HYBRID DISCOUNTINIOS GALERKIN METHODS FOR ANISOTROPIC DIFFUSION EQUATION

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Abstract

We examine two Hybrid Discontinuous Galerkin methods for anisotropic diffusion equation and perform a priori and posteriori error analysis with rectangular mesh. A posteriori error analysis supports a tendency of a priori error analysis.

Key Words: *anisotropic diffusion equation, hybrid discontinuous galerkin method, error analysis*

1 Introduction

$$-\nabla \cdot \mathrm{A(x)}\nabla \mathrm{u(x)} = \mathrm{f(x)}\ in\ x \in \Omega, u(x) = 0\ on\ x\ \partial\Omega \tag{1}$$

$$\mathrm{A(x)} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} A_{\parallel}(x) & 0 \\ 0 & A_{\perp}(x) \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Equation (1) represents anisotropic diffusion phenomenon in such as plasma physics. In this equation, diffusion coefficient has different values toward vector field F $(\mathrm{A}_{\parallel}, A_{\perp}(= \beta\mathrm{A}_{\parallel}))$. We developed plasma simulation code using this equation. In next work, we consider to introduce n-tree adaptive mesh technique. Hybrid Discontinuous Galerkin (HDG) method can directly be applied to any mesh with floating node such as n-tree mesh. For this reason, we investigate HDG method for anisotropic diffusion equation.

2 Hybrid Discontinuous Galerkin Method

For precise analysis, we define function spaces as follows. We introduce a mesh $\mathrm{K_h} = \{K\}$, where, $\mathrm{h_K} = \mathrm{diam\ K}$, $\mathrm{h} = max_{K \in K_h}$, $\mathrm{E_h} = \{e \in \partial K \mid K \in K_h\}$. A broken Sobolev space is introduced as $\mathrm{H^k}(K_h) = \{v \in L^2(\Omega) : v|_K \in H^k(K)\}$. The so-called skeleton is defined as $\Gamma_\mathrm{h} \coloneqq \bigcup_{e \in E_h} e$. We set $\mathrm{V} \coloneqq \mathrm{H}^2(K_h) \times L_D^2(\Gamma_h)$. The inner products are defined by $(\mathrm{u,v})_{K_h} = \sum_{K \in K_h} \int_K uv\, dx\ \ (\forall u, v \in L^2(\Omega))$ and $\langle \hat{\mathrm{u}}, \hat{\mathrm{v}} \rangle_{\partial K} = \sum_{K \in K_h} \int_{\partial K} \hat{u}\hat{v}\, dx\ \ (\forall \hat{u}, \hat{v} \in L^2(\Gamma_\mathrm{h}))$. Polynomial function space on each meshes are $\mathrm{P^k}(K_h)$, $\mathrm{P^k}(E_h)$. Furthermore, $\mathrm{V_h^{\mathit{k}}} \coloneqq P^k(K_h)$, $\mathrm{V_h^{\mathit{k}}} \coloneqq P^l(E_h) \cap L_D^2(\Gamma_h)$, $\mathrm{V_h^{k,l}} \coloneqq V_h^k \times V_h^l$.

We consider the following HDG equation:

$$\text{Find } \mathrm{B_h}(\vec{u}_h,\vec{v}_h) = (f, v_h)\ \forall \vec{v}_h \in V_h^k \qquad (2)$$

$$B_h(\vec{u}_h,\vec{v}_h) = (A\nabla u_h, \nabla v_h)_{K_h} + \langle (A\nabla u_h)_n, \hat{v}_h - v_h\rangle_{\partial K_h} + \langle (A\nabla v_h)_n, \hat{u}_h - u_h\rangle_{\partial K_h} + \tau\langle \Lambda(\hat{u}_h - u_h), \hat{v}_h - v_h\rangle_{\partial K_h}$$

We introduce following HDG norm. C_1, C_2, C_3 are positive constants.

$$\left|\left|\left|u\right|\right|\right|_{HDG} := \sum_{K\in K_h}\left\{\left|\left|\nabla u\right|\right|_K^2 + 1/h \left|\left|u - u_F\right|\right|_{\partial K}^2 + h\left|\left|\nabla u\right|\right|_{\partial K}^2\right\} \qquad (3)$$

(Coercivity) $$|\mathrm{B_h}(\vec{u}_h,\vec{u}_h)| \ge C_1 \left|\left|\left|\vec{u}_h\right|\right|\right|_{HDG}^2 \qquad (4)$$

(Boundness) $$\mathrm{B_h}(\vec{u}_h,\vec{v}_h) \le C_2 \left|\left|\left|\vec{u}_h\right|\right|\right|_{HDG} \left|\left|\left|\vec{v}_h\right|\right|\right|_{HDG} \qquad (5)$$

(Consistency) $$\mathrm{B_h}(\vec{u},\vec{v}_h) = (f, v_h)\ \ \left(= B_h(\vec{u}_h,\vec{v}_h)\right) \qquad (6)$$

(Symmetry) $$\mathrm{B_h}(\vec{u}_h,\vec{v}_h) = B_h(\vec{v}_h,\vec{u}_h) \qquad (7)$$

From coercivity and boundness, there exists a unique weak solution of (2). Furthermore, for consistency and symmetry, L2 error estimate is shown by Aubin nitsch's trick.

(L2 norm error estimate) $$\left|\left|u - u_h\right|\right|_{L^2} \le C_3 h^{k+1} \qquad (8)$$

3 Numerical Results

We consider k=1 and rectangular mesh, where the calculation region is $[0,1]^2$. Let a vector field F satisfies the relation $\tan\theta = F_y/F_x = (2x-1)/(2y-1), A_{\parallel} = 1$. We suppose $\mathrm{u(x,y)} = \sin(\pi x)\sin(\pi y)$ and derive a source term f. We calculate u_h by HDG method using derived A(x) and f. Table 1 shows L2 error. Therein, CG is continuous Galerkin method and C in HDG_C_* means continuous Skelton and D in HDG_D_* means discontinuous Skelton. HDG_*_A2 means case of $\Lambda = A_n \cdot A_n$ and HDG_*_I means case of $\Lambda = 1$, respectity.

Table 1 error $\left|\left|u - u_h\right|\right|_{L2}$

β = 1e-3		τ = 10^8			
h	CG	HDG_C_I	HDG_C_A2	HDG_D_I	HDG_D_A2
0.333333	5.23E-01	7.42E-01	7.42E-01	7.51E-01	7.51E-01
0.142857	1.68E-01	2.08E-01	2.08E-01	2.08E-01	2.08E-01
0.066666	5.84E-02	5.23E-02	5.23E-02	5.21E-02	5.21E-02
0.032258	1.92E-02	1.36E-02	1.36E-02	1.36E-02	1.36E-02
0.015873	5.96E-03	3.61E-03	3.61E-03	3.61E-03	3.61E-03
0.007874	1.74E-03	9.56E-04	9.56E-04	9.55E-04	9.55E-04
0.003922	4.83E-04	2.52E-04	2.52E-04	2.50E-04	2.50E-04
Order	1.99E+00	1.84E+00	1.84E+00	1.84E+00	1.84E+00

4 Conclusions

We performed a priori and a posteriori error analysis with rectangular mesh. A posteriori error analysis supports a tendency of a priori error analysis**.** In the future work, we will study HDG method for anisotropic diffusion problem on rectangular adaptive mesh with floating node.

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