(n + \frac{1}{2}),$$

where $n = 0, 1, 2, \ldots$

Ch. 7 in *"Quantum Mechanics,"* by A. Goswami.

Ch. 2 in *"Modern Quantum Mechanics,"* by J. Sakurai.

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Quantum Harmonic Oscillator

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\psi(x) + [\epsilon - \xi^2]\psi(x) = 0,$$

which has Hermite-Gaussian solutions,

$$\psi(\xi) = \mathrm{H}_n(\xi)e^{-\xi^2/2}, \qquad E = \frac{1}{2}\hbar\omega\epsilon = \hbar\omega(n+\frac{1}{2}),$$

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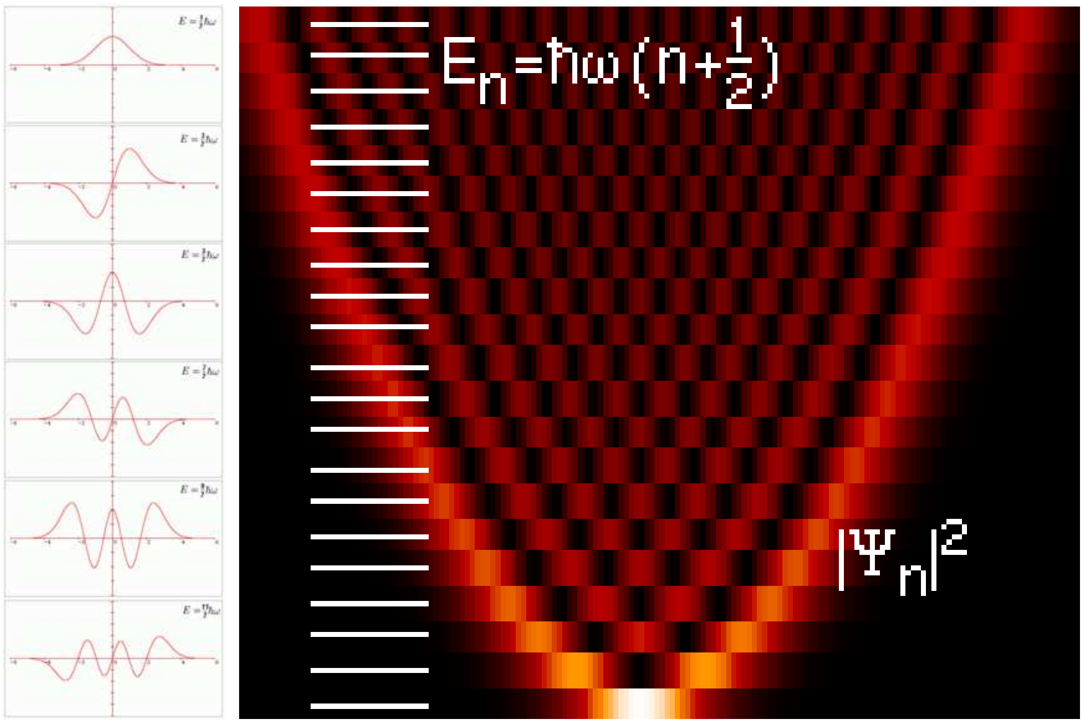

Simple Harmonic Oscillator: operator method

- one-dimensional harmonic oscillator, $\hat{H} = \frac{p^2}{2m} + \frac{1}{2}kx^2$, where $[\hat{x}, \hat{p}] = i\hbar$
- define *annihilation* operator (destruction, lowering, or step-down operators):

$$\hat{a} = \sqrt{m\omega/2\hbar}\hat{x} + i\hat{p}/\sqrt{2m\hbar\omega}.$$

- define *creation* operator (raising, or step-up operators):

$$\hat{a}^\dagger = \sqrt{m\omega/2\hbar}\hat{x} - i\hat{p}/\sqrt{2m\hbar\omega}.$$

- note that $\hat{a}$ and $\hat{a}^\dagger$ are not hermitian operators, but $(\hat{a}^\dagger)^\dagger = \hat{a}$.
- the commutation relation for $\hat{a}$ and $\hat{a}^\dagger$ is $[\hat{a}, \hat{a}^\dagger] = 1$.
- the oscillator Hamiltonian can be written as,

$$\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}) = \hbar\omega(\hat{N} + \frac{1}{2}),$$

where $\hat{N}$ is called the number operator, which is hermitian.

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Simple Harmonic Oscillator: operator method

- the number operator, $\hat{N} = \hat{a}^\dagger \hat{a}$,
- $[\hat{H}, \hat{a}] = -\hbar\omega\hat{a}$, and $[\hat{H}, \hat{a}^\dagger] = \hbar\omega\hat{a}^\dagger$.
- the eigen-energy of the system, $\hat{H}|\Psi\rangle = E|\Psi\rangle$, then

$$\hat{H}\hat{a}|\Psi\rangle = (E - \hbar\omega)\hat{a}|\Psi\rangle, \qquad \hat{H}\hat{a}^\dagger|\Psi\rangle = (E + \hbar\omega)\hat{a}^\dagger|\Psi\rangle.$$

- for any hermitian operator, $\langle\Psi|\hat{Q}^2|\Psi\rangle = \langle\hat{Q}\Psi|\hat{Q}\Psi\rangle \geq 0$.
- thus $\langle\Psi|\hat{H}|\Psi\rangle \geq 0$.
- ground state (lowest energy state), $\hat{a}|\Psi_0\rangle = 0$.
- energy of the ground state, $\hat{H}|\Psi_0\rangle = \frac{1}{2}\hbar\omega|\Psi_0\rangle$.
- excited state, $\hat{H}|\Psi_n\rangle = \hat{H}(\hat{a}^\dagger)^n|\Psi_0\rangle = \hbar\omega(n + \frac{1}{2})(\hat{a}^\dagger)^n|\Psi_0\rangle$.
- eigen-energy for excited state, $E_n = (n + \frac{1}{2})\hbar\omega$.

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Simple Harmonic Oscillator: operator method

- normalization of the eigenstates, $(\hat{a}^\dagger)^n|\Psi_0\rangle = c_n|\Psi_n\rangle$, where $c_n = \sqrt{n}$.
- $\hat{a}|\Psi_n\rangle = \sqrt{n}|\Psi_{n-1}\rangle$,
- $\hat{a}^\dagger|\Psi_n\rangle = \sqrt{n+1}|\Psi_{n+1}\rangle$,
- x-representation, $\Psi_n(x) = \langle x|\Psi_n\rangle$.
- ground state, $\langle x|\hat{a}|\Psi_0\rangle = 0$, i.e.

$$[\sqrt{\frac{m\omega}{2\hbar}}x + \hbar\frac{1}{\sqrt{2m\hbar\omega}}\frac{\mathrm{d}}{\mathrm{d}x}]\Psi_0(x) = 0,$$

- define a dimensionless variable $\xi = \sqrt{m\omega/\hbar}x$, we obtain

$$(\xi + \frac{\mathrm{d}}{\mathrm{d}\xi})\Psi_0 = 0,$$

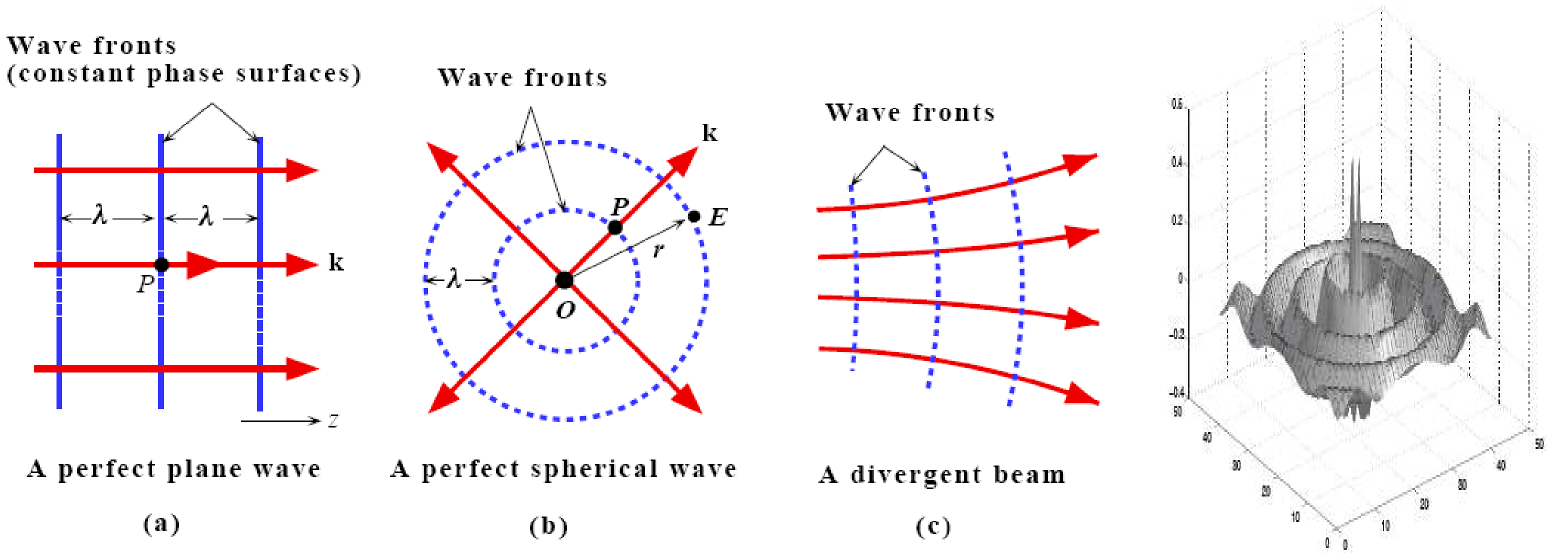
with the solution $\Psi_0(\xi) = c_0\exp(-\xi^2/2)$.

brain-storms

- Damped harmonic oscillator: $\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{b}{m}\frac{\mathrm{d}x}{\mathrm{d}t} + {\omega_0}^2 x = 0$,
  where b is an experimentally determined damping constant satisfying the relationship $F = -bv$. An example of a system obeying this equation would be a weighted spring underwater if the damping force exerted by the water is assumed to be linearly proportional to v.

- Mode expansion of the field in other bases, e.x. spherical wave:

$$E(r) = \frac{A}{|r - r_0|}\text{exp}(-ik|r - r_0|),$$



How to quantize fields?

Postulates of Quantum Mechanics

Postulate 1: An isolated quantum system is described by a vector in a Hilbert space. Two vectors differing only by a multiplying constant represent the same physical state.

- quantum state: $|\Psi\rangle = \sum_i \alpha_i |\psi_i\rangle$,
- completeness: $\sum_i |\psi_i\rangle\langle\psi_i| = I$,
- probability interpretation (projection): $\Psi(x) = \langle x|\Psi\rangle$,
- operator: $\hat{A}|\Psi\rangle = |\Phi\rangle$,
- representation: $\langle\phi|\hat{A}|\psi\rangle$,
- adjoint of $\hat{A}$: $\langle\phi|\hat{A}|\psi\rangle = \langle\psi|\hat{A}^\dagger|\phi\rangle^*$,
- hermitian operator: $\hat{H} = \hat{H}^\dagger$,
- unitary operator: $\hat{U}\hat{U}^\dagger = \hat{U}^\dagger\hat{U} = I$.

Ch. 1-5 in *"The Principles of Quantum Mechanics,"* by P. Dirac.

Ch. 1 in *"Mathematical Methods of Quantum Optics,"* by R. Puri.

Operators

- For a unitary operator, $\langle\psi_i|\psi_j\rangle = \langle\psi_i|\hat{U}^\dagger\hat{U}\psi_j\rangle$, the set of states $\hat{U}|\psi\rangle$ preserves the scalar product.
- $\hat{U}$ can be represented as $\hat{U} = \exp(i\hat{H})$ if $\hat{H}$ is hermitian.
- normal operator: $[\hat{A}, \hat{A}^\dagger] = 0$, the eigenstates of only a normal operator are *orthonormal*.
  i.e. hermitian and unitary operators are normal operators.
- The sum of the diagonal elements $\langle\phi|\hat{A}|\psi\rangle$ is call the *trace* of $\hat{A}$,

$$\mathrm{Tr}(\hat{A}) = \sum_i \langle\phi_i|\hat{A}|\phi_i\rangle,$$

  The value of the trace of an operator is independent of the basis.
- The eigenvalues of a hermitian operator are real, $\hat{H}|\Psi\rangle = \lambda|\Psi\rangle$, where λ is real.
- If $\hat{A}$ and $\hat{B}$ do not commute then they do not admit a common set of eigenvectors.

Postulates of Quantum Mechanics

Postulate 2: To each dynamical variable there corresponds a unique hermitian operator.
Postulate 3: If $\hat{A}$ and $\hat{B}$ are hermitian operators corresponding to classical dynamical variables a and b, then the commutator of $\hat{A}$ and $\hat{B}$ is given by

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A} = i\hbar\{a, b\},$$

where $\{a, b\}$ is the classical Poisson bracket.
Postulate 4: Each act of measurement of an observable $\hat{A}$ of a system in state $|\Psi\rangle$ collapses the system to an eigenstate $|\psi_i\rangle$ of $\hat{A}$ with probability $|\langle\phi_i|\Psi\rangle|^2$.
The average or the expectation value of $\hat{A}$ is given by

$$\langle\hat{A}\rangle = \sum_i \lambda_i |\langle\phi_i|\Psi\rangle|^2 = \langle\Psi|\hat{A}|\Psi\rangle,$$

where λ_i is the eigenvalue of $\hat{A}$ corresponding to the eigenstate $|\psi_i\rangle$.

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Uncertainty relation

- Non-commuting observable do not admit common eigenvectors.
- Non-commuting observables can not have definite values simultaneously.
- Simultaneous measurement of non-commuting observables to an arbitrary degree of accuracy is thus *incompatible*.
- variance: $\Delta\hat{A}^2 = \langle\Psi|(\hat{A} - \langle\hat{A}\rangle)^2|\Psi\rangle = \langle\Psi|\hat{A}^2|\Psi\rangle - \langle\Psi|\hat{A}|\Psi\rangle^2$.

$$\Delta A^2 \Delta B^2 \geq \frac{1}{4}[\langle\hat{F}\rangle^2 + \langle\hat{C}\rangle^2],$$

where

$$[\hat{A}, \hat{B}] = i\hat{C}, \qquad \text{and} \qquad \hat{F} = \hat{A}\hat{B} + \hat{B}\hat{A} - 2\langle\hat{A}\rangle\langle\hat{B}\rangle.$$

- Take the operators $\hat{A} = \hat{q}$ (position) and $\hat{B} = \hat{p}$ (momentum) for a free particle,

$$[\hat{q}, \hat{p}] = i\hbar \rightarrow \langle\Delta\hat{q}^2\rangle\langle\Delta\hat{p}^2\rangle \geq \frac{\hbar^2}{4}.$$

Uncertainty relation

- Schwarz inequality: $\langle\phi|\phi\rangle\langle\psi|\psi\rangle \geq \langle\phi|\psi\rangle\langle\psi|\phi\rangle$.
- Equality holds if and only if the two states are *linear dependent*, $|\psi\rangle = \lambda|\phi\rangle$, where λ is a complex number.
- uncertainty relation,

$$\Delta A^2 \Delta B^2 \geq \frac{1}{4}[\langle\hat{F}\rangle^2 + \langle\hat{C}\rangle^2],$$

where

$$[\hat{A}, \hat{B}] = i\hat{C}, \qquad \text{and} \qquad \hat{F} = \hat{A}\hat{B} + \hat{B}\hat{A} - 2\langle\hat{A}\rangle\langle\hat{B}\rangle.$$

- the operator $\hat{F}$ is a measure of correlations between $\hat{A}$ and$\hat{B}$.
- define two states,

$$|\psi_1\rangle = [\hat{A} - \langle\hat{A}\rangle]|\psi\rangle, \qquad |\psi_2\rangle = [\hat{B} - \langle\hat{B}\rangle]|\psi\rangle,$$

the uncertainty product is minimum, i.e. $|\psi_1\rangle = -i\lambda|\psi_2\rangle$,

$$[\hat{A} + i\lambda\hat{B}]|\psi\rangle = [\langle\hat{A}\rangle + i\lambda\langle\hat{B}\rangle]|\psi\rangle = z|\psi\rangle.$$

- the state $|\psi\rangle$ is a *minimum uncertainty state*.

Uncertainty relation

- if *Re*$(\lambda) = 0$, $\hat{A} + i\lambda\hat{B}$ is a normal operator, which have orthonormal eigenstates.
- the variances,

$$\Delta\hat{A}^2 = -\frac{i\lambda}{2}[\langle\hat{F}\rangle + i\langle\hat{C}\rangle], \qquad \Delta\hat{B}^2 = -\frac{i}{2\lambda}[\langle\hat{F}\rangle - i\langle\hat{C}\rangle],$$

- set $\lambda = \lambda_r + i\lambda_i$,

$$\Delta\hat{A}^2 = \frac{1}{2}[\lambda_i\langle\hat{F}\rangle + \lambda_r\langle\hat{C}\rangle], \qquad \Delta\hat{B}^2 = \frac{1}{|\lambda|^2}\Delta\hat{A}^2, \qquad \lambda_i\langle\hat{C}\rangle - \lambda_r\langle\hat{F}\rangle = 0.$$

- if $|\lambda| = 1$, then $\Delta\hat{A}^2 = \Delta\hat{B}^2$, *equal variance minimum uncertainty states*.
- if $|\lambda| = 1$ along with $\lambda_i = 0$, then $\Delta\hat{A}^2 = \Delta\hat{B}^2$ and $\langle\hat{F}\rangle = 0$, **uncorrelated** *equal variance minimum uncertainty states*.
- if $\lambda_r \neq 0$, then $\langle\hat{F}\rangle = \frac{\lambda_i}{\lambda_r}\langle\hat{C}\rangle$, $\quad \Delta\hat{A}^2 = \frac{|\lambda|^2}{2\lambda_r}\langle\hat{C}\rangle$, $\quad \Delta\hat{B}^2 = \frac{1}{2\lambda_r}\langle\hat{C}\rangle$.
If $\hat{C}$ is a positive operator then the minimum uncertainty states exist only if $\lambda_r > 0$.

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Momentum as a generator of Translation

- For an infinitesimal translation by dx, and the operator that does the job by $\mathcal{T}(dx)$,

$$\mathcal{T}(dx)|x\rangle = |x + dx\rangle,$$

- the infinitesimal translation should be unitary, $\mathcal{T}^{\dagger}(dx)\mathcal{T}(dx) = 1$,
- two successive infinitesimal translations, $\mathcal{T}(dx_1)\mathcal{T}(dx_2) = \mathcal{T}(dx_1 + dx_2)$,
- a translation in the opposite direction, $\mathcal{T}(dx_1) = \mathcal{T}^{-1}(dx)$,
- identity operation, $dx \to 0$, then $\lim_{dx\to 0}\mathcal{T}(dx) = 1$,
- define a Hermitian operator,

$$\mathcal{T}(dx) = \exp(-i\hat{K}\cdot dx) \approx 1 - i\hat{K}\cdot dx,$$

Ch. 2 in *"Modern Quantum Mechanics,"* by J. Sakurai.

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Momentum as a generator of Translation

- define a Hermitian operator,

$$\mathcal{T}(dx) = \exp(-i\hat{K}\cdot dx) \approx 1 - i\hat{K}\cdot dx,$$

- we have the communication relation,

$$[\hat{x}, \lceil\S] = dx, \qquad \text{or} \qquad [\hat{x}_i, \hat{K}_j] = i\delta_{ij},$$

- L. De Brogie's relation,

$$\frac{2\pi}{\lambda} = \frac{p}{\hbar},$$

- define $\hat{K} = \hat{p}/\hbar$, then

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij},$$

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Momentum Operator in the Position basis

- the definition of momentum as the generator of infinitesimal translations,

$$\begin{aligned}
(1 - \frac{i\hat{p}\Delta x}{\hbar})|\alpha\rangle &= \int dx \mathcal{T}(\Delta x)|x\rangle\langle x|\alpha\rangle \\
&= \int dx|x + \Delta x\rangle\langle x|\alpha\rangle \\
&= \int dx|x\rangle\langle x - \Delta x|\alpha\rangle \\
&= \int dx|x\rangle(\langle x|\alpha\rangle - \Delta x \frac{\partial}{\partial x}\langle x|\alpha\rangle)
\end{aligned}$$

- comparison of both sides,

$$\hat{p}|\alpha\rangle = \int dx|x\rangle(-i\hbar\frac{\partial}{\partial x}\langle x|\alpha\rangle),$$

- or

$$\langle x|\hat{p}|\alpha\rangle = -i\hbar\frac{\partial}{\partial x}\langle x|\alpha\rangle$$

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Uncertainty relation for $\hat{q}$ and $\hat{p}$

- take the operators $\hat{A} = \hat{q}$ (position) and $\hat{B} = \hat{p}$ (momentum) for a free particle,

$$[\hat{q}, \hat{p}] = i\hbar \to \langle \Delta\hat{q}^2\rangle\langle \Delta\hat{p}^2\rangle \geq \frac{\hbar^2}{4}.$$

- define two states, $|\psi_1\rangle = [\hat{A} - \langle\hat{A}\rangle]|\psi\rangle \equiv \hat{\alpha}|\psi\rangle, \qquad |\psi_2\rangle = [\hat{B} - \langle\hat{B}\rangle]|\psi\rangle \equiv \hat{\beta}|\psi\rangle$.
- for uncorrelated minimum uncertainty states,

$$\hat{\alpha}|\psi\rangle = -i\lambda\hat{\beta}|\psi\rangle, \qquad \langle\psi|\hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\alpha}|\psi\rangle = 0,$$

  where λ is a real number.
- if $\hat{A} = \hat{q}$ and $\hat{B} = \hat{p}$, we have $(\hat{q} - \langle\hat{q}\rangle)|\psi\rangle = -i\lambda(\hat{p} - \langle\hat{p}\rangle)|\psi\rangle$.
- the wavefunction in the q-basis is, i.e. $\hat{p} = -i\hbar\partial/\partial q$,

$$\psi(q) = \langle q|\psi\rangle = \frac{1}{(2\pi\langle\Delta\hat{q}^2\rangle)^{1/4}}\text{exp}[\frac{i\langle\hat{p}\rangle q}{\hbar} - \frac{(q-\langle\hat{q}\rangle)^2}{4\langle\Delta\hat{q}^2\rangle}],$$

- in the p-basis, $\psi(p) = \langle p|\psi\rangle = \frac{1}{(2\pi\langle\Delta\hat{p}^2\rangle)^{1/4}}\text{exp}[-\frac{i}{\hbar}(\langle\hat{q}\rangle(p-\langle\hat{p}\rangle) - \frac{(p-\langle\hat{p}\rangle)^2}{4\langle\Delta\hat{p}^2\rangle}]$.

Minimum Uncertainty State

- $(\hat{q} - \langle \hat{q} \rangle)|\psi\rangle = -i\lambda(\hat{p} - \langle \hat{p} \rangle)|\psi\rangle$
- if we define $\lambda = e^{-2r}$, then

$$(e^r \hat{q} + ie^{-r}\hat{p})|\psi\rangle = (e^r \langle \hat{q} \rangle + ie^{-r}\langle \hat{p} \rangle)|\psi\rangle,$$

- the minimum uncertainty state is defined as an *eigenstate* of a non-Hermitian operator $e^r \hat{q} + ie^{-r}\hat{p}$ with a c-number eigenvalue $e^r \langle \hat{q} \rangle + ie^{-r}\langle \hat{p} \rangle$.
- the variances of $\hat{q}$ and $\hat{p}$ are

$$\langle \Delta \hat{q}^2 \rangle = \frac{\hbar}{2} e^{-2r}, \qquad \langle \Delta \hat{p}^2 \rangle = \frac{\hbar}{2} e^{2r}.$$

- here r is referred as the *squeezing parameter*.

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Gaussian Wave Packets

- in the x-space,

$$\Psi(x) = \langle x|\Psi\rangle = [\frac{1}{\pi^{1/4}\sqrt{d}}]\text{exp}[ikx - \frac{x^2}{2d^2}]$$

, which is a plane wave with wave number k and width d.

- the expectation value of $\hat{X}$ is zero for symmetry,

$$\langle\hat{X}\rangle = \int_{-\infty}^{\infty} \mathrm{d}x\langle\Psi|x\rangle\hat{X}\langle x|\Psi\rangle = 0.$$

- variation of $\hat{X}$, $\langle\Delta\hat{X}^2\rangle = \frac{d^2}{2}$.
- the expectation value of $\hat{P}$, $\langle\hat{P}\rangle = \hbar k$, i.e. $\langle x|\hat{P}|\Psi\rangle = -i\hbar\frac{\partial}{\partial x}\langle x|\Psi\rangle$.
- variation of $\hat{P}$, $\langle\Delta\hat{P}^2\rangle = \frac{\hbar^2}{2d^2}$.
- the Heisenberg uncertainty product is, $\langle\Delta\hat{X}^2\rangle\langle\Delta\hat{P}^2\rangle = \frac{\hbar^2}{4}$.
- a Gaussian wave packet is called a *minimum uncertainty wave packet*.

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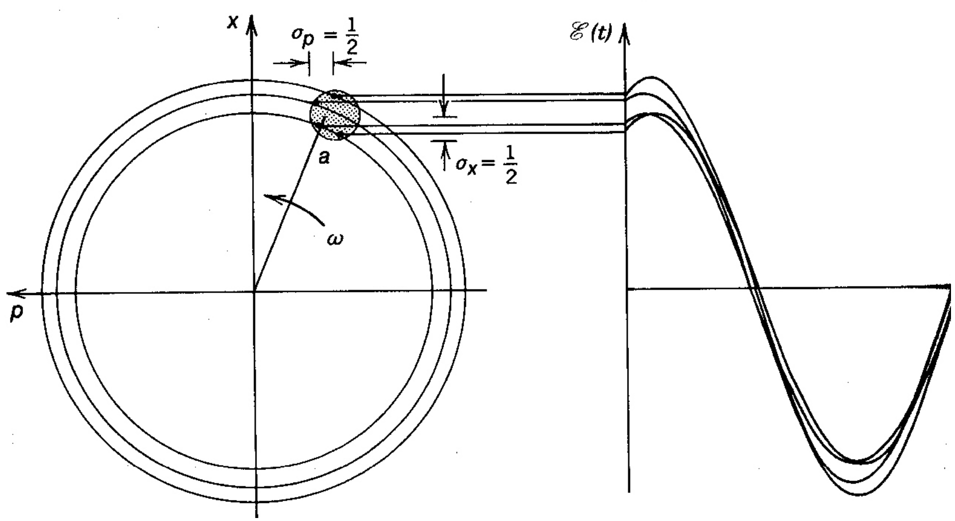
Phase diagram for EM waves

Electromagnetic waves can be represented by

$$\hat{E}(t) = E_0[\hat{X}_1 \sin(\omega t) - \hat{X}_2 \cos(\omega t)]$$

where

$$\begin{aligned} \hat{X}_1 &= \text{amplitude quadrature} \\ \hat{X}_2 &= \text{phase quadrature} \end{aligned}$$



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Quadrature operators

- the electric and magnetic fields become,

$$\begin{aligned}\hat{E}_x(z,t) &= \sum_j \left(\frac{\hbar\omega_j}{\epsilon_0 V}\right)^{1/2}[\hat{a}_j e^{-i\omega_j t} + \hat{a}_j^\dagger e^{i\omega_j t}]\sin(k_j z),\\ &= \sum_j c_j[\hat{a}_{1j}\cos\omega_j t + \hat{a}_{2j}\sin\omega_j t]u_j(r),\end{aligned}$$

- note that $\hat{a}$ and $\hat{a}^\dagger$ are not hermitian operators, but $(\hat{a}^\dagger)^\dagger = \hat{a}$.
- $\hat{a}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger)$ and $\hat{a}_2 = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger)$ are two Hermitian (quadrature) operators.
- the commutation relation for $\hat{a}$ and $\hat{a}^\dagger$ is $[\hat{a}, \hat{a}^\dagger] = 1$,
- the commutation relation for $\hat{a}$ and $\hat{a}^\dagger$ is $[\hat{a}_1, \hat{a}_2] = \frac{i}{2}$,
- and $\langle\Delta\hat{a}_1^2\rangle\langle\Delta\hat{a}_2^2\rangle \geq \frac{1}{16}$.

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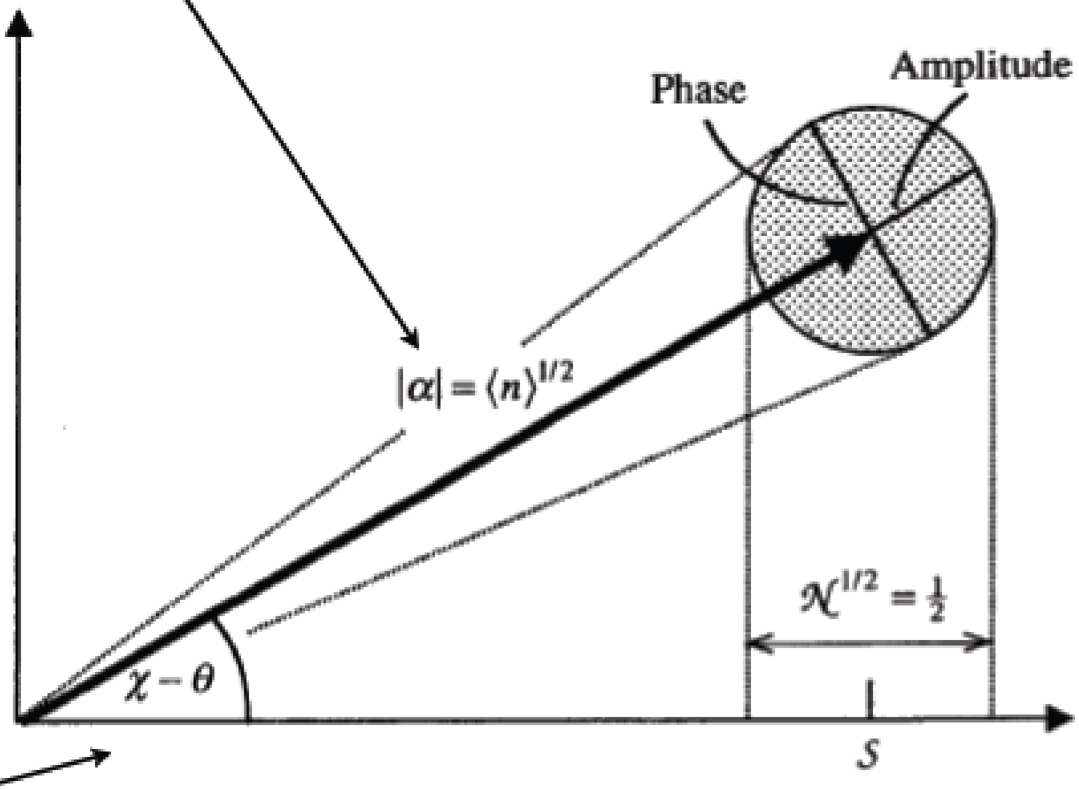
Phase diagram for coherent states

mean number of photons

$$< \hat{N} >=< \alpha|\hat{N}|\alpha >=< \alpha|\hat{a}^\dagger\hat{a}|\alpha >= |\alpha|^2$$



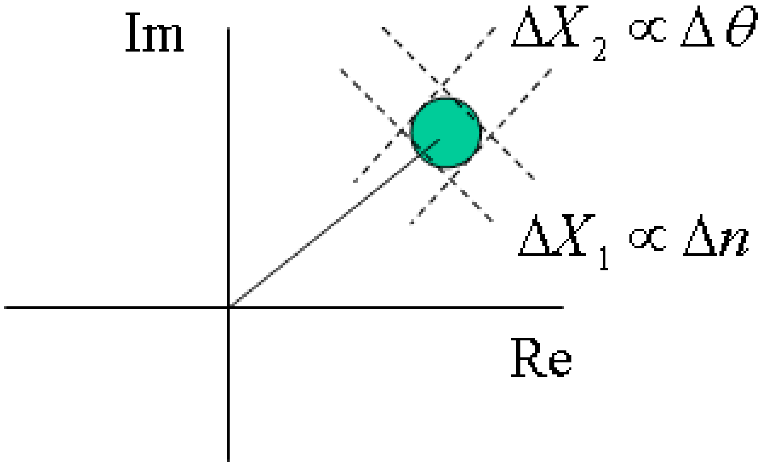
phase of the field

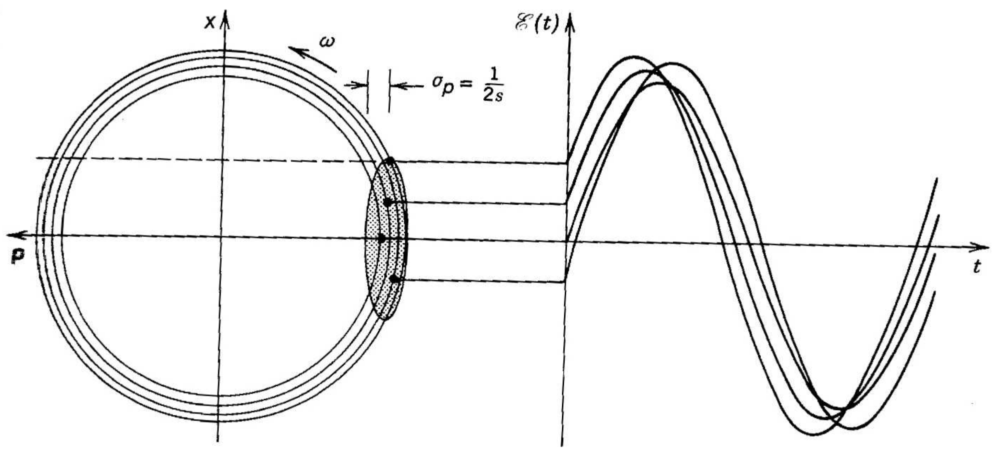
$$\alpha = |\alpha| \exp(i\theta)$$

Coherent and Squeezed States

Uncertainty Principle: $\Delta\hat{X}_1\Delta\hat{X}_2 \geq 1$.

1. Coherent states: $\Delta\hat{X}_1 = \Delta\hat{X}_2 = 1$,
2. Amplitude squeezed states: $\Delta\hat{X}_1 < 1$,
3. Phase squeezed states: $\Delta\hat{X}_2 < 1$,
4. Quadrature squeezed states.





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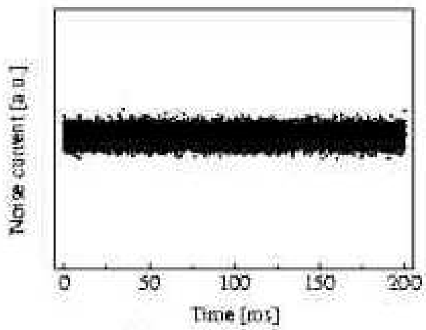
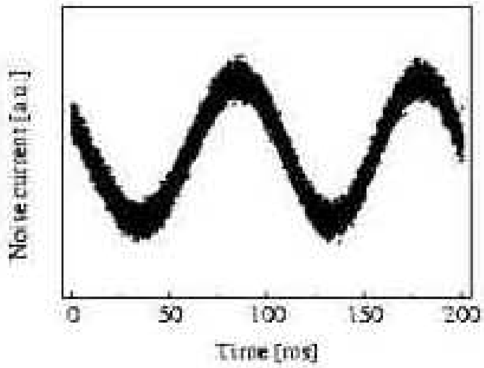
Vacuum, Coherent, and Squeezed states

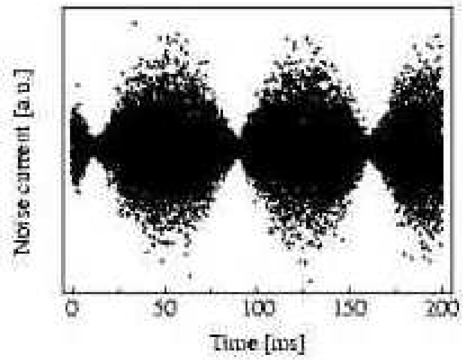
vacuum

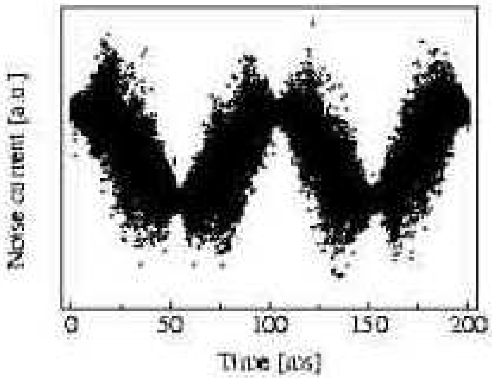


coherent

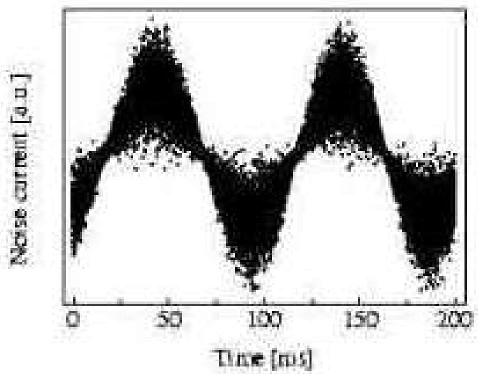


squeezed-vacuum

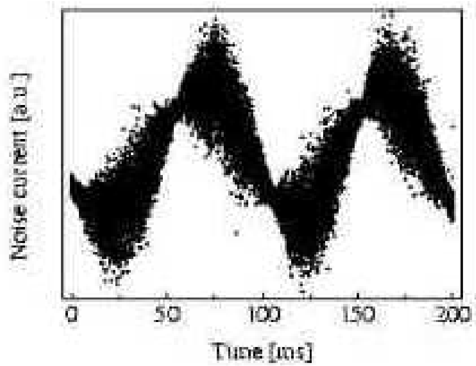




amp-squeezed



phase-squeezed



quad-squeezed

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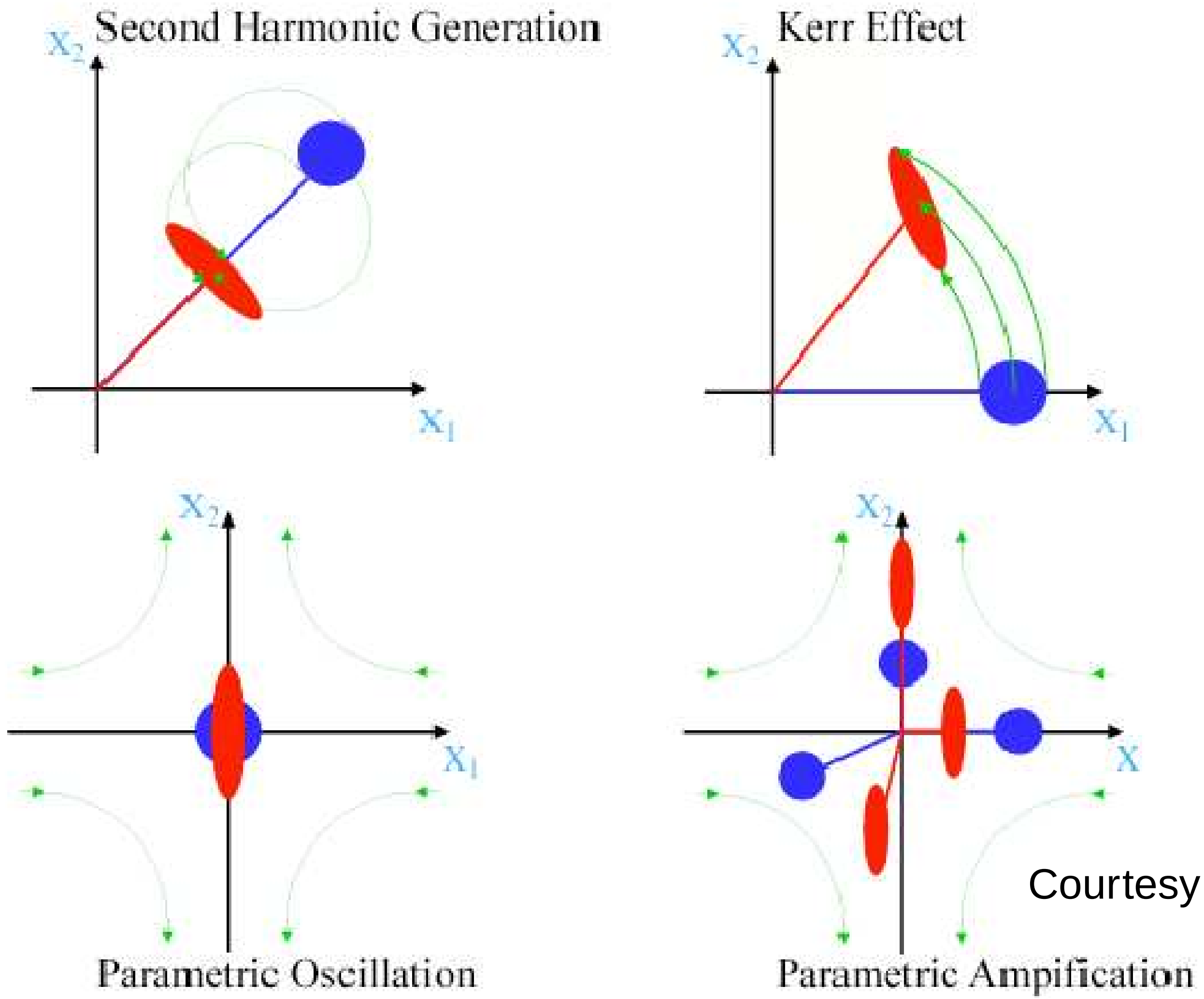
Generations of Squeezed States

Nonlinear optics:

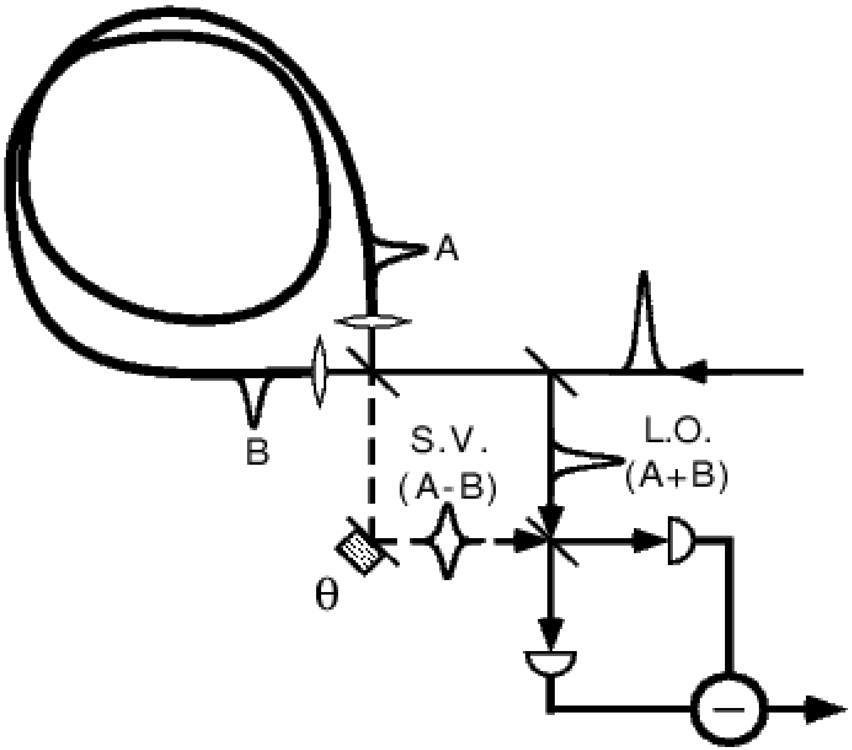


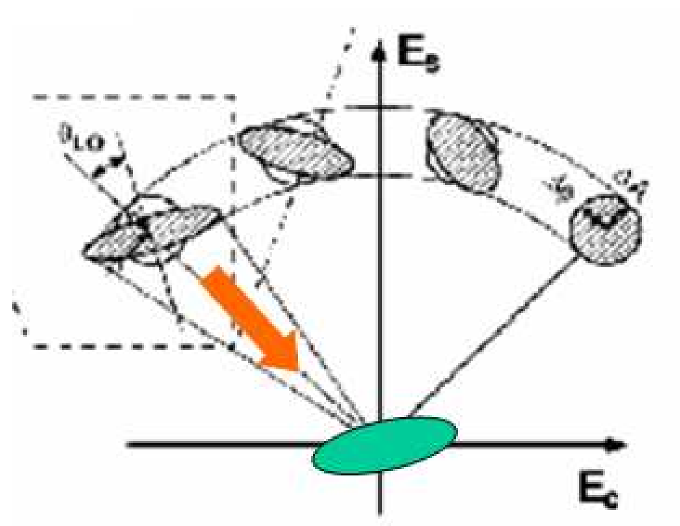
Courtesy of P. K. Lam

Generation and Detection of Squeezed Vacuum

1. Balanced Sagnac Loop (to cancel the mean field),

2. Homodyne Detection.





M. Rosenbluh and R. M. Shelby, *Phys. Rev. Lett.* **66**, 153(1991).

Schrödinger equation

Postulate 5: The time evolution of a state $|\Psi\rangle$ is governed by the Schrödinger equation,

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\Psi(t)\rangle = \hat{H}(t)|\Psi(t)\rangle,$$

where $\hat{H}(t)$ is the Hamiltonian which is a hermitian operator associated with the total energy of the system.
The solution of the Schrödinger equation is,

$$|\Psi(t)\rangle = \overleftarrow{T}\exp[-\frac{i}{\hbar}\int_{t_0}^{t}\mathrm{d}\tau\hat{H}(\tau)]|\Psi(0)\rangle \equiv \hat{U}_S(t,t_0)|\Psi(t_0)\rangle,$$

where $\overleftarrow{(}T)$ is the time-ordering operator.

Schrödinger picture:

$$|\Psi(r,t)\rangle = \sum_i \alpha_i(t)|\psi_i(r)\rangle.$$

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Time Evolution of a Minimum Uncertainty State

- the Hamiltonian for a free particle, $\hat{H} = \frac{\hat{p}^2}{2m}$, then

$$\hat{U} = \exp(-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}t).$$

- the Schrödinger wavefunction,

$$\begin{aligned}\Psi(q,t) = \langle q|\hat{U}|\Psi(0)\rangle &= \int_{-\infty}^{\infty} \mathrm{d}p\langle|p\rangle\Psi(p,0)\exp(-\frac{i}{\hbar}\frac{p^2}{2m}t),\\ &= \frac{1}{(2\pi)^{1/4}(\Delta q + i\hbar t/2m\Delta q)^{1/2}}\exp[-\frac{q^2}{4(\Delta q)^2 + 2i\hbar t/m}],\end{aligned}$$

where $\Delta q = \hbar/2\langle\hat{p}^2\rangle^{1/2}$, and $\langle q|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp(\frac{ipq}{\hbar})$.

- even though the momentum uncertainty $\langle\Delta\hat{p}^2\rangle$ is preserved,
- the position uncertainty increases as time develops,

$$\langle\Delta\hat{q}^2(t)\rangle = (\Delta\hat{q})^2 + \frac{\hbar^2 t^2}{4m^2(\Delta q)^2}$$

Gaussian Optics

- Wave equation: In free space, the vector potential, A, is defined as $A(r,t) = \vec{n}\psi(x,y,z)e^{j\omega t}$, which obeys the vector wave equation,

$$\nabla^2\psi + k^2\psi = 0.$$

- The paraxial wave equation: $\psi(x,y,z) = u(x,y,z)e^{-jkz}$, one obtains

$$\nabla_T^2 u - 2jk\frac{\partial u}{\partial z} = 0,$$

where $\nabla_T \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}$.

- This solution is proportional to the impulse response function (Fresnel kernel),

$$h(x,y,z) = \frac{j}{\lambda z}e^{-jk[(x^2+y^2)/2z]},$$

i.e. $\nabla_T^2 h(x,y,z) - 2jk\frac{\partial h}{\partial z} = 0$.

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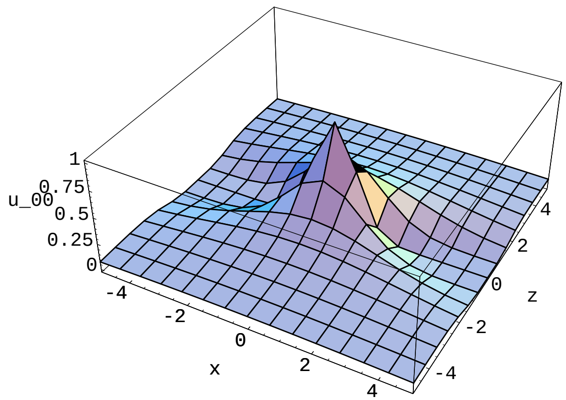
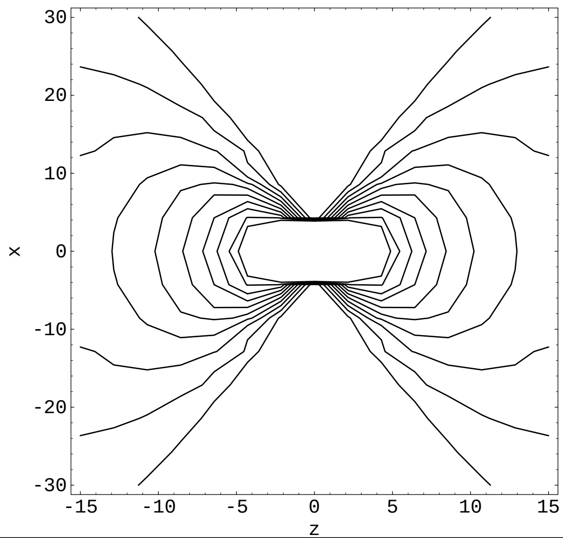
Gaussian Optics

- The solution of the scalar paraxial wave equation is,

$$u_{00}(x,y,z) = \frac{\sqrt{2}}{\sqrt{\pi} w} exp(j\phi) exp(-\frac{x^2+y^2}{w^2}) exp[-\frac{jk}{2R}(x^2+y^2],$$

- beam width $w^2(z) = \frac{2b}{k}(1+\frac{z^2}{b^2} = w_0^2[1+(\frac{\lambda z}{\pi w_0^2})^2],$
- radius of phase front $\frac{1}{R(z)} = \frac{z}{z^2+b^2} = \frac{z}{z^2+(\pi w_0^2/\lambda)^2},$
- $phasedelay \tan\phi = \frac{z}{b} = \frac{z}{\pi w_0^2/\lambda},$
- with the minimum beam radius $w_0 = \sqrt{2bk}$.





Heisenberg equation

- The solution of the Schrödinger equation is,
  $|\Psi(t)\rangle = \overleftarrow{T}\,\text{exp}[-\frac{i}{\hbar}\int_{t_0}^{t} \text{d}\tau \hat{H}(\tau)]|\Psi(0)\rangle \equiv \hat{U}_S(t,t_0)|\Psi(t_0)\rangle$.

- The quantities of physical interest are the expectation values of operators,

$$\langle\Psi(t)|\hat{A}|\Psi(t)\rangle = \langle\Psi(t_0)|\hat{A}(t)|\Psi(t_0)\rangle,$$

  where

$$\hat{A}(t) = \hat{U}_S^\dagger(t,t_0)\hat{A}\hat{U}_S(t,t_0).$$

- The time-dependent operator $\hat{A}(t)$ evolves according to the Heisenberg equation,

$$i\hbar\frac{\text{d}}{\text{d}t}\hat{A}(t) = [\hat{A},\hat{H}(t)].$$

- Schrödinger picture: time evolution of the states.

- Heisenberg picture: time evolution of the operators.

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Interaction picture

- Consider a system described by $|\Psi(t)\rangle$ evolving under the action of a hamiltonian $\hat{H}(t)$ decomposable as,

$$\hat{H}(t) = \hat{H}_0 + \hat{H}_1(t),$$

where $\hat{H}_0$ is time-independent.

- Define

$$|\Psi_I(t)\rangle = \mathrm{exp}(i\hat{H}_0 t/\hbar)|\Psi(t)\rangle,$$

then $|\Psi_I(t)\rangle$ evolves accords to

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\Psi_I(t)\rangle = \hat{H}_I(t)|\Psi_I(t)\rangle,$$

where

$$\hat{H}_I(t) = \mathrm{exp}(i\hat{H}_0 t/\hbar)\hat{H}_1(t)\mathrm{exp}(-i\hat{H}_0 t/\hbar).$$

- The evolution is in the **interaction picture** generated by $\hat{H}_0$.

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Paradoxes of Quantum Theory

- Geometric phase
- Measurement theory
- Schrödinger’s Cat paradox
- Einstein-Podolosky-Rosen paradox
- Local Hidden Variables theory

Quantum Zeno effect (watchdog effect)

- multi-time joint probability: $P(\{|\phi_i\rangle, t_i\})$, the probability that a system in a state $|\phi_0(t_0)\rangle$ at t_0 is found in the state $|\phi_i\rangle$ at t_i, where $i = 1, \ldots, n$.
- at t_1: the state is $\hat{U}_S(t_1, t_0)|\phi_0(t_0)\rangle$.
- projection on $|\phi_1\rangle$ is

$$|\phi_1(t_1)\rangle = |\phi_1\rangle\langle\phi_1|\hat{U}_S(t_1, t_0)|\phi_0(t_0)\rangle.$$

- the sate $|\phi_1(t_1)\rangle$ then evolves till time t_2 to $\hat{U}_S(t_2, t_1)|\phi_1(t_1)\rangle$, with the projection,

$$|\phi_2(t_2)\rangle = |\phi_2\rangle\langle\phi_2|\hat{U}_S(t_2, t_1)|\phi_1(t_1)\rangle.$$

- continuing till time t_n,

$$P(\{|\phi_i\rangle, t_i\}) = |\prod_{i=1}^{n}\langle\phi_i|\hat{U}_S(t_i, t_{i-1})|\phi_{i-1}\rangle|^2.$$

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Quantum Zeno effect (watchdog effect)

- consider a time-independent hamiltonian, $\hat{U}_S(t_i, t_j) = \exp[-i\hat{H}(t_i - t_j)/\hbar]$.
- let the observation be spaced at equal time intervals, $t_i - t_{i-1} = t/n$.
- the probability that at each time t_i the system is observed in its initial state $|\phi_0\rangle$ is,

$$P(\{|\phi_0\rangle, t_i\}) = |\langle\phi_0|\exp[-i\hat{H}t/n\hbar]|\phi_0\rangle|^{2n}.$$

- let $t/n \ll 1$,

$$|\langle\phi_0|\exp[-i\hat{H}t/n\hbar]|\phi_0\rangle|^2 \approx 1 - (\frac{t}{n\hbar})^2 \Delta\hat{H}^2,$$

where $\Delta\hat{H}^2 = \langle\phi_0|\hat{H}^2|\phi_0\rangle - \langle\phi_0|\hat{H}|\phi_0\rangle^2$.

Quantum Zeno effect (watchdog effect)

- the joint probability for n equally spaced observations becomes,

$$P(\{|\phi_0\rangle, t_i\}) = [1 - (\frac{t}{n\hbar})^2 \Delta \hat{H}^2]^n.$$

- for *unobserved in between*, the probability is,

$$P(\{|\phi_0\rangle, t\}) = 1 - (\frac{t^2}{\hbar^2}) \Delta \hat{H}^2.$$

- the probability of finding the system in its initial state at a given time is **increased** if it is observed repeatedly at intermediate times.
- for $n \gg 1$,

$$P(\{|\phi_0\rangle, t_i\}) = [1 - (\frac{t}{n\hbar})^2 \Delta \hat{H}^2]^n \approx \text{exp}[-t^2 \Delta \hat{H}^2 / n\hbar^2],$$

the system under observation does not evolve.

- this effect was invoked to predict the inhibition of decay of an unstable system.

Time-dependent perturbation theory

- with the interaction picture, $\hat{H} = \hat{H}_0 + \hat{H}_1$.
- the state, $\Psi(r,t) = \sum_n C_n(t) u_n(r) e^{-i\omega_n t}$ with the energy eigenvalue $\hat{H}_0 u_n(r) = \hbar\omega_n u_n(r)$.
- the wavefunction has the initial value, $\Psi(r,0) = u_i(r)$, i.e. $C_i(0) = 1, C_{n\neq i} = 0$.
- the equation of motion for the probability amplitude $C_n(t)$ is,

$$\begin{aligned}\dot{C}_n(t) &= -\frac{i}{\hbar}\sum_m \langle n|\hat{H}_1|m\rangle e^{i\omega_{nm}t} C_m(t),\\ &\approx \dot{C}_n^{(1)}(t) = -i\hbar^{-1}\langle n|\hat{H}_1|i\rangle e^{i\omega_{ni}t}.\end{aligned}$$

- if $\hat{H}_1 = V_0$ time independent, we have

$$C_n(t) \approx C_n^{(1)}(t) = -i\hbar^{-1}\langle n|\hat{H}_1|i\rangle \frac{e^{i\omega_{ni}t} - 1}{i\omega_{ni}} = -i\hbar^{-1}\langle n|\hat{H}_1|i\rangle e^{i\omega_{ni}t/2}\frac{\sin(\omega_{ni}t/2)}{\omega_{ni}/2}$$

Ch. 3 in *"Elements of Quantum Optics,"* by P. Meystre and M. Sargent III.

Ch. 5 in *"Modern Quantum Mechanics,"* by J. Sakurai.

Rotational-Wave Approximation

- if $\hat{H}_1 = V_0 \cos \nu t$, we have

$$C_n(t) \approx C_n{}^{(1)}(t) = -i\frac{V_{ni}}{2\hbar}[\frac{e^{i(\omega_{ni}+\nu)t}-1}{i(\omega_{ni}+\nu)} + \frac{e^{i(\omega_{ni}-\nu)t}-1}{i(\omega_{ni}-\nu)}],$$

  where $V_{ni} = \langle n|\hat{H}_1|i\rangle$.

- if near resonance $\omega_{ni} \approx \nu$, we can neglect the terms with $\omega_{ni} + \nu$. This is called the **rotational-wave approximation**.

- making the rotational-wave approximation,

$$|C_n^{(1)}|^2 = \frac{|V_{ni}|^2}{4\hbar^2}\frac{\sin^2[(\omega_{ni}-\nu)t/2]}{(\omega_{ni}-\nu)^2/4}.$$

- we have the same transition probability as the dc case, provided we substitute $\omega_{ni} - \nu$ for ω_{ni}.

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Fermi-Golden rule

- the total transition probability from an initial state to the final state is,

$$P_T \approx \int \mathsf{D}(\omega)|C_n^{(1)}|^2 \mathrm{d}\omega,$$

where $\mathsf{D}(\omega)$ is the density of state factor.

- Fermi-Golden rule,

$$P_T = \int \mathrm{d}\omega \mathsf{D}(\omega) \frac{|V(\omega)|^2}{4\hbar^2} t^2 \frac{\sin^2[(\omega_{ni}-\nu)t/2]}{[(\omega_{ni}-\nu)t/2]^2}.$$

- consider resonance condition $\omega = \nu$,

$$\begin{aligned} P_T &\approx \mathsf{D}(\nu)\frac{|V(\nu)|^2}{4\hbar^2} t^2 \int \mathrm{d}\omega \frac{\sin^2[(\omega_{ni}-\nu)t/2]}{[(\omega_{ni}-\nu)t/2]^2}, \\ &= \frac{\pi}{2\hbar^2}\mathsf{D}(\nu)|V(\nu)|^2 t. \end{aligned}$$

- the transition rate, $\Gamma = \frac{\mathrm{d}P_T}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}|C_n^{(1)}|^2 = \frac{\pi}{2\hbar^2}\mathsf{D}(\nu)|V(\nu)|^2$, which is a constant in time.

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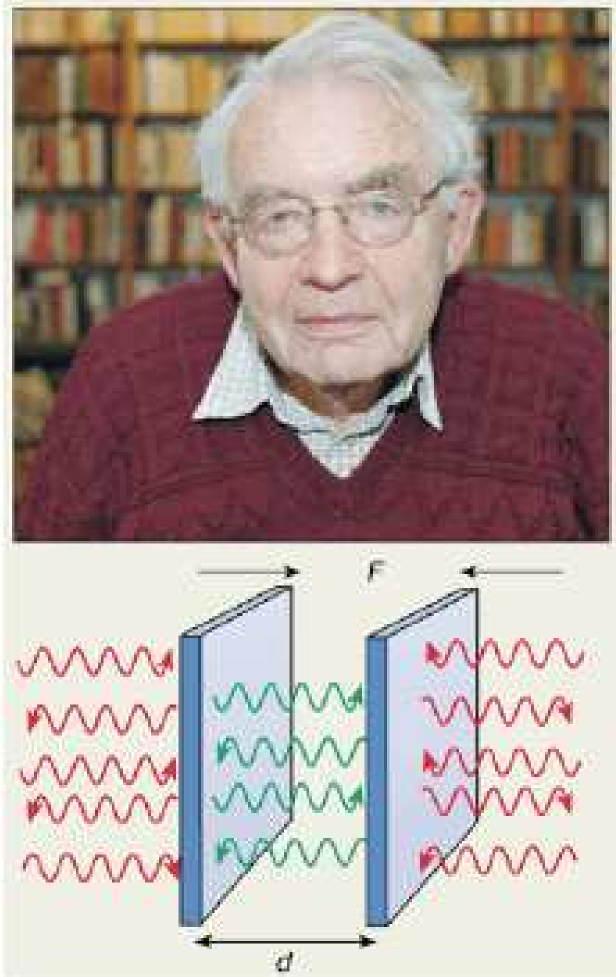
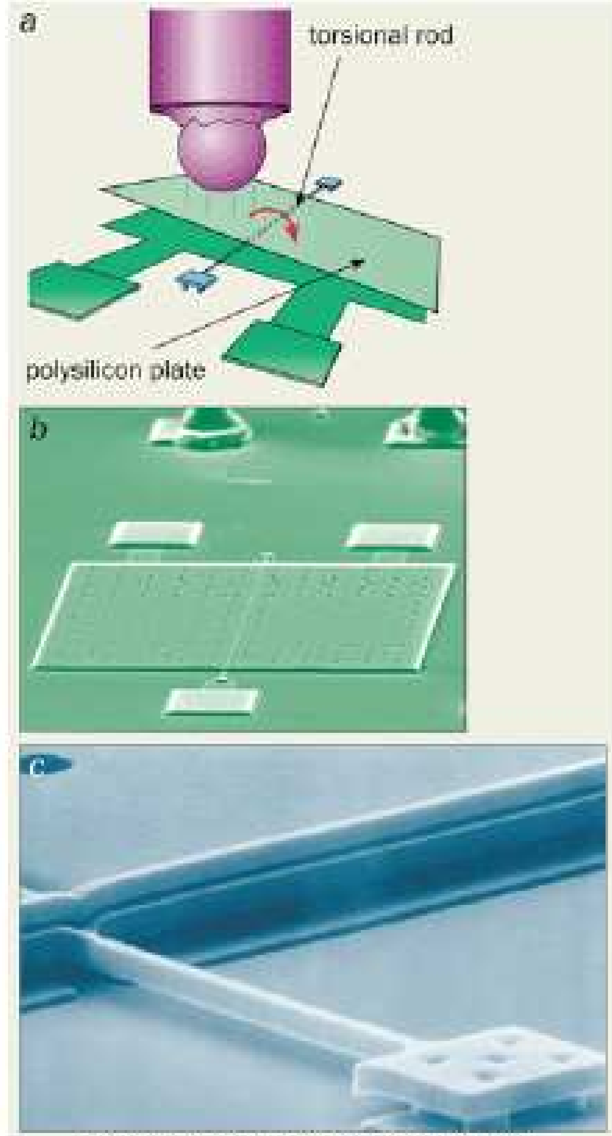
Casimir effect

Hendrik Casimir (1909-2000)



there is a force between two metal slabs if brought in close vicinity

important for micromechanical devices (MEMS)



http://physicsweb.org/articles/world/15/9/6

force is due to vacuum fluctuations of the electromagnetic field

S. K. Lamoreaux, "Demonstration of the Casimir Force in the 0.6 to 6 μm Range" *Phys. Rev. Lett.* 78, 5–8 (1997)