Brief Review to “Ultrafast Optics”

Foreword: This review intends to provide the fundamental background necessary for this selective course, especially for those who have never heard of the term “Ultrafast Optics”. Another purpose is to quickly recall your memory if you have taken IPT 543000-Ultrafast Optics offered by Prof. S.-D. Yang. However, I’m assuming you are still familiar with Maxwell’s equations and basic elements of a laser (gain medium, population inversion, resonant cavity). If not, please review them on your own.

- What is “Ultrafast”?

The time-scales

<table>
<thead>
<tr>
<th>Time (s)</th>
<th>Milli-</th>
<th>Micro-</th>
<th>Nano-</th>
<th>Pico-</th>
<th>Femto-</th>
<th>Atto-</th>
<th>Zepto-</th>
<th>Yocto-</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>1 kHz</td>
<td>1 MHz</td>
<td>1 GHz</td>
<td>1 THz</td>
<td>1 PHz</td>
<td>1 EHz</td>
<td>1 ZHz</td>
<td>1 YHz</td>
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Let’s begin with the time scale definitions:

(Just for fun, why don’t you find out the prefixes of the rest scales?)

Ok, now we know how time scales, let’s look at a pulse train in the real world as shown in the figure below: by what standard do we distinguish it as ultrafast, fast, or even slow? Typical laser pulse periods T (inverse of the repetition frequency, 1/f_{rep}) fall within the range of several ms to hundreds of ps. Is that ultrafast? Looking closely on a single “pulse”, the carrier frequency is very very fast!!! So is it the “ultrafast” we are talking about? Or is it the duration of the pulse envelope that we are considering?

(By the way, are you familiar with the terms “carrier” and “envelope”?)

Chen-Bin Huang
(In determining pulse durations, pulse intensities are used rather than fields, why?)

A pulse train and a closer look of a single pulse.

- **Pulse train and Fourier transform**

Now, let’s see how to generate an optical pulse train. It is conventional to express the time-domain electric-field as

\[
e(t) = \text{Re}\{a(t)\exp[j\omega_0 t]\}
\]

(1)

where \(a(t)\) represents the complex envelope function and \(\omega_0\) for the carrier frequency. The benefit of taking the carrier out is so that the envelope function lies in the baseband, much easier to work with. Using a simple example, we consider a periodic envelope function comprised of a lot of time harmonics with base frequency \(f_{\text{rep}}\), corresponding amplitudes \(A_m\), and phases \(\phi_m\)

\[
a(t) = \sum_{m=-\infty}^{\infty} A_m \exp[j(m2\pi f_{\text{rep}}t) + \phi_m(t)].
\]

(2)

Note that we don’t have to worry about the real carrier frequency (center wavelength) and the formulation above remains valid. For a well-defined pulse train to persist, the phase relations between different harmonics must remain fixed. This is termed
“mode-locking” that you must have heard of.

(Question: What does the term “coherence” mean? From Eq. 2, can you provide a realistic argument for coherent t and non-coherent fields?)

So why is mode-locking so important? Let’s look at a simple case: suppose we have 21 frequency components, each separated by $f_{\text{rep}}=1$ GHz and having equal amplitude. When the phases are random (but still time-invariant), we see the envelope is really just trashy in figure (a) below. But when the phases are all equal (transform-limited case), wow, what a dramatic difference! Note, however, phase randomness does not alter the periodicity of the pulse envelope.

![Diagram of time-domain intensity](https://via.placeholder.com/150)

Time-domain intensity of 21 equi-amplitude frequency components with (a) random spectral phases and (b) flat phase.

Analytically we may obtain the periodic envelope function for a transform-limited pulse envelope with N frequency components of equal amplitude as
\[
a_{mL}(t) \propto \frac{\sin[N\omega_{rep}t]}{\sin[\omega_{rep}t/2]}.
\]

We can see that at \(t=0\) and multiples of pulse period \(T=1/f_{rep}\), the field values are enhanced by \(N\) times. At these temporal positions, the pulse intensities \(I(t) \propto |a(t)|^2\) are enhanced by \(N^2\)! To this end, you see that optical pulse train can be derived through mode-locking. The narrower the pulses, the higher the pulse peak intensity. This is the ultimate target and the initial purpose of ultrafast optics.

*(So, have you ever thought about how to generate a single pulse in the real world and what are the consequences?)*

Recall from your memory, suppose we are given a periodic function such as \(a(t)\) considered above, can we inversely obtain the amplitudes \(A_m\) ? Of course we can, otherwise we don’t need to be here. The frequency amplitudes can be obtained from the time-domain as

\[
A_m = \frac{1}{T} \int_0^T a(t) \exp[-jm2\pi f_{rep}t] dt
\]

In the case of aperiodic envelope function, we may assume the period goes to infinity, this results in the much implemented Fourier transformation within this course

\[
A(\omega) = \int_{-\infty}^{\infty} a(t)e^{-j\omega t} dt
\]

\[
a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\omega)e^{j\omega t} d\omega
\]

where \(A(\omega)\) is known as the Fourier transform of \(a(t)\), and \(a(t)\) is obtained by performing inverse Fourier transform of \(A(\omega)\).

Some convenient Fourier-transform relationships: verify these identities on your
own:

1. Reality condition: if \( f(t) \) is real, then \( F(-\omega) = F^*(\omega) \).

2. If \( f(t) \) is real AND \( f(-t) = f(t) \), then \( F(\omega) = F^*(\omega) \).

3. Scaling formula: if \( h(t) = f(at) \), then \( H(\omega) = \frac{1}{|a|} F\left(\frac{\omega}{a}\right) \).

4. Time delay formula: if \( h(t) = f(t-\tau) \), then \( H(\omega) = F(\omega) \exp[-j\omega\tau] \).

5. Frequency offset relation: if \( h(t) = f(t) \exp[j\omega_0t] \), then \( H(\omega) = F(\omega - \omega_0) \).

6. The convolution relation: if \( h(t) = f(t) \otimes g(t) \equiv \int f(\tau) g(t-\tau) d\tau \), then \( H(\omega) = F(\omega) G(\omega) \).

7. Parseval’s theorem: time-integrated intensity is equal to frequency-integrated intensity: \( \int_a(t) a^*(t) dt = \frac{1}{2\pi} \int A(\omega) A^*(\omega) d\omega \). For this reason, the quantity \( |A(\omega)|^2 = A(\omega) A^*(\omega) \) is referred as the power spectral density.

8. Delta function: if \( f(t) = \delta(t) \), by definition \( \int_{-\infty}^{\infty} \delta(t) dt = 1 \), we see that \( F(\omega) = 1 \).

9. Time harmonic: if \( f(t) = e^{j\omega_0 t} \), then \( F(\omega) = 2\pi \delta(\omega - \omega_0) \).

10. **Gaussian**: if \( a(t) = \exp[-\frac{t^2}{t_p^2}] \), then \( A(\omega) = t_p \sqrt{\pi} \exp[-\frac{\omega^2 t_p^2}{4}] \) using the very useful identity \( \int_{-\infty}^{\infty} e^{-u^2} du = \sqrt{\pi} \) and after completing the square.

11. **Derivative properties**: \( \frac{\partial}{\partial t} \leftrightarrow j\omega \) and \( \frac{\partial^2}{\partial t^2} \leftrightarrow -\omega^2 \); \( \frac{\partial}{\partial \omega} \leftrightarrow -jt \) and \( \frac{\partial^2}{\partial \omega^2} \leftrightarrow -t^2 \). These will be **heavily used** in our later derivations in the second- and third-order nonlinearities.

We frequently express the optical field in terms of the slowly-varying envelope as
\[ e(t) = \Re \{a(t)e^{j\omega_0 t}\} = \frac{1}{2}\{a(t)e^{j\omega t} + a^*(t)e^{-j\omega t}\} \]  \hspace{1cm} (6)

The spectrum \(E(\omega)\) is expressed by Fourier transform of the electric field

\[ E(\omega) = \frac{1}{2}\{A(\omega - \omega_0) + A^*(-\omega - \omega_0)\} \]  \hspace{1cm} (7)

The optical field can be obtained by direct inverse Fourier transform of the double-sided spectrum or from the single-sided spectrum through

\[ e(t) = \Re \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\omega - \omega_0)e^{j\omega t} \, d\omega \right] \]  \hspace{1cm} (8)

● **Mode-locking**

 Basically mode-locking schemes can be divided into two major categories: active or passive. Can you tell the difference?

 For active mode-locking, one typically uses gain/loss control (such as with an intensity modulator) or phase control (with an optical phase modulator).

![Schematic of loss modulation for intensity mode-locking.](image)
Active mode-locking using intensity modulator from a frequency-domain analysis.

Now we examine active mode-locking case where a sinusoidal signal is applied through a phase modulator $e^{i\phi(t)} = e^{i\delta \cos(\omega_m t)}$. This case can be qualitatively understood using the *instantaneous frequency* caused by the phase modulation as

$$\omega_{inst} = \omega_0 + \frac{\partial \phi(t)}{\partial t} = \omega_0 - \delta_m \omega_m \sin(\omega_m t). \tag{9}$$

*Instantaneous frequencies* are zero at time locations for maximum or minimum phase modulations ($t=0, \pm \frac{\pi}{\omega_m}, \pm \frac{2\pi}{\omega_m}, \ldots$), corresponding to up- and down-chirp. Physically, mode-locking happen at one of these two locations due to the band-pass nature of the gain medium.

*(What does Eq. 9 mean? In analogy, how would you express frequency-dependent time delays?)*
For passive mode-locking, there are even more types: such as using \{fast, slow\} saturable absorber, Kerr-lens mode-locking (artificial fast saturable absorber), additive pulse mode-locking,…etc.
• **Field and intensity correlations**

Ultrafast pulses are so short that no existing electronics are capable of resolving them. So the common approach is to measure the ultrashort pulse by itself. A simple setup is shown below. This is termed as “auto-correlation”: auto means the photo-detector detects the resulting interfering fields from two replicas of the same pulse with certain amount of time delay $\tau$. 

![Schematic of fast-saturable absorber mode-locking.](image1)

![Schematic of Kerr-lens mode-locking (from Wikipedia).](image2)
The output field is

\[ e_D(t) = \frac{1}{2} \Re\{a(t)e^{j\omega t} + a(t-\tau)e^{j\omega(t-\tau)}\} \quad (10) \]

and the detected power is

\[ P_{\text{out}}(t) = \langle e_D^2(t) \rangle = \frac{1}{4} \left[ |a(t)|^2 + |a(t-\tau)|^2 + [a(t)a(t-\tau)^* e^{j\omega \tau} + c.c.] \right] \quad (11) \]

since we agree that we are using a slow detector, the acquired signal is time-integrated as

\[ \int P_{\text{out}}(t)dt = \frac{1}{2} \int |a(t)|^2 dt + \frac{1}{4} \int [a(t)a^*(t-\tau)e^{j\omega \tau} + c.c.]dt \quad (12) \]

We define the first-order correlation function as

\[ \Gamma_a(\tau) = \frac{1}{\Delta T} \int_{-\Delta T/2}^{\Delta T/2} dt a(t)a^*(t-\tau) = \langle a(t)a^*(t-\tau) \rangle, \quad (13) \]

so the output power can be expressed as

\[ \langle P_{\text{out}} \rangle = \frac{1}{2} \Gamma_a(0)[1 + G_1(\tau)] = \frac{1}{2} \Gamma_a(0)\left[1 + \frac{\Gamma_a(\tau)}{\Gamma_a(0)} \cos[\omega_0 \tau + \Phi(\tau)]\right] \quad (14) \]

and the resulting electric field auto-correlation trace will show fast fringes within a slowly-varying envelope as function of delay. Typically we define the correlation time \( \tau_c \) as twice the delay for which the maximum value drops by half.
(To refresh your skills using Matlab, try to plot this as a function of delay with an input pulse of 5 ps FWHM duration, centered at 1.55 um.)

(One bigger question, can we characterize optical pulses using this configuration?)

(How is the Fourier transform of correlation different from that of convolution?)

Evidently field auto-correlation is equivalent to measuring the power spectrum of the pulse, and no phase information may be retrieved. We need more complicated means, so let’s try higher-order nonlinear effect: intensity auto-correlation. As the name implies, here we want to perform the auto-correlation between the pulse intensities rather than fields. This can be accomplished by generating the second harmonic of the input fields via second order optical nonlinearity. A schematic sketch for a non-collinear auto-correlator is shown below. The second harmonic field is given by

\[
e_{\text{SHG}}(t) \propto \text{Re}\{e_{\text{out}}^2(t)\} \\
\propto \text{Re}\{[a^2(t) + a^2(t-\tau)e^{-j\omega_0\tau} + 2a(t)a(t-\tau)e^{-j\omega_0\tau}]e^{2j\omega_0\tau}\}.
\]

Again we are only capable of detecting the time averaged power of the second-harmonic as

\[
\langle P_{\text{SHG}}(t) \rangle = \langle |e_{\text{SHG}}(t)|^2 \rangle \propto \langle |a(t)|^4 \rangle \left(1 + \frac{2\langle |a(t)|^2|a(t-\tau)|^2 \rangle}{\langle |a(t)|^4 \rangle} \right).
\]

We now define the normalized intensity auto-correlation function

\[
G_2(\tau) = \frac{\langle I(t)I(t-\tau) \rangle}{\langle |I(t)|^2 \rangle}.
\]
Intensity autocorrelation traces are even functions of delay, so no phase information can be uniquely retrieved. Also, since they are always symmetric to delay, one cannot tell the pulse asymmetry from it.

It is nice to learn the difference between the intensity autocorrelation traces of a pulse and continuous noise. Shown below, we can see that the background of noise autocorrelation trace never drops to zero, but still shows a peak at zero delay. This peak is due to the correlation time, which can be as short as the inverse of the bandwidth. This width has nothing to do with the actual smooth-profiled pulse width, but only to the substructure in the intensity.
(Try to derive the above equation on your own, it's very helpful)
(What is the Fourier transform of intensity auto-correlation?)

• Dispersion

We review what happens to a pulse propagating in a dispersive material (where the refractive index is frequency dependent). We often express the frequency-dependent phase as a function of the frequency-dependent propagation constant and propagation length \( \psi(\omega) = -\beta(\omega)z \). For the input pulse \( e_{in}(t) \) with spectrum \( E_{in}(\omega) \), we have the output pulse as

\[
e_{out}(z,t) = \frac{1}{2\pi} \int d\omega E_{in}(\omega) e^{j\omega t} e^{j\psi(\omega)} = \text{Re}\left\{ \frac{e^{j\omega t}}{2\pi} \int d\tilde{\omega} A_{in}(\tilde{\omega}) e^{j\tilde{\omega}t} e^{-j\beta(\omega)z} \right\}, \tag{18}
\]

where \( \tilde{\omega} = \omega - \omega_0 \).

We now express the propagation constant in its Taylor series near the carrier frequency as

\[
\beta(\omega) = \beta_0 + \beta_1 \tilde{\omega} + \frac{\beta_2}{2!} \tilde{\omega}^2 + \frac{\beta_3}{3!} \tilde{\omega}^3 + \ldots \tag{19}
\]

where \( \beta_n = \frac{\partial^n}{\partial \omega^n} \beta(\omega) \bigg|_{\omega=\omega_0} \). The consequence of dispersion is now evident when we express the output pulse as

\[
e_{out}(z,t) = \text{Re}\left\{ \frac{e^{j(\omega_0 t - \beta_0 z)}}{2\pi} \int d\tilde{\omega} A_{in}(\tilde{\omega}) e^{j[\tilde{\omega}t - (\beta_0 \tilde{\omega} + \frac{1}{2!} \beta_1 \tilde{\omega}^2 + \ldots)z]} \right\} = \text{Re}\{ e^{j(\omega_0 t - \beta_0 z)} a_{out}(z,t) \} \tag{20}
\]

and the output pulse is the product of the carrier term and the envelope function \( a_{out}(z,t) \). The carrier propagates at the phase velocity of \( v_p = \omega_0 / \beta_0 \), which is unaffected by the variation of \( \beta \) or \( \psi \) with \( \omega \).

We first look at the case where the propagation constant is only up to the first order (linear in frequency), that is, \( \beta(\omega) = \beta_0 + \beta_1 \tilde{\omega} \). We see that the output envelope is...
simply a delayed version of the input

\[ a_{\text{out}}(z,t) = a_{\text{in}}(t - \beta_1 z), \quad (21) \]

and the envelope function travels at group velocity (the velocity of energy transport) of

\[ v_g = \frac{1}{\beta_1} = \left( \frac{\partial \beta}{\partial \omega} \right)_{\omega = \omega_0}^{-1}. \quad (22) \]

For \( \beta(\omega) \) of higher order, the output envelope function will be distorted.

*Why don’t you try to simulate how a 200 fs transform-limited Gaussian input pulse distorts after propagation in a single-mode fiber of 1m, 5m and 50m? For SMF, use \(|\beta_2|=20 \text{ ps}^2/\text{km}\) (Also try to see what happens when non-negligible cubic spectral phase is impressed onto the spectrum?)*