The 18th International Conference on Finite Elements in Flow Problems (FEF2015)
16-18 March 2015, Taipei, Taiwan

GEOMETRICALLY CONSISTENT METHODS BASED ON VORTICITY-VELOCITY COUPLING

Maxim A. Olshanskii[1*] **and Leo Rebholz**[2]

[1] Department of Mathematics,
University of Houston
Houston, TX 77004, USA
molshan@math.uh.edu
[2] Department of Mathematical Sciences
Clemson University
Clemson, SC 29634, USA
rebholz@clemson.edu

Key words: *Incompressible Navier-Stokes equations, vorticity, multiscale, conservation laws.*

Fluid flow vorticity is an important dynamic variable, and many physical phenomena can be described in terms of vorticity more readily than in terms of primitive variables. Vorticity plays a fundamental role in understanding the physics of laminar, transitional and turbulent flows, in mathematical analysis of fluid equations, and in computational fluid dynamics in general.

The advantages of using the vorticity equation for numerical simulations include the following: it allows access to the physically relevant variables of vortex dominated flows, boundary conditions can be easier to implement in external flows, and it provides a unified framework for treating flows in inertial and non-inertial frames of reference. Recently, it was also shown that vorticity equation may be efficiently exploited to model subgrid scales dynamics in a variational multiscale approach [1] and that vorticity computed from the adjoint equation rather than a curl of a Galerkin velocity approximation is a suitable variable for correct helicity and vorticity balances [2, 3]. Hence, for the expense of solving for velocity and vorticity in a coupled fashion one may benefit from computing numerical solutions that are more physically relevant and consistent to the geometry of the Navier-Stokes equations.

In this presentation, we shall discuss Galerkin methods based on the vorticity equation

$$\frac{\partial \mathbf{w}}{\partial t} - \nu\Delta\mathbf{w} + (\mathbf{u}\cdot\nabla)\mathbf{w} - (\mathbf{w}\cdot\nabla)\mathbf{u} = \nabla\times\mathbf{f}, \tag{1}$$

coupled with the momentum equation through the Lamb vector

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + \mathbb{D}(\mathbf{u})\mathbf{u} + \dfrac{1}{2}\mathbf{w}\times\mathbf{u} + \nabla p = \mathbf{f} \\ \qquad\qquad\qquad\qquad\qquad\qquad \mathrm{div}\mathbf{u} = 0, \end{cases} \tag{2}$$

with $\mathbb{D}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + [\nabla\mathbf{u}]^T)$. We shall consider a new vorticity boundary condition for solid walls and discuss conservation properties of a finite element method based on (1)-(2). Numerical examples will be given to illustrate the accuracy gained by solving the coupled system comparing to more traditional methods in pressure-velocity variables. Following [4, 5], we shall show that velocity and vorticity equations in (1)-(2) can be efficiently decoupled on every time step of numerical integration, leading to stable and fast time-stepping method.

Another way of explicit inclusion of the Lamb vector in the momentum equation is based on the identity:

$$(\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{w}\times\mathbf{u} + \nabla P,$$

where $P = \frac{1}{2}\mathbf{u}\cdot\mathbf{u} + p$ is the Bernoulli pressure. Likewise, the vorticity equation can be reformulated using

$$(\mathbf{u}\cdot\nabla)\mathbf{w} - (\mathbf{w}\cdot\nabla)\mathbf{u} = 2\mathbb{D}(\mathbf{w})\,\mathbf{u} - \nabla\eta,$$

with the helical density $\eta := \mathbf{u}\cdot\mathbf{w}$. The latter formulation enables one to solve directly for helical density and to enforce the divergence-free of vorticity in a numerical method explicitly [4]. We shall show results for these alternative formulations as well.

The support of the U.S. Army Research Office through the project No 65294-MA is acknowledged.

REFERENCES

[1] Bensow RE, Larson MG (2010) Residual based VMS subgrid modeling for vortex flows, *Comput. Methods Appl. Mech. Engrg.* **199** 802-809.

[2] Olshanskii MA and Rebholz LG (2010) A note on helicity balance of the Galerkin method for the 3D Navier-Stokes equations, *Comp. Meth. Appl. Mech. Eng.* **199** 1032-1035.

[3] Evans JA, Hughes TJR (2013) Isogeometric divergence-conforming B-splines for the unsteady Navier-Stokes equations, *J. Comput. Phys.*, **241** 141-167.

[4] Olshanskii MA, Rebholz LG (2010) Velocity-Vorticity-Helicity formulation and a solver for the Navier-Stokes equations, *J. Comput. Phys.*, **229**, 4291-4303.

[5] Benzi M, Olshanskii MA, Rebholz L, Wang Z (2012) Assessment of a vorticity based solver for the Navier-Stokes equations, *Comput. Methods Appl. Mech. Engrg.*, **247**, 216-225.