Third-Order Ultrafast Nonlinear Optics

Foreword: We now consider the nonlinear optics of ultrashort pulses in media in which the refractive index depends on the pulse temporal intensity. Some of the materials were introduced briefly in section describing mode locking of solid-state lasers. Note that because ultrafast nonlinearities are best incorporated into the wave equation in the time domain, while dispersive effects are most easily included in the frequency domain, derivations of the nonlinear propagation equations are rather intricate. For details on the mathematical derivations, you are directed to these two references[1,2].

• Nonlinear wave propagation equation

We start from the basic wave equation for isotropic, nonmagnetic, source-free medium derived from Maxwell's Equations:

$$\nabla\times\nabla\times\mathbf{E}=\nabla(\nabla\bullet\mathbf{E})-\nabla^2\mathbf{E}=-\mu_0\frac{\partial^2\mathbf{D}}{\partial t^2}. \quad (1)$$

We define

$$\mathbf{D}=\varepsilon_0\mathbf{E}+\mathbf{P}_{(1)}+\mathbf{P}_{NL}=\mathbf{D}_{(1)}+\mathbf{P}_{NL}, \quad (2)$$

and $\mathbf{D}_{(1)}=\varepsilon_{(1)}\mathbf{E}=n_{(1)}^2\varepsilon_0\mathbf{E}$.

From $\nabla\bullet\mathbf{D}=0$, we obtain

$$\varepsilon_{(1)}(\nabla\bullet\mathbf{E})=-\mathbf{E}\bullet\nabla\varepsilon_{(1)}-\nabla\bullet\mathbf{P}_{NL}. \quad (3)$$

Here we consider weak nonlinearity and isotropic medium, so Eq. (1) is reduced to a much simplified scalar equation

[1] H. A. Haus, *Waves and Fields in Optoelectronics*, Prentice-Hall, Englewood Cliffs, NJ, 1984.
[2] G. P. Agrawal, *Nonlinear Fiber Optics*, Academic Press, San Diego, CA, 1995.

$$\nabla^2 E - \mu_0 \varepsilon_{(1)} \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2}. \tag{4}$$

In a scalar treatment, we denote

$$D = \varepsilon_0 n^2 E \text{ where } n = n_0 + \delta n_{(1)} + \delta n_{NL}, \tag{5}$$

so that

$$D \approx \varepsilon_0 (n_0^2 + 2n_0 \delta n_{(1)} + 2n_0 \delta n_{NL}) E \approx (\varepsilon_{(1)} + 2\varepsilon_0 n_0 \delta n_{NL}) E. \tag{6}$$

And thus we take the nonlinear polarization as

$$P_{NL} \approx 2\varepsilon_0 n_0 \delta n_{NL} E, \tag{7}$$

δn_{NL} usually assumes the form proportional to the instantaneous optical intensity.

Plane wave propagation in uniform nonlinear media

We define the optical field and nonlinear polarization as

$$E = \mathrm{Re}\{a(z,t) e^{j(\omega_0 t - \beta_0 z)}\} \text{ and } P_{NL} = \mathrm{Re}\{\tilde{P}_{NL}(z,t) e^{j\omega_0 t}\}, \tag{8}$$

where $\beta_0 = \omega_0 n_0 / c$ is the propagation constant for the carrier at ω_0. The Fourier transform of the time-domain envelope function is

$$A(z, \tilde{\omega}) = \int a(z,t) e^{-j\tilde{\omega} t} dt, \tag{9}$$

with $\tilde{\omega} = \omega - \omega_0$.

Now we insert Eq. (8) back into Eq. (4) and invoke slowly varying envelope approximation both in time and space, we obtain

$$\int \frac{d\tilde{\omega}}{2\pi} \left[\frac{\partial A}{\partial z} + \frac{j}{2\beta_0} [\beta^2(\tilde{\omega}) - \beta_0^2] A \right] e^{j\tilde{\omega} t} = \frac{-j\omega_0 \mu_0 c}{2n_0} \tilde{P}_{NL} e^{j\beta_0 z}, \tag{10}$$

where $\beta^2(\tilde{\omega}) = \omega^2 \mu_0 \varepsilon_{(1)}$ and can be expanded using the Taylor series around the carrier frequency and incorporating propagating loss α as

$$\beta(\tilde{\omega}) = \beta_0 + \beta_1 \tilde{\omega} + \frac{\beta_2}{2} \tilde{\omega}^2 - \frac{j\alpha}{2}. \tag{11}$$

Assuming the nonlinear polarization is proportional to the instantaneous intensity ($\delta n_{NL} = n_2 |a|^2$), we arrive at the nonlinear field propagation equation in a uniform media

$$\left\{ \frac{\partial}{\partial z} + \beta_1 \frac{\partial}{\partial t} - \frac{j\beta_2}{2} \frac{\partial^2}{\partial t^2} + \frac{j\omega_0 n_2}{c} |a|^2 + \frac{\alpha}{2} \right\} a(z,t) = 0 . \quad (12)$$

Here $|a|^2$ is normalized to give the optical intensity. This equation, together with its guided-wave sibling discussed below, serves as the point of departure for much of our discussion of nonlinear pulse propagation. Effects resulting from higher order terms that have been ignored in our derivation will be discussed later.

Nonlinear propagation in waveguides

We consider nonlinear propagation for the case of weakly guiding waveguides. A prominent example is the standard step-index glass optical fiber[3], consisting of a central circular core region, on the order of several μm in diameter, surrounded by a lower index cladding region. The index difference between the core and the cladding is around 10^{-3}.

For optical field propagating in a waveguide, we may no longer ignore the transversal derivatives. After all, it is the transversal layer structure enabling the wave guiding phenomenon, so Eq. (4) is now

$$\nabla_T^2 E + \frac{\partial^2 E}{\partial z^2} - \mu_0 \varepsilon_{(1)} \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P_{NL}}{\partial t^2} . \quad (13)$$

For weak nonlinear polarization, we may treat the right hand side as a source term that perturbs the original equation with the right hand side being zero. Now we express the source-less equation in the frequency-domain as

3 C. K. Kao, the pioneer of optical fiber, is awarded with the Noble Prize in Physic in 2009.

$$\nabla_T^2 \tilde{E} + \frac{\partial^2 \tilde{E}}{\partial z^2} + \omega^2 \mu_0 \varepsilon_{(1)}(x, y, \omega) \tilde{E} = 0, \tag{14}$$

note that the spatial and frequency dependence of the dielectric function is now included.

We now define the guided field in the frequency-domain as the multiplication of the transverse spatial function and the propagation constant

$$\tilde{E}(x, y, z, \omega) = u(x, y, \omega) e^{-j\beta(\omega)z}, \tag{15}$$

and Eq. (14) is now reduced to a simpler eigenvalue equation

$$\nabla_T^2 u + (\omega^2 \mu_0 \varepsilon_{(1)} - \beta^2) u = 0. \tag{16}$$

Solving this equation provides the solution to both the transverse spatial mode profile and the corresponding propagation constant as a function of frequency.

Here we adopt a transverse integration procedure to gain further insights into the relation among the propagation constant, mode profile and the index variation[4]:

$$\iint u^* \nabla_T^2 u dxdy + \iint (\omega^2 \mu_0 \varepsilon_{(1)} - \beta^2) |u|^2 dxdy = 0. \tag{17}$$

Using integration by parts, we obtain

$$\iint u^* \nabla_T^2 u dxdy = \iint \nabla_T \bullet (u^* \nabla_T u) dxdy - \iint |\nabla_T u|^2 dxdy. \tag{18}$$

The first term on the right-hand side vanishes using divergence theorem, and the effective propagation constant can be found as

$$\beta^2(\omega) = \frac{\iint [\omega^2 \mu_0 \varepsilon_{(1)} |u(x, y, \omega)|^2 - |\nabla_T u(x, y, \omega)|^2] dxdy}{\iint |u(x, y, \omega)|^2 dxdy}. \tag{19}$$

And we can see that the effective propagation constant is always smaller to the plane wave case due to the transverse spatial confinement. We also note the upper and lower bounds for the propagation constant

$$\omega^2 \mu_0 \varepsilon_{(1)}^{cladding} < \beta^2 < \omega^2 \mu_0 \varepsilon_{(1)}^{core}. \tag{20}$$

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[4] A. W. Snyder and J. D. Love, *Optical Waveguide Theory*, Chapman and Hall, London, 1983.

We now turn back to the time-domain formulation, Eq. (13). Similar to what we performed for plane waves, we now define

$$E = \mathrm{Re}\{a(z,t)e^{j(\omega_0 t-\beta_0 z)}u(x,y)\} \text{ and } P_{NL} = \mathrm{Re}\{\tilde{P}_{NL}(x,y,z,t)e^{j\omega_0 t}\}. \quad (21)$$

So Eq. (13) can be expressed as

$$\int \frac{d\tilde{\omega}}{2\pi}\{\nabla_T^2 - \beta_0^2 - 2j\beta_0\frac{\partial}{\partial z} + \frac{\partial^2}{\partial z^2} + \omega^2\mu_0\varepsilon_{(1)}\}Aue^{j\omega t} = \mu_0\frac{\partial^2 P_{NL}}{\partial t^2}e^{j\beta_0 z}. \quad (22)$$

Invoking slow-varying envelope approximation (*SVEA*)

$$\int \frac{d\tilde{\omega}}{2\pi}\{\frac{\partial A}{\partial z} + j[\beta(\omega)-\beta_0]\}ue^{j\tilde{\omega}t} = \frac{-j\mu_0\omega_0^2}{2\beta_0}\tilde{P}_{NL}e^{j\beta_0 z}. \quad (23)$$

Using the similar Taylor expansion on the propagation constant as before

$$\frac{\partial a}{\partial z} + \beta_1\frac{\partial a}{\partial t} - \frac{j\beta_2}{2}\frac{\partial^2 a}{\partial t^2} + j\kappa|a|^2 a + \frac{\alpha}{2}a = 0, \quad (24a)$$

$$\kappa = \frac{\omega_0^2}{\beta_0 c^2}\frac{\iint n_0 n_2 |u|^4 dxdy}{\iint |u|^2 dxdy}. \quad (24b)$$

To obtain the instantaneous power, we make the definition of

$$|a_p|^2 = |a|^2\iint |u|^2 dxdy. \quad (25)$$

Finally, we arrive at the guided nonlinear propagation equation as

$$\frac{\partial a_p}{\partial z} + \beta_1\frac{\partial a_p}{\partial t} - \frac{j\beta_2}{2}\frac{\partial^2 a_p}{\partial t^2} + j\gamma|a_p|^2 a_p + \frac{\alpha}{2}a_p = 0, \quad (26a)$$

$$\gamma = \frac{\omega_0^2 n_0 n_2}{\beta_0 c^2 A_{eff}} \text{ and } A_{eff} = \frac{\{\iint |u|^2 dxdy\}^2}{\iint |u|^4 dxdy}. \quad (26b)$$

• The non-linear Schrödinger equation (NLSE)

Using the results from our previous section, we discuss some properties of the simplified nonlinear propagation equation. For simplicity, here in this section, *a* is used to express a_p (normalized to give power) in the previous section.

Pulse propagation

By ignoring all the dispersion, nonlinear and loss terms in Eq. (26), the simple pulse propagation is then

$$\frac{\partial a}{\partial z} + \beta_1 \frac{\partial a}{\partial t} = 0 . \qquad (27)$$

The solution to this simple equation is of the form $a(t - z/v_g)$, which denotes the pulse envelope is propagating at a group velocity $v_g = 1/\beta_1$.

Propagation with loss

With all the terms in Eq. (26), if we write $a(z,t) = a'(z,t)e^{-\frac{\alpha z}{2}}$, then Eq. (26) is just

$$\frac{\partial a'}{\partial z} + \beta_1 \frac{\partial a'}{\partial t} - \frac{j\beta_2}{2}\frac{\partial^2 a'}{\partial t^2} + j\gamma e^{-\alpha z}\left|a'\right|^2 a' = 0 . \qquad (28)$$

Without nonlinear term, the dispersion is independent of loss. Loss only reduces the nonlinear strength throughout the length of propagation.

Propagation without loss: NLSE

We now drop the loss term and define two new variables so that we are moving along with the pulse at the group velocity

$$z' = z$$
$$\tau = \frac{t - z/v_g}{T_0} , \qquad (29)$$

where T_0 is an arbitrary characteristic time factor, very often set as the input pulse duration. So now Eq. (26) is reduced to

$$\frac{\partial a}{\partial z'} - \frac{j\beta_2}{2T_0^2}\frac{\partial^2 a}{\partial \tau^2} + j\gamma\left|a\right|^2 a = 0 . \qquad (30)$$

We now define a **characteristic dispersion length** L_D

$$L_D \equiv \frac{T_0^2}{|\beta_2|} \text{ and then normalize } \zeta = \frac{z'}{L_D}, \tag{31}$$

which tells how long the pulse is significantly broadened due to dispersion in the absence of nonlinearity. We also re-normalize the field amplitude as

$$a(z',\tau) = \sqrt{P_c}\, a(z',\tau) \text{ where } \gamma P_c L_D = 1. \tag{32}$$

As we will learn latter, this yields a stationary solution (fundamental soliton).

We arrive at the final ***dimensionless NLSE*** as

$$j\frac{\partial a}{\partial \zeta} = -\frac{\mathrm{sgn}(\beta_2)}{2}\frac{\partial^2 a}{\partial \tau^2} + |a|^2 a\,. \tag{33}$$

This equation is especially useful in describing the fundamental soliton.

Characteristic propagation regimes

For a given input pulse with duration T_0, peak power P_{peak}, and fiber length L, we first define the **characteristic nonlinear length** as

$$L_{NL} \equiv \frac{1}{\gamma P_{peak}}. \tag{34}$$

L_{NL} gives the idea the length where nonlinearity significantly broadens the bandwidth of the power spectrum for a transform-limited input pulse in the absence of dispersion. Together with the definition of L_D, we may now categorize the pulse propagation behaviors into four regimes: (a) $L \ll L_D$ and $L \ll L_{NL}$; (b) $L_D \ll L \ll L_{NL}$; (c) $L_{NL} \ll L \ll L_D$; and (d) $L_D \ll L$ and $L_{NL} \ll L$.

Another, related way to classify the nonlinear propagation problem is based on the peak input power. In particular, we define a dimensionless parameter N, such that

$$N^2 \equiv \frac{L_D}{L_{NL}} = \frac{T_0^2/|\beta_2|}{1/\gamma P_{peak}} = \frac{P_{peak}}{P_c}. \tag{35}$$

N characterizes the relative strengths of the nonlinearity and dispersion: for

N^2>>1, the nonlinearity dominates the initial propagation; while for N^2<<1, dispersion dominates. As long as N remains constant, the relative balance between nonlinearity and dispersion remains unchanged, even when the pulse width, power, and dispersion, etc. are individually varied.

• Self-phase modulation (SPM)

We now consider self-phase modulation (SPM), in which the pulse intensity transiently modifies the refractive index through the optical Kerr effect, which in turn affects the temporal phase of that pulse. We start with the nonlinear propagation equation in the form

$$\frac{\partial a}{\partial z} = \frac{j\beta_2}{2}\frac{\partial^2 a}{\partial t^2} - j\gamma|a|^2 a\,. \tag{36}$$

In the following we consider two distinct cases: the first where dispersion is neglected, and the second where SPM occurs simultaneously with normal dispersion. The case where SPM occurs together with anomalous dispersion is discussed in later section for solitons.

SPM without dispersion

With no dispersion, Eq. (36) is simplified to

$$\frac{\partial a}{\partial z} = -j\gamma|a|^2 a\,, \tag{37}$$

and the solution is easily found as

$$a(z,t) = a(0,t)\exp[-j\gamma|a(0,t)|^2 z]\,, \tag{38}$$

and we can see that the pulse *retains it shape* while accumulating a nonlinear temporal phase as it propagates!

As you have learned in Ultrafast Optics, a nonlinear temporal phase brings about a variation in the instantaneous frequency across the pulse, described by

$$\omega_{inst}(z,t) = \omega_0 + \Delta\omega(z,t), \tag{39a}$$

where

$$\Delta\omega(z,t) = \frac{\partial(\Delta\phi)}{\partial t} = -\frac{\partial}{\partial t}(\gamma|a|^2 z). \tag{39b}$$

Thus, the instantaneous frequency is modulated according to the time derivative of the intensity. The magnitude of the frequency modulation scales with the magnitude of the peak nonlinear phase shift.

Discussions:

1. For most material, γ is positive, so SPM always gives an up-chirp.
2. The optical bandwidth enhancement factor is directly dependent on the maximum phase accumulation (the peak intensity and propagation length). For a Gaussian input pulse, this enhancement factor is $\frac{BW_{SPM}}{BW_{in}} \approx 1.03 \cdot \Delta\phi_{\max}$.
3. The power spectra exhibit peaks at the frequency extremes. This behavior can be understood based on the instantaneous frequency picture. At the frequency extremes, a relatively broad range of times contributes to the same instantaneous frequency, leading to the observed peaks.
4. The power spectra exhibit a deep modulation. This can also be understood based on the instantaneous frequency picture. The amplitude of the power spectrum depends on the interference, hence the relative phase, between the contributions. We can obtain a relation for the number of maxima and minima as

$$\begin{aligned} N_{peak} &\approx \frac{\Delta\phi_{\max}}{\pi} + 1 \\ N_{\min} &\approx \frac{\Delta\phi_{\max}}{\pi} \end{aligned}. \tag{40}$$

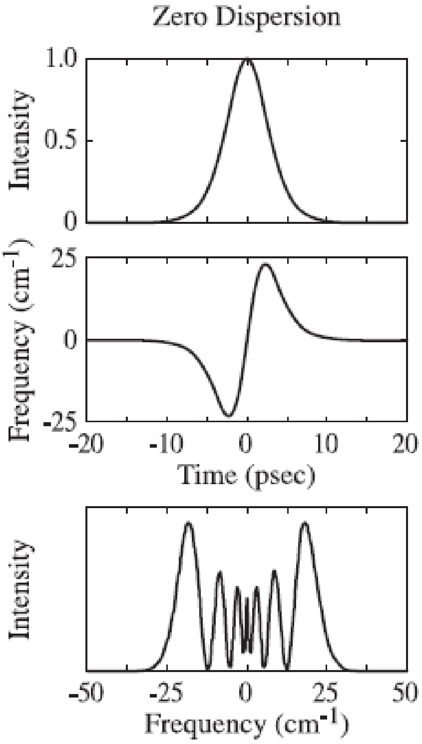


Figure showing the pulse intensity profile, instantaneous frequency and the resulting SPM power spectrum. The pulse generates a peak nonlinear phase accumulation of 6π.

Dispersion-less SPM but with loss

With loss, we can write $a(z,t) = a'(z,t)\exp(-\alpha z/2)$, and the propagation equation is now

$$\frac{\partial a'}{\partial z} = -j\gamma |a'|^2 a' e^{-\alpha z}, \tag{41}$$

and the solution can be easily obtained as

$$a'(z = L,t) = a'(0,t)\exp[\frac{-j\gamma |a'(0,t)|^2}{\alpha}(1 - e^{-\alpha L})]. \tag{42}$$

SPM with normal dispersion

Now we assume the medium has normal dispersion and nonlinearity, but the loss is ignored. We provide some discussion using the following figure. Evolution of temporal intensity profiles and power spectra for *sech(t/to)* input pulse under SPM with normal dispersion, for $N = 5$ to give the same spectrum broadening as compared to the case without dispersion.

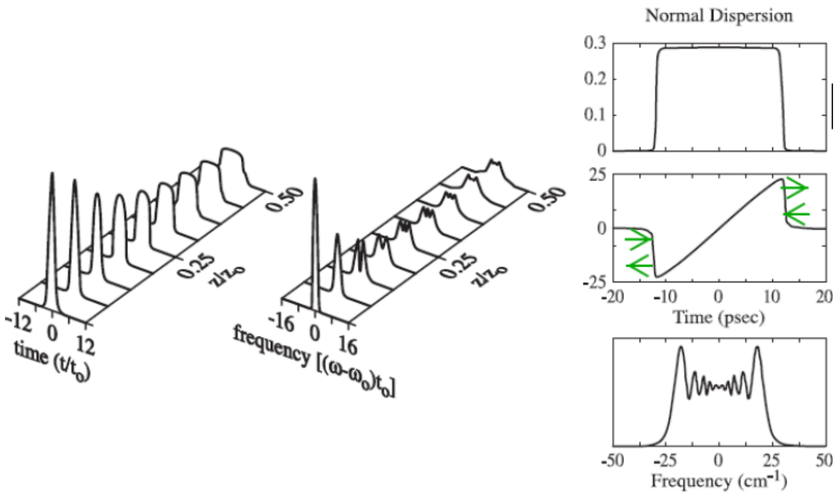


Figure showing the case for normal dispersion plus SPM.

1. Initially, SPM dominates, and the spectrum begins to broaden. However, once the spectrum broadens, dispersion plays an increasingly strong role. As the pulse broadens, its intensity is reduced, and therefore, further spectral broadening through SPM proceeds more and more slowly.
2. Nonlinear propagation eventually produces a nearly flat-topped intensity profile. At the same time, the shape of the chirp becomes remarkably linear. Furthermore, a very large fraction of the energy in the pulse falls within the linear chirp region.
3. The power spectra are substantially smoother, and most of the deep modulation is washed out. This can be explained by the observation that although there are still two different time instants contributing to any particular $\Delta\tilde{\omega}$, the relative strengths of the contributions from the two time instants are now very unbalanced.

Cross-phase modulation (XPM)

Now we assume there are two optical fields with same polarization and collinearly propagating within the fiber. We denote $a(z,t)=a_1(z,t)e^{j\tilde{\omega}_1 t}+a_2(z,t)e^{j\tilde{\omega}_2 t}$ and plug into Eq. (36). After collecting the corresponding terms, we have

$$\begin{aligned}\frac{\partial a_1}{\partial z}&=\frac{j\beta_2}{2}\frac{\partial^2 a_1}{\partial t^2}-j\gamma[|a_1|^2+2|a_2|^2]a_1\\ \frac{\partial a_2}{\partial z}&=\frac{j\beta_2}{2}\frac{\partial^2 a_2}{\partial t^2}-j\gamma[2|a_1|^2+|a_2|^2]a_2\end{aligned}, \tag{43}$$

and we can see that XPM arising from another interacting field gives an extra factor of two in the nonlinear phase. Besides this, other frequency components are generated as well.

• Pulse compression

As we have discussed earlier, SPM along with normal dispersion provides spectral broadening with a smoother spectrum. If one can compensate the nonlinear spectral phase accumulated, there is a great potential for ultrashort pulse generation. Mathematically this can be simply stated as

$$a_{SPM}(t)=\int\frac{d\omega}{2\pi}|A_{SPM}(\omega)|\exp[j\Psi_{SPM}(\omega)]. \tag{44}$$

If somehow we can send the SPM pulses to a pure dispersive delay line so that the output field is

$$a_{SPM}(t)=\int\frac{d\omega}{2\pi}|A_{SPM}(\tilde{\omega})|\exp\{j[\Psi_{SPM}(\omega)+\Psi_{DDL}(\omega)]\}, \tag{45}$$

where ideally we wish to have $\Psi_{DDL}(\omega)=-\Psi_{SPM}(\omega)$ so that we may obtain compressed, bandwidth-limited optical pulse. We shall see many nice experimental demonstrations in class slides.

A very extensive numerical study over the optimum compression issue for *sech(t/t0)* pulses has been performed[5]. The normalized length and the peak power is defined as

$$z_0 = \frac{\pi}{2} L_D = \frac{\pi t_0^2}{2|\beta_2|} ,\quad N^2 = \frac{L_D}{L_{NL}} = \gamma P_p L_D \tag{46}$$

and a simple relation over maximum compression ratio is obtained that serves as experimental guideline

$$\frac{\Delta t_{compress}}{\Delta t_{in}} \approx \frac{1.6}{N} .\quad z_{opt} \approx \frac{1.6 z_0}{N} \approx 2.5\sqrt{L_D L_{NL}} \tag{47}$$

• Modulational instability

We move onto the anomalous dispersion regime ($\beta_2 < 0$, red shifted waves travel slower). We first look into a phenomenon drastically different from the normal dispersion regime called modulational instability. Suppose we have a CW light of $a(0,t) = \sqrt{P_0}$, the solution to the nonlinear propagation equation is then

$$a_0(z,t) = \sqrt{P_0} e^{-j\gamma P_0 z} . \tag{48}$$

Now if there is a small intensity variation (perturbation) to the CW light

$$a(z,t) = [\sqrt{P_0} + \tilde{a}(z,t)] e^{-j\gamma P_0 z} . \tag{49}$$

We can obtain the equation for the perturbed wave as

$$\frac{\partial \tilde{a}}{\partial z} = \frac{j\beta_2}{2} \frac{\partial^2 \tilde{a}}{\partial t^2} - j\gamma P_0 (\tilde{a} + \tilde{a}^*) . \tag{50}$$

A generalized solution to this equation is

5 W. J. Tomlinson, R. H. Stolen, C. V. Shank, JOSAB 1, 139-149 (1984).

$$\tilde{a}(z,t) = A\cos(Kz - \Omega t) + jB\sin(Kz - \Omega t). \tag{51}$$

Substituting Eq. (51) into Eq. (50) we obtain

$$\begin{pmatrix} K & \dfrac{\beta_2\Omega^2}{2} \\ \dfrac{\beta_2\Omega^2}{2} + 2\gamma P_0 & K \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = 0. \tag{52}$$

From det()=0, we arrive at an important relation

$$K = \pm\frac{\beta_2\Omega}{2}\sqrt{\Omega^2 + \frac{4\gamma P_0}{\beta_2}}. \tag{53}$$

We can see that for normal dispersion regime, $\beta_2>0$ makes K real so the perturbation is retained. For anomalous dispersion regime and for high power, however, this gives an imaginary K value and the perturbation grows exponentially!

For anomalous dispersion, we can see that perturbation grows for all frequencies below a gain cutoff Ω_c

$$\Omega_c^2 = \frac{4\gamma P_0}{\beta_2}, \tag{54}$$

and we denote the gain as $g=Im(K)$

$$g_{\max} = \gamma P_0 \text{, which happens at } \Omega_{\max} = \pm\frac{\Omega_c}{\sqrt{2}}. \tag{55}$$

Both the maximum attainable gain and the frequency where the maximum gain occurs grow with power.

• Solitons

Pulse propagation behavior can be drastically different in the anomalous dispersion regime. An especially important finding is the existence of solitons, where a pulse retains it's shape during propagation due to the cancellation between SPM and dispersion.

We begin with the dimensionless NLSE, and since β_2 is negative, we have

$$j\frac{\partial a}{\partial \zeta} = \frac{1}{2}\frac{\partial^2 a}{\partial \tau^2} + |a|^2 a\ . \tag{56}$$

A simple solution to this dimensionless equation is of the form

$$a(\zeta,\tau) = \text{sech}(\tau)\exp[-\frac{j\zeta}{2}]\,, \tag{57}$$

and the solution in real units is given as

$$a(z,t) = \sqrt{P_c}\,\text{sech}(\frac{t}{T_0})\exp[\frac{-j|\beta_2|z}{2T_0^2}]\,. \tag{58}$$

Comments:

1. The definition of $P_c = \frac{|\beta_2|}{\gamma T_0^2}$ provides the formation of fundamental solitons (*N=1*).
2. Fundamental solitons propagate without shape distortions.
3. Although the pulse shape is retained, a constant phase proportional to the propagation distance is accumulated across the entire pulse.
4. Using the NLSE scaling law, we know that pulses of the form $a(\zeta,\tau) = A\text{sech}(A\tau)\exp[\frac{-jA^2\zeta}{2}]$ are also solitons.
5. Pulse area of soliton is independent of amplitude, $Area = \pi\sqrt{\frac{|\beta_2|}{\gamma}}$.
6. Pulse energy is inversely proportional to the duration, $U = \frac{2|\beta_2|}{\gamma T_0}$.
7. Higher-order solitons of order N take the form of $a(0,\tau) = N\sec h(\tau)$. Higher order solitons undergo periodic pulse shape evolutions, and the soliton period is defined as $z_0 = \frac{\pi}{2}L_D$.
8. Adiabatic soliton compression: when gradual gain or dispersion alternation is incorporated along the longitudinal direction, solitons retain it's shape (order)

by decreasing it's duration. This is of particular interest in pulse compression and nonlinear comb broadening. The pulse compression ratio is simply defined by the ratio of the GVD of the input to the output end as

$$\eta_c = \frac{T(0)}{T(z)} = \frac{\beta_2(0)}{\beta_2(z)}. \tag{59}$$

• Higher-order effects

For pulses with duration much shorter than 1ps, the approximations in our previous analysis for the NLSE will breakdown. Here we look into more accurate derivation of the nonlinear propagation equation.

Nonlinear envelope equation in uniform media

We first define the electric field and the nonlinear polarization as

$$E(x,y,z,t) = \mathrm{Re}[\int \frac{d\tilde{\omega}}{2\pi} A(x,y,z,\tilde{\omega}) e^{j(\omega_0+\tilde{\omega})t} e^{-j(\beta_0+\beta_1\tilde{\omega})z}], \tag{60a}$$

$$P_{NL}(x,y,z,t) = \mathrm{Re}[\int \frac{d\tilde{\omega}}{2\pi} Q(x,y,z,\tilde{\omega}) e^{j(\omega_0+\tilde{\omega})t} e^{-j(\beta_0+\beta_1\tilde{\omega})z}], \tag{60b}$$

where Q is the Fourier transform of the nonlinear polarization envelope function.

Now we insert these into the scalar wave equation, we get

$$\left(\frac{\partial^2}{\partial z^2} - 2j(\beta_0+\beta_1\tilde{\omega})\frac{\partial}{\partial z} + (\omega_0+\tilde{\omega})^2 \mu_0 \varepsilon_{(1)} - (\beta_0+\beta_1\tilde{\omega})^2 + \nabla_T^2\right) A = -\mu_0(\omega_0+\tilde{\omega})^2 Q, \tag{61}$$

we now express the propagation constant as

$$\beta(\tilde{\omega}) = (\omega_0+\tilde{\omega})\sqrt{\mu_0\varepsilon_{(1)}} = \beta_0 + \beta_1\tilde{\omega} + \tilde{D}, \tag{62a}$$

$$\tilde{D} = \frac{\beta_2}{2}\tilde{\omega}^2 + \frac{\beta_3}{6}\tilde{\omega}^3 + \dots . \tag{62b}$$

After some arrangement we can obtain

$$\begin{aligned}&(1+\frac{\tilde{\omega}}{\omega_0})\left[\frac{\partial}{\partial z} + j\tilde{D}\right]A + \frac{j}{2\beta_0}\nabla_T^2 A + \frac{j\mu_0\omega_0 c}{2n_0}(1+\frac{\tilde{\omega}}{\omega_0})^2 Q \\ &= -\frac{j}{2\beta_0}(\frac{\partial^2}{\partial z^2} + \tilde{D}^2)A - (\frac{\omega_0\beta_1}{\beta_0} - 1)\frac{\tilde{\omega}}{\omega_0}\left[\frac{\partial}{\partial z} + j\tilde{D}\right]A\end{aligned} . \tag{63}$$

We invoke slowly-varying envelope approximation (SVEA) in *space* and *time*, so that the right-hand side terms can be omitted. Now we have

$$\left[\frac{\partial}{\partial z} + j\tilde{D}\right]A + \frac{j}{2\beta_0}\frac{1}{(1+\frac{\tilde{\omega}}{\omega_0})}\nabla_T^2 A = -\frac{j\mu_0\omega_0 c}{2n_0}(1+\frac{\tilde{\omega}}{\omega_0})Q . \tag{64}$$

We now perform an inverse Fourier transform and denote the in a retarded time frame $t' = t - \beta_1 z$, we arrive at the nonlinear envelope equation (**NEE**) as

$$\frac{\partial a}{\partial z} + jDa + \frac{j}{2\beta_0}\left(1 - \frac{j}{\omega_0}\frac{\partial}{\partial t'}\right)^{-1}\nabla_T^2 a = \frac{-j\mu_0\omega_0 c}{2n_0}\left(1 - \frac{j}{\omega_0}\frac{\partial}{\partial t'}\right)q , \tag{65a}$$

$$D = \frac{-\beta_2}{2}\frac{\partial^2}{\partial t'^2} + \frac{j\beta_3}{6}\frac{\partial^3}{\partial t'^3} + \dots \tag{65b}$$

From the SVEA conditions, NEE is valid for as long as all the individual terms contributing to the evolution of E are small within a wavelength.

Comments:

1. We have not yet specified the form of nonlinear polarization, so NEE is a generalized formulation.
2. Effects of higher-order dispersion can be incorporated through $\tilde{D}$, which is desired.
3. The time-derivative of the nonlinear polarization envelope gives rise to intensity-dependent group-velocity, and will be discussed later.
4. The transverse Laplacian accounts for spatial variations; the time-derivative

before it couples spatial and temporal degrees of freedom.

Nonlinear envelope equation in waveguide

Following closely to the preceding section and our treatment for the waveguides, we may define the electric field as

$$E(x,y,z,t) = \mathrm{Re}[\int \frac{d\tilde{\omega}}{2\pi} A(z,\tilde{\omega}) u(x,y) e^{j(\omega_0+\tilde{\omega})t} e^{-j(\beta_0+\beta_1\tilde{\omega})z}], \qquad (66)$$

using the same approximation as previous section, we arrive at

$$\left[\frac{\partial}{\partial z} + j\tilde{D}\right] Au = \frac{-j\mu_0\omega_0^2}{2\beta_0}\left(1+\frac{\tilde{\omega}}{\omega_0}\right)Q . \qquad (67)$$

The time-domain equation takes the form of

$$u\frac{\partial a}{\partial z} + jDau = \frac{-j\mu_0\omega_0^2}{2\beta_0}\left(1-\frac{j}{\omega_0}\frac{\partial}{\partial t'}\right)q . \qquad (68)$$

With the same treatment for waveguides, and using the normalization factor so that $|a_p|^2$ denotes the power in the guide mode, we have the NEE for single-mode waveguide as

$$\frac{\partial a_p}{\partial z} + jDa_p + j\gamma\left(1-\frac{j}{\omega_0}\frac{\partial}{\partial t'}\right)|a_p|^2 a_p = 0 . \qquad (69)$$

This equation is very similar to NLSE, but with the added higher-order dispersion and the time-derivative terms.

Delayed nonlinear response and the Raman effect

Up to now we have only considered nonlinear polarization due to electronic response interacting with optical field. Nonlinear index changes due to electronic processes are very often faster than an optical cycle and taken as instantaneous. There are other processes at a slower response rate that also contribute to nonlinear refractive index changes: such as vibration of nuclei, rotation of molecules, and laser induced reordering. Raman scattering is a particularly well known effect arising from the nuclear contribution to the nonlinear index. In spontaneous Raman scattering, the interaction of a monochromatic laser beam with the material leads to generation of a small amount of light that is down-shifted in frequency and scattered out of the incident beam.

Here we consider the self-action effects involving the nonlinear propagation of a single ultrashort pulse. We restrict our attention to linear polarization field and isotropic medium, such as optical fiber. The result is that the instantaneous nonlinear index is now expressed as

$$n_2|a(t)|^2 \rightarrow n_{2e}|a(t)|^2 + n_{2R}\int dt'\, f(t-t')|a(t')|^2 \ , \qquad (70a)$$

$$q = 2n_0\varepsilon_0 a(t)\left\{n_{2e}|a(t)|^2 + n_{2R}\int dt'\, f(t-t')|a(t')|^2\right\}, \qquad (70b)$$

where *f(t)* is commonly referred as the Raman response function.

Some assumptions to the Raman response function:

1. Causality: *f(t)*=0 for *t*<0.
2. *f(t)* is real, equivalent to assuming no loss is introduced.
3. *f(t)* is normalized so that $\int dt f(t) = 1$. With this step, we may express Eq. (70a) as

$$n_2|a(t)|^2 \rightarrow n_2\left\{(1-\alpha)|a(t)|^2 + \alpha\int dt'\, f(t-t')|a(t')|^2\right\} \ , \qquad (71)$$

so that α parameterizes the contribution to the nonlinear index by the delayed nuclear process for short pulses. For very long pulses, we can see that Eq. (71) simply reduces to $n_2|a(t)^2|$.

To gain more understanding, we now look at a very simplified case of Eq. (65a) in one-dimension and by setting dispersion and the time-derivative terms all to zero:

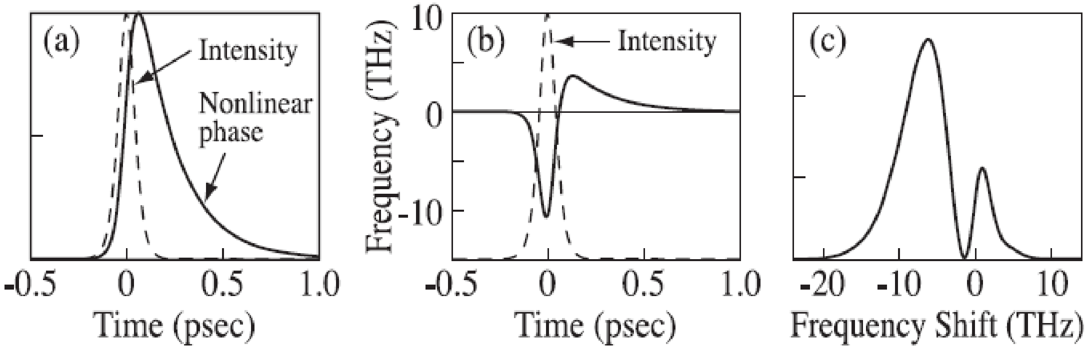
Only delayed nonlinear index considered (α=1):

$$\frac{\partial a}{\partial z} = \frac{-j\omega_0 n_2}{c} a(t) \int dt' \, f(t-t') \left| a(t') \right|^2 . \tag{72}$$

The solution is simply

$$a(z,t) = a(0,t) e^{j\Delta\phi(z,t)} \text{ where } \Delta\phi(z,t) = \frac{-j\omega_0 n_2}{c} \int dt' \, f(t-t') \left| a(0,t') \right|^2 , \tag{73}$$

and we can see that the effect of Raman response function is to provide a delayed time-varying instantaneous frequency shift shown in the figure below. As we can see, most of the energy of the pulse center will be red-shifted.



We can also understand the down-converting phenomenon in the frequency-domain. Again we look at the simplified version of Eq. (65a), but to make a more general analysis, now retaining the dispersion term and the instantaneous nonlinear response as

$$\frac{\partial a}{\partial z} + jDa = \frac{-j\omega_0 n_2}{c} a(t) \left\{ (1-\alpha) \left| a(t) \right|^2 + \alpha \int dt' \, f(t-t') \left| a(t') \right|^2 \right\} . \tag{74}$$

Now we consider two CW fields, with $\tilde{\omega}_p > \tilde{\omega}_s$ and each denoting the frequency offset from the carrier frequency ω_0

$$a(z,t) = a_p(z)e^{j\tilde{\omega}_p t} + a_s(z)e^{j\tilde{\omega}_s t} . \quad (75)$$

We substitute this into Eq. (74) and only retain the terms associated with the field at frequency $\tilde{\omega}_s$, we have

$$\frac{\partial a_s}{\partial z} + jDa_s = \frac{-j\omega_0 n_2}{c} a_s \left\{ |a_s|^2 + 2|a_p|^2 + \alpha[\mathscr{F}(\tilde{\omega}_s - \tilde{\omega}_p) - 1]|a_p|^2 \right\} , \quad (76)$$

where $\mathscr{F}(\omega)$ is the Fourier transform of *f(t)*.

We may now obtain an equation describing the evolution of the signal power by multiplying Eq. (76) with a^*_s and adding the resulting equation with it's complex conjugate, we obtain

$$\frac{\partial |a_s|^2}{\partial z} = g_{Raman} |a_p|^2 |a_s|^2 , \quad (77a)$$

$$g_{Raman} = \frac{4\pi n_2 \alpha}{\lambda} \mathrm{Im}\{\mathscr{F}(\tilde{\omega}_s - \tilde{\omega}_p)\} . \quad (77b)$$

The coefficient g_{Raman} is called the Raman gain.

Comments:

1. For a real *f(t)*, we have $\mathscr{F}(-u) = \mathscr{F}^*(u)$.
2. Due to causality, *f(t)* is single sided, we have $\mathrm{Im}\{\mathscr{F}(u)\} > 0, u < 0$ and $\mathrm{Im}\{\mathscr{F}(u)\} < 0, u > 0$.
3. We can see that indeed the signal field at lower frequency is amplified at the expense of the pump field being attenuated during propagation. This is the frequency domain view of the down-converting process, in consistency with the time-domain explanation using the delayed instantaneous frequency picture.

Pulses long compared to f(t):

Eq. (65a) is now simplified to

$$\frac{\partial a}{\partial z} + jDa = \frac{-j\omega_0 n_2}{c}\left\{|a|^2 - T_R \frac{\partial |a|^2}{\partial t}\right\}a\,, \tag{78}$$

where $T_R \approx 3\,fs$ for fused silica.

The Raman response function of fused silica is particularly important because of the numerous studies and practical applications of nonlinear pulse propagation in optical fibers. The delayed component of the nonlinear response can have a particularly noticeable effect in the case of solitons. Two important examples include the **soliton self-frequency shift** and **fission of higher order solitons,** described within the course slides. One important application of supercontinuum (SC) generation using holey fiber will also be discussed there.

Self-steepening effect

Here we describe the effect of the time-derivative term in Eq. (69). We look at the case where dispersion is set to zero and only instantaneous nonlinear response is considered. From our discussion of NLSE for pure SPM, we know the pulse should maintain its shape during propagation without the additional time-derivative term. We now write a simplified version of the NEE as

$$\frac{\partial a}{\partial z} + j\gamma\left(1 - \frac{j}{\omega_0}\frac{\partial}{\partial t'}\right)|a|^2 a = 0\,. \tag{79}$$

Here we have dropped the expression a_p for simplicity, but bear in mind the normalization of the field squared still gives the instantaneous power.

We express the envelope as $a(z,t') = |a(z,t')|e^{j\phi(z,t')}$ and substitute into Eq.

(79). The real part that describes the evolution of the envelope gives

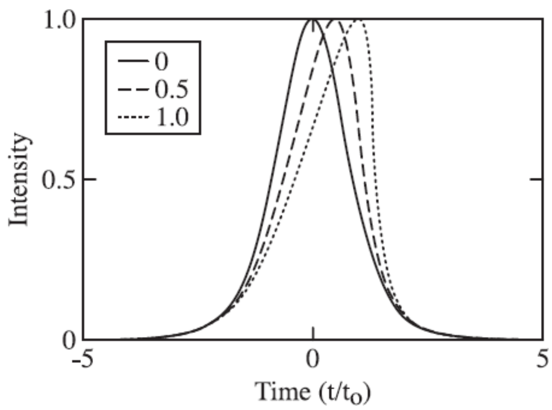
$$\frac{\partial|a|}{\partial z}+\frac{3\gamma}{\omega_0}|a|^2\frac{\partial|a|}{\partial t'}=0\,. \qquad (80)$$

The solution is simply of the form

$$|a|\propto f(t'-\frac{z}{u})\,, \qquad (81a)$$

$$u=\frac{\omega_0}{3\gamma|a|^2}\,. \qquad (81b)$$

Thus we can see that the pulse will be distorted even when dispersion is zero, mainly due to the intensity-dependent group velocity u as shown in Eq. (81). In the figure below, we clearly see the effect of this "shock" term for positive nonlinear index: the pulse peak is delayed relative to the wings, and steepening of pulse peaks towards the trailing edge becomes more evident as the pulse propagates.



Comments:

1. Our model predicts that the self-steepening may continue until $\partial|a|^2/\partial t'\to\infty$, where the optical shock front is formed.

2. We can make a rough estimate of the length required for shock front to appear by taking the extra delay of the peak equals to the pulse duration t_0:

$$L_{shock}\approx\frac{\omega_0 t_0}{3\gamma P_0}\,. \qquad (82)$$

3. As compared to the solitons characteristic nonlinear length, we obtain the

ratio

$$\frac{L_{shock}}{L_{NL}} \approx \frac{\omega_0 t_0}{3}. \tag{83}$$

which states that we do not have to worry about the shock term unless we have few-cycle pulses.