Fourier Series

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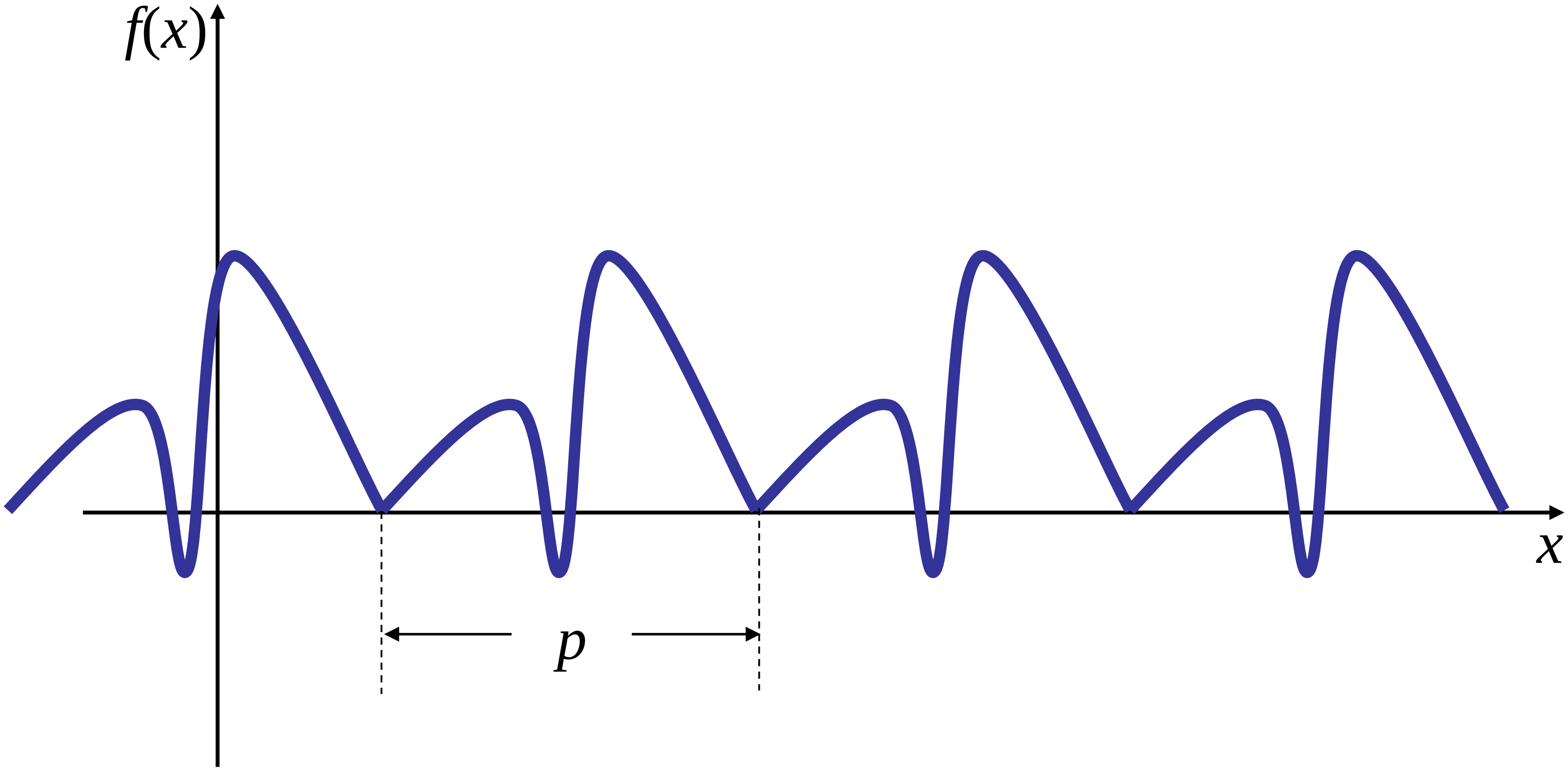
Periodic Function

- A function $f(x)$ is called *periodic*, if it is defined for every x in the domain of f and if there is some positive number p such that

$$f(x+p) = f(x)$$

- The number p is called a *period* of $f(x)$.
- If a periodic function f has a smallest period p, this is often called the *fundamental period* of $f(x)$.

Periodic Function



Periodic Function

- For any integer n,

$$f(x+np) = f(x), \text{ for all } x$$

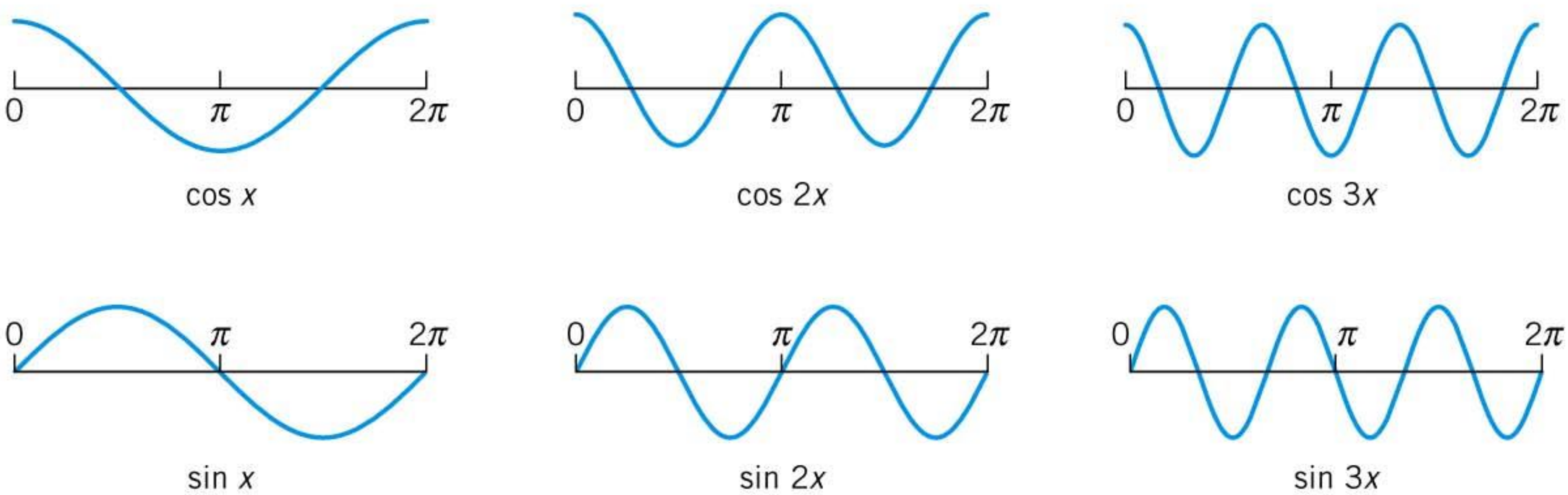
- If $f(x)$ and $g(x)$ have period p, then so does

$$h(x) = af(x) + bg(x),$$

  a and b are constants.

Trigonometric Functions

- Consider the following periodic functions with period 2π: $1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots, \sin nx, \cos nx, \ldots$



Cosine and sine functions having the period 2π

Trigonometric Series

- The trigonometric series is of the form

$$a_0 \cdot 1 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x \cdots$$

$$= a_0 \cdot 1 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

  $a_0, a_1, a_2, a_3, \ldots, b_1, b_2, b_3, \ldots$ are real constants and are called the *coefficients* of the series.

- If the series converges, its sum will be a function of period 2π.

Fourier Series

- Assume $f(x)$ is a periodic function of period 2π that can be represented by the trigonometric series:

$$f(x) = a_0 \cdot 1 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

  That is, we assume that the series converges and has $f(x)$ as its sum.
- **Question**: how to compute the coefficients?

Properties of Trigonometric Functions

- Recall some basic properties of trigonometric functions

$$\int_{-\pi}^{\pi}\cos nx\,dx=\int_{-\pi}^{\pi}(1\cdot\cos nx)\,dx=0$$

$$\int_{-\pi}^{\pi}\sin nx\,dx=\int_{-\pi}^{\pi}(1\cdot\sin nx)\,dx=0$$

$$\int_{-\pi}^{\pi}\cos mx\sin nx\,dx$$

$$=\frac{1}{2}\int_{-\pi}^{\pi}\sin(n-m)x\,dx+\frac{1}{2}\int_{-\pi}^{\pi}\sin(n+m)x\,dx=0$$

Properties of Trigonometric Functions

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)x dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)x dx = \begin{cases} \pi & n = m \\ 0 & n \neq m \end{cases}$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos(n-m)x dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos(n+m)x dx = \begin{cases} \pi & n = m \\ 0 & n \neq m \end{cases}$$

Determination of the Constant Term a_0

- Integrating both sides from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)dx$$

$$= 2\pi a_0$$

$$\boxed{a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx}$$

Determination of the Coefficients a_n of the Cosine Term

- Multiply by $\cos mx$ with fixed positive integer and then integrate both sides from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx$$

$$= \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right) \right] \cos mx\,dx$$

$$= \pi a_m$$

$$\boxed{a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos mx\,dx}$$

Determination of the Coefficients b_n of the Sine Term

- Multiply by $\sin mx$ with fixed positive integer and then integrate both sides from –π to π:

$$\int_{-\pi}^{\pi} f(x)\sin mx dx$$

$$= \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)\right]\sin mx dx$$

$$= \pi b_m$$

$$\boxed{b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx dx}$$

Summary

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

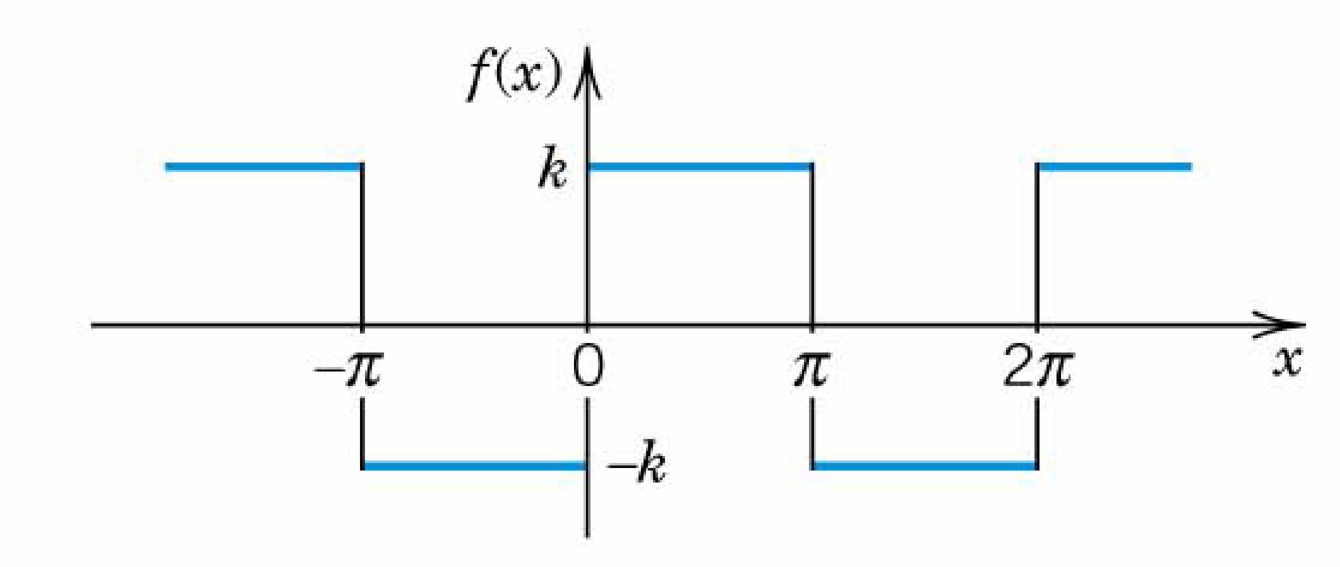
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

Fourier Series: Square Wave

- Ex. Find the Fourier coefficients of the periodic function $f(x)$

$$f(x)=\begin{cases}-k & -\pi<x<0\\ k & 0<x<-\pi\end{cases},\qquad f(x+2\pi)=f(x)$$



Fourier Series: Square Wave

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -k\,dx + \int_{0}^{\pi} k\,dx\right]$$

$$= \frac{1}{2\pi}\left[-k\pi + k\pi\right] = 0$$

Q: Is $f(x)$ even or odd?!

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\cos nx\,dx + \int_{0}^{\pi} k\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-k}{n}\sin nx\Big|_{-\pi}^{0} + \frac{k}{n}\sin nx\Big|_{0}^{\pi}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{k}{n}\cos nx\Big|_{-\pi}^{0} - \frac{k}{n}\cos nx\Big|_{0}^{\pi}\right]$$

$$\boxed{\cos(-n\pi) = \cos(n\pi)}$$

$$= \frac{1}{\pi}\left[\frac{k}{n}(1-\cos n\pi) - \frac{k}{n}(\cos n\pi - 1)\right]$$

$$= \frac{2k}{n\pi}(1-\cos n\pi)$$

$$\boxed{1-\cos n\pi = \begin{cases} 2 & n = 1,3,\cdots \\ 0 & n = 2,4,\cdots \end{cases}}$$

Fourier Series: Square Wave

$$b_1 = \frac{4k}{\pi}, b_2 = 0, b_3 = \frac{4k}{3\pi}, b_4 = 0, b_5 = \frac{4k}{5\pi}, \cdots$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

$$= \sum_{n=1}^{\infty} \frac{2k}{n\pi} \left(1 - \cos n\pi \right) \sin nx$$

$$= \frac{4k}{\pi} \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \right)$$

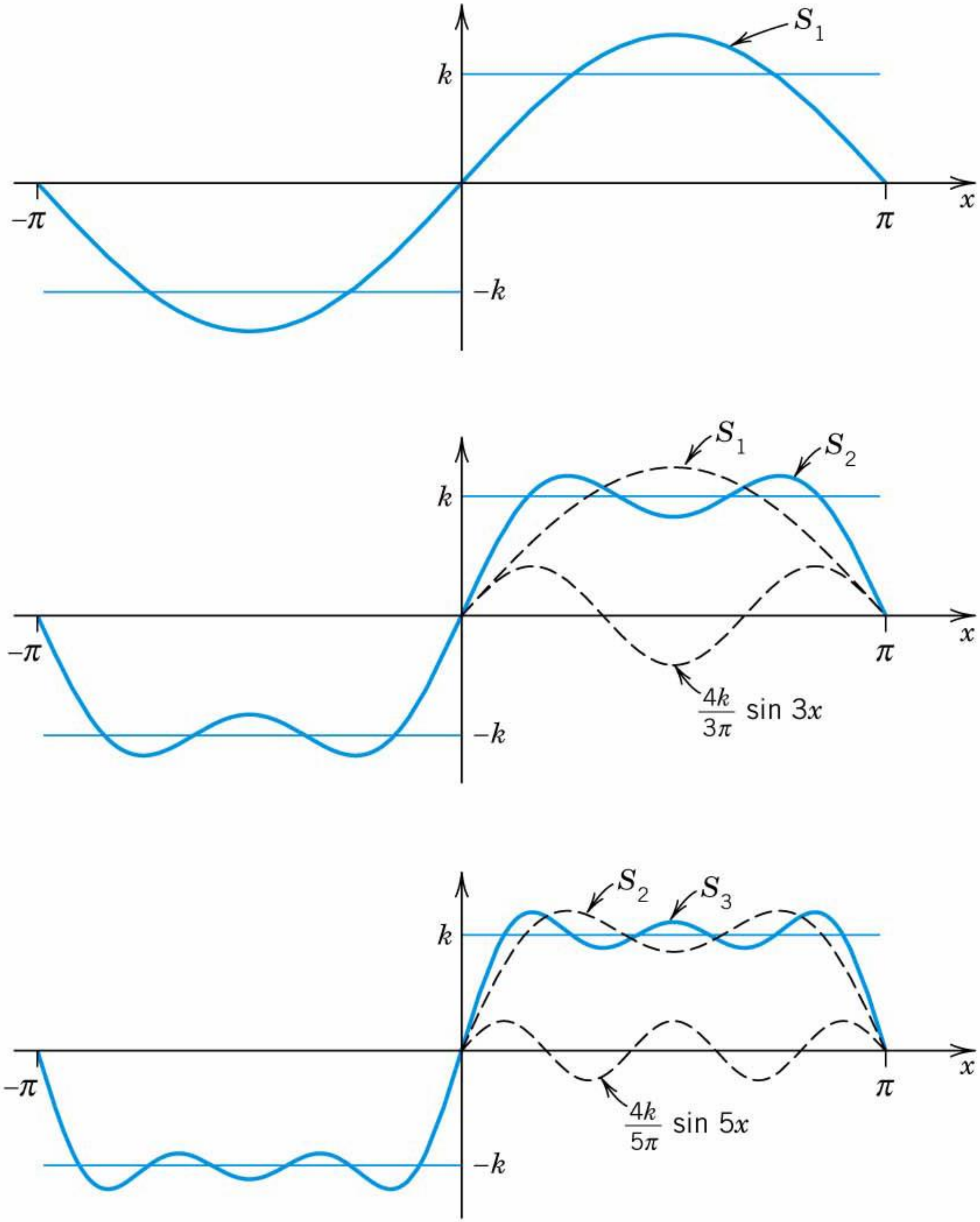
Fourier Series: Square Wave

- Consider the series expansion of finite sine terms:

$$S_1 = \frac{4k}{\pi}\sin x$$

$$S_2 = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right)$$

$$S_3 = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x\right)$$



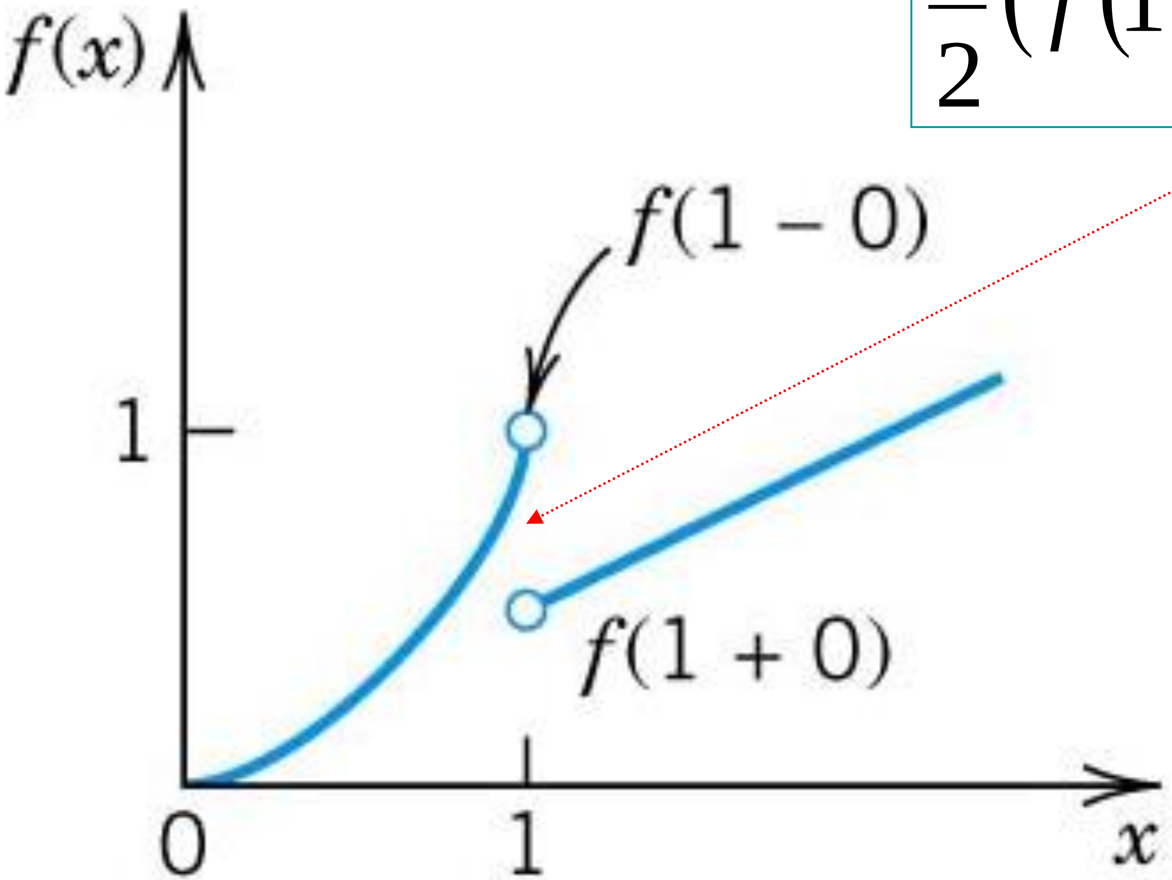
S_1
k
x
$-\pi$
π
$-k$
S_1
S_2
k
x
$-\pi$
π
$\frac{4k}{3\pi}$ sin $3x$
$-k$
S_2
S_3
k
x
$-\pi$
π
$\frac{4k}{5\pi}$ sin $5x$
$-k$

Representation by Fourier Series

- **Thm**. If a periodic function $f(x)$ with period 2π is peicewise continuous in the interval $-\pi \leqq x \leqq \pi$ and has a left-hand derivative and right-hand derivative at each point of that interval, then Fourier series of $f(x)$ is convergent. Its sum is $f(x)$, except at a point of x_0 at which $f(x)$ is discontinuous and the sum of the series is the average of the left- and right-hand limits of $f(x)$ at x_0.

Representation by Fourier Series

$$\frac{1}{2}\left(f(1-0)+f(1+0)\right)$$



Functions of Any Period $p=2L$

- A function $f(x)$ of period $p=2L$ has a Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2\pi}{2L}nx + b_n \sin\frac{2\pi}{2L}nx\right)$$

$$= a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

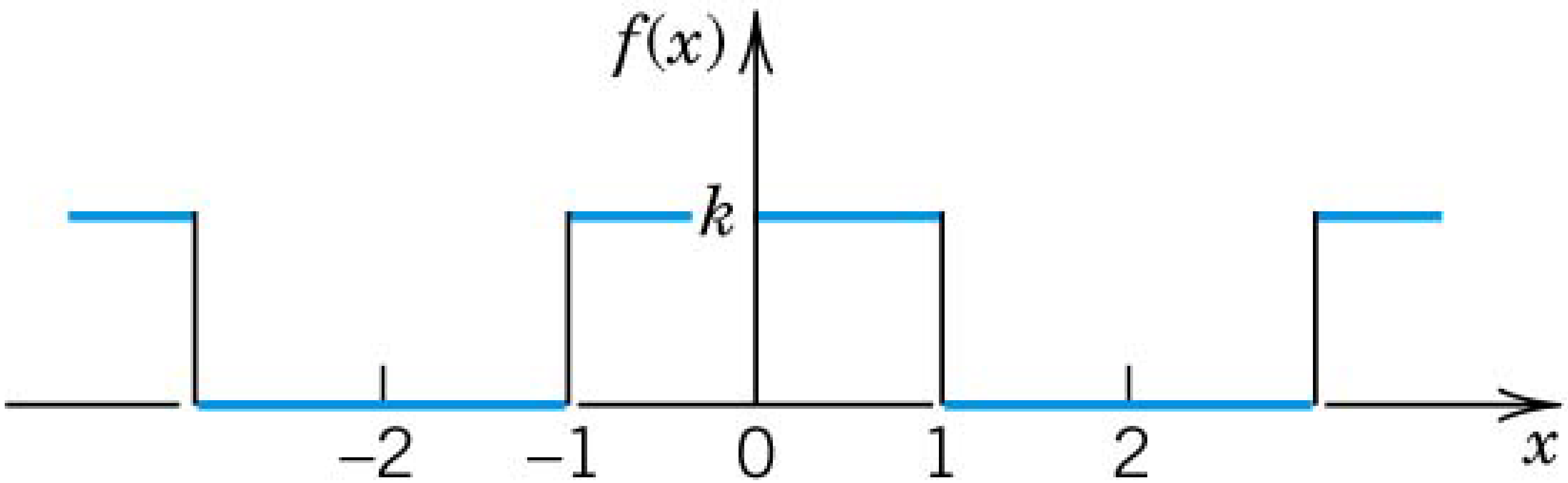
$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx \qquad n = 1,2,3,\cdots$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

Fourier Series: Periodic Square Wave

- **Ex**. Find the Fourier series of the function

$$f(x)=\begin{cases}0 & -2<x<-1\\ k & -1<x<1\\ 0 & 1<x<2\end{cases}\quad ,\ p=2L=4, L=2$$



Fourier Series: Periodic Square Wave

$$a_0 = \frac{1}{4}\int_{-2}^{2} f(x)dx = \frac{1}{4}\int_{-1}^{1} k\,dx = \frac{k}{2}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx = \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi x}{2}dx$$

$$= \frac{k}{n\pi}\sin\frac{n\pi}{2}x\Big|_{-1}^{1} = \frac{2k}{n\pi}\sin\frac{n\pi}{2}$$

$$a_n = \frac{2k}{n\pi}, \quad n = 1,5,9,\cdots \quad a_n = -\frac{2k}{n\pi}, \quad n = 3,7,11,\cdots$$

Fourier Series: Periodic Square Wave

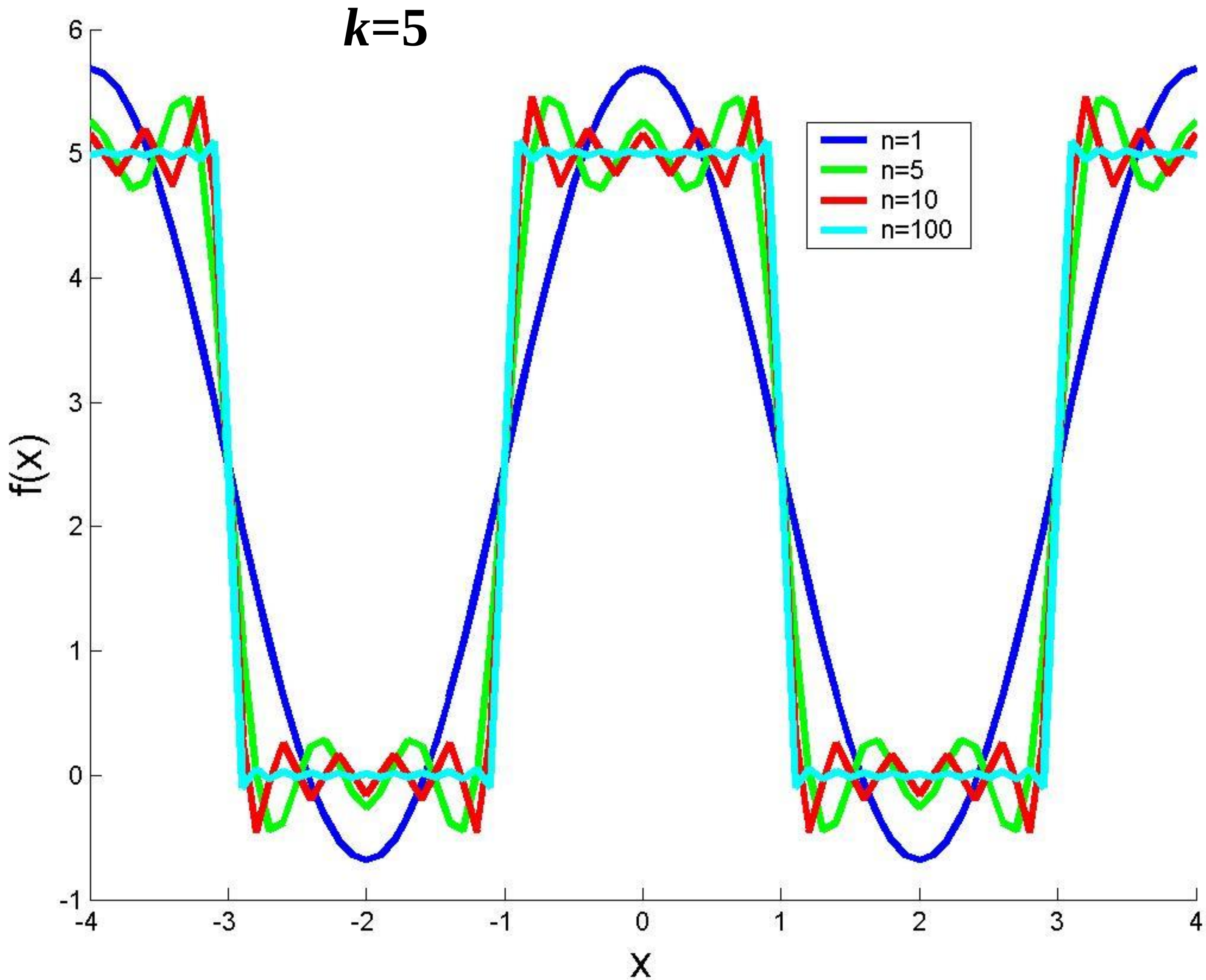
$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_{-1}^{1} k\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{-k}{n\pi}\cos\frac{n\pi}{2}x\Big|_{-1}^{1} = \frac{-k}{n\pi}\cos\frac{n\pi}{2} + \frac{k}{n\pi}\cos-\frac{n\pi}{2}$$

$$= \frac{k}{n\pi}\cos\frac{n\pi}{2} - \frac{k}{n\pi}\cos\frac{n\pi}{2} = 0$$

$$f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x - \cdots\right)$$

Fourier Series: Periodic Square Wave

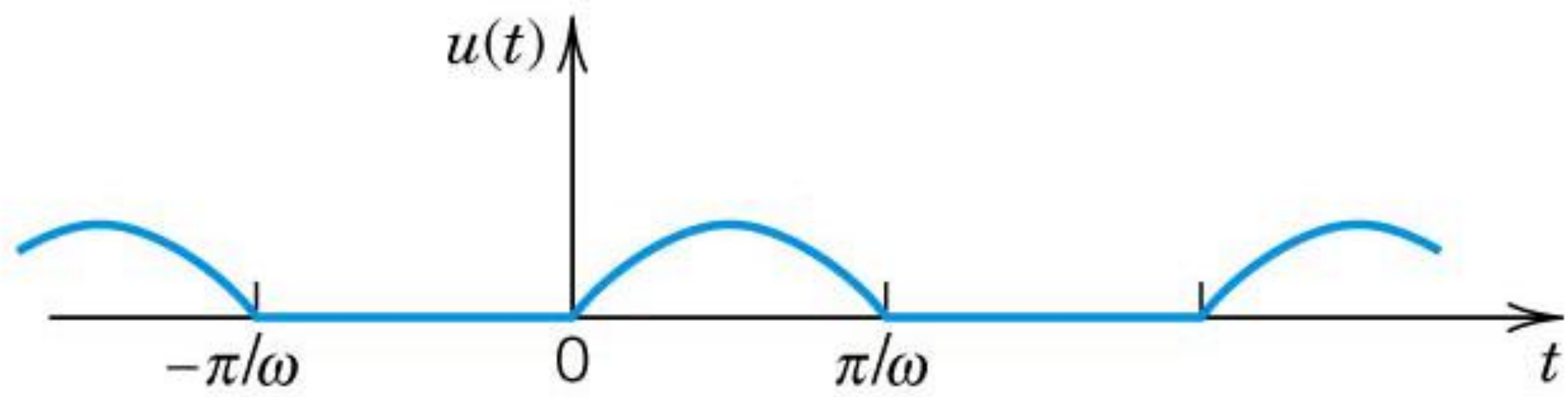


Ph.D.

Fourier Series: Half-wave Rectifier

- **<u>Ex</u>**. A sinusoidal voltage $E\sin\omega t$, is passed through a half-wave rectifier that clips the negative portion of the wave. Find the Fourier series of the resulting periodic function:

$$u(t)=\begin{cases} 0 & -L<t<0 \\ E\sin\omega t & 0<t<L \end{cases},\ p=2L=\frac{2\pi}{\omega},\ L=\frac{\pi}{\omega}$$



Fourier Series: Half-wave Rectifier

$$b_n = \frac{\omega}{\pi}\int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} u(t)\sin n\omega t dt = \frac{\omega}{\pi}\int_0^{\frac{\pi}{\omega}} E\sin\omega t\sin n\omega t dt$$

$$= \frac{E\omega}{2\pi}\int_0^{\frac{\pi}{\omega}}\left[\cos(1-n)\omega t - \cos(1+n)\omega t\right]dt$$

$$= \frac{E\omega}{2\pi}\left[\frac{\sin(1-n)\omega t}{(1-n)\omega} - \frac{\sin(1+n)\omega t}{(1+n)\omega}\right]\Bigg|_0^{\frac{\pi}{\omega}}$$

$$= \frac{E}{2\pi}\left[\frac{\sin(1-n)\pi}{(1-n)} - \frac{\sin(1+n)\pi}{(1+n)}\right]$$

$$= \begin{cases} \frac{E}{2} & n = 1 \\ 0 & n = 2,3,4,\cdots \end{cases}$$

$$\boxed{\begin{array}{l} \lim\limits_{n\to 1}\frac{\sin(1-n)\pi}{(1-n)} = \lim\limits_{n\to 1}\frac{-\pi\cos(1-n)\pi}{-1} = \pi \\ \frac{d}{dn}\sin(1-n)\pi = -\pi\cos(1-n)\pi \end{array}}$$

Fourier Series: Half-wave Rectifier

$$a_0 = \frac{\omega}{2\pi}\int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} u(t)dt = \frac{\omega}{2\pi}\int_0^{\frac{\pi}{\omega}} E\sin\omega t dt$$

$$= \frac{-E}{2\pi}\cos\omega t\Big|_0^{\frac{\pi}{\omega}} = \frac{-E}{2\pi}(\cos\pi - \cos 0) = \frac{E}{\pi}$$

$$a_n = \frac{\omega}{\pi}\int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} u(t)\cos n\omega t dt = \frac{\omega}{\pi}\int_0^{\frac{\pi}{\omega}} E\sin\omega t\cos n\omega t dt$$

$$= \frac{E\omega}{2\pi}\int_0^{\frac{\pi}{\omega}}\left[\sin(1+n)\omega t + \sin(1-n)\omega t\right]dt$$

$$= \frac{E\omega}{2\pi}\left[-\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega}\right]\Bigg|_0^{\frac{\pi}{\omega}}$$

Fourier Series: Half-wave Rectifier

$$a_n = \frac{E}{2\pi}\left[\frac{1-\cos(1+n)\pi}{(1+n)} + \frac{1-\cos(1-n)\pi}{(1-n)}\right]$$

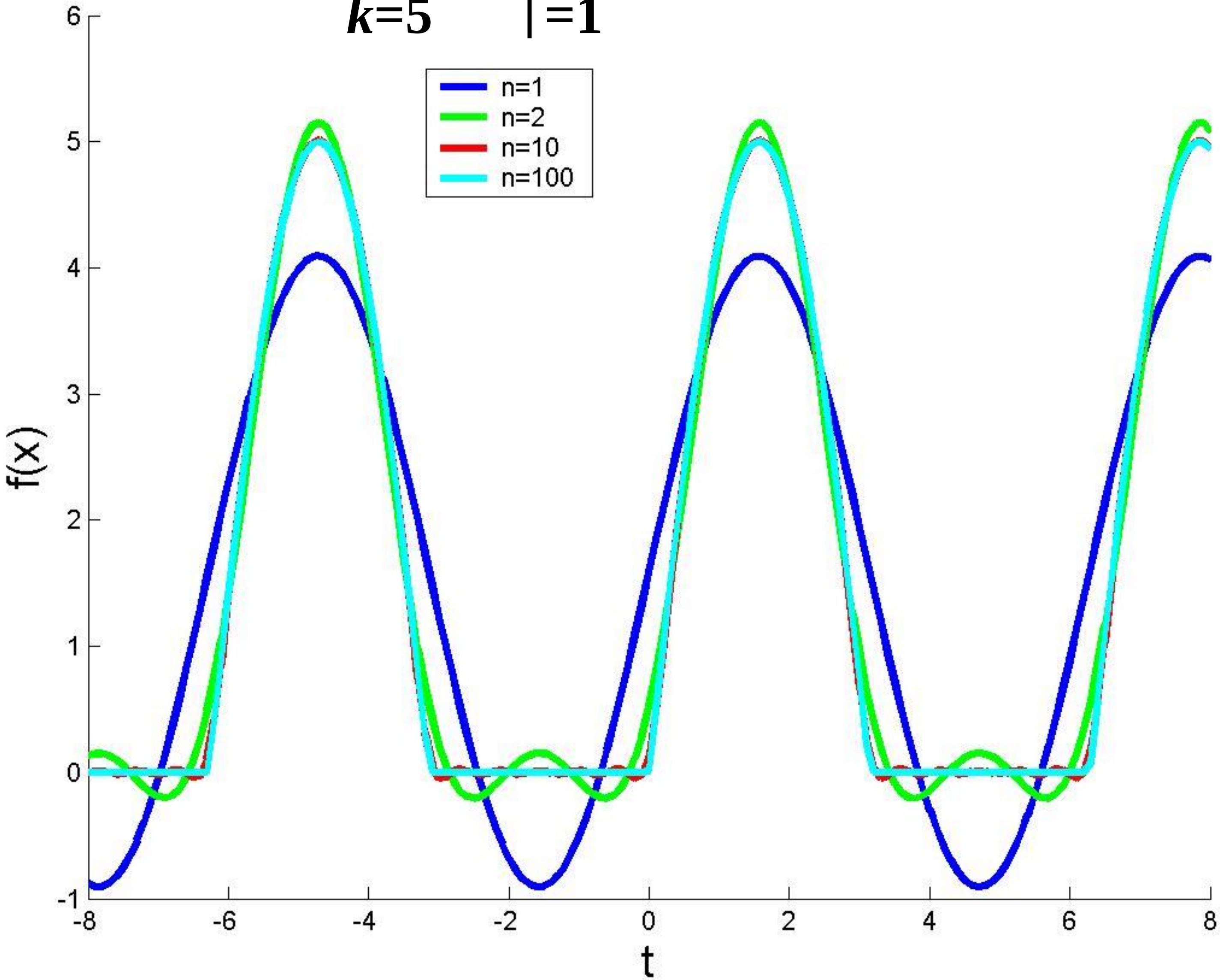
$$= \begin{cases} 0 & n = 1,3,5,\cdots \\ \dfrac{E}{2\pi}\left[\dfrac{2}{(1+n)} + \dfrac{2}{(1-n)}\right] & n = 2,4,6,\cdots \end{cases}$$

$$= \begin{cases} 0 & n = 1,3,5,\cdots \\ \dfrac{-2E}{(n-1)(n+1)\pi} & n = 2,4,6,\cdots \end{cases}$$

$$\boxed{u(t) = \frac{E}{\pi} + \frac{E}{2}\sin\omega t - \frac{2E}{\pi}\left(\frac{1}{1\cdot 3}\cos 2\omega t + \frac{1}{3\cdot 5}\cos 4\omega t\right)}$$

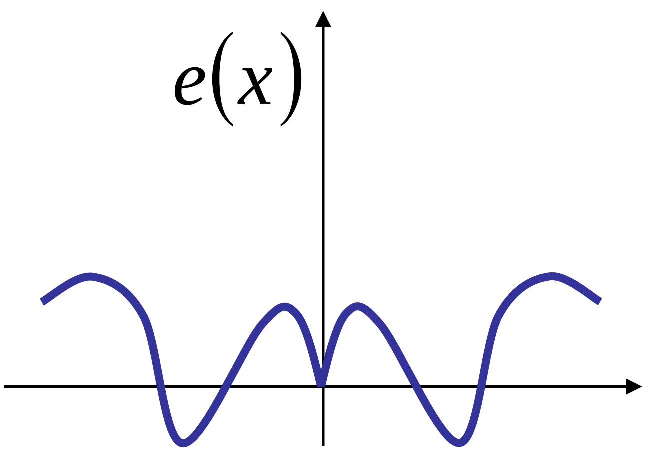
Fourier Series: Half-wave Rectifier

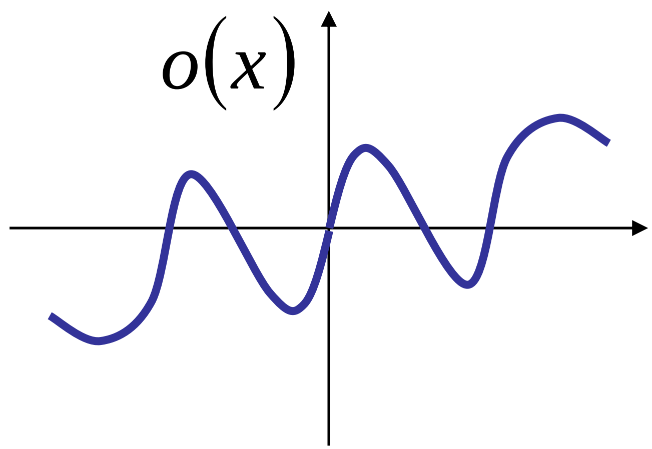
***k*=5 ⌉=1**



Even and Odd Functions

- A function $e(x)$ is even, if $e(-x)=e(x)$.
- A function $o(x)$ is odd, if $o(-x)=-o(x)$.





Even and Odd Functions: Facts

- If $e(x)$ is an even function, then

$$\int_{-L}^{L} e(x)dx = 2\int_{0}^{L} e(x)dx$$

- If $o(x)$ is an odd function, then

$$\int_{-L}^{L} o(x)dx = 0$$

- The product of an even and an odd function is odd

$$e(-x)o(-x) = -e(x)o(x)$$

Fourier Cosine Series

- **Thm**. The Fourier series of an even function of period $2L$ is a *Fourier cosine series*

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{L}x$$

with coefficients

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{L}\int_{0}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

Fourier Sine Series

- **Thm**. The Fourier series of an odd function of period $2L$ is a *Fourier sine series*

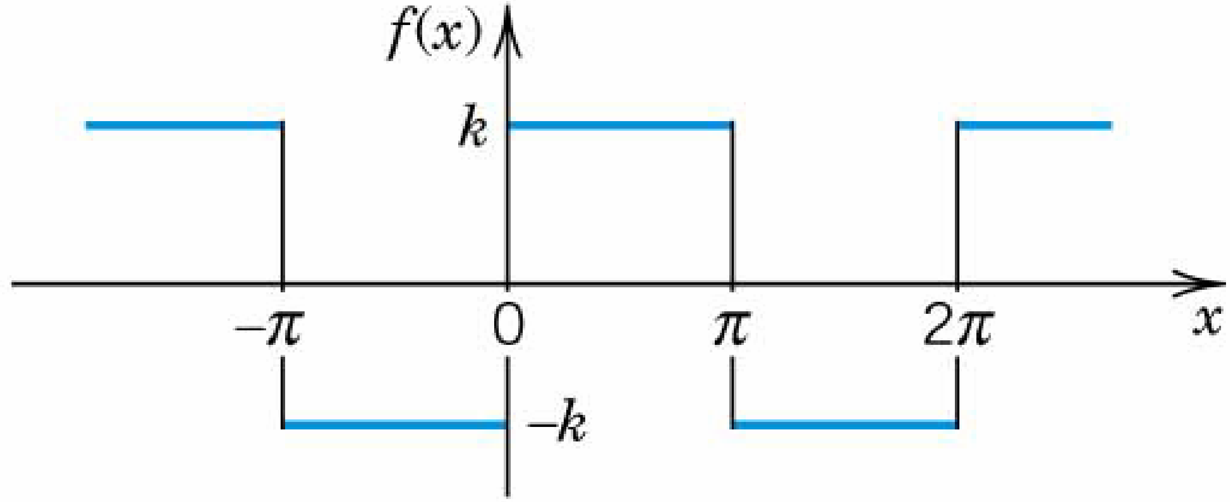
$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

with coefficients

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx = \frac{2}{L}\int_{0}^{L} f(x) \sin \frac{n\pi x}{L} dx$$

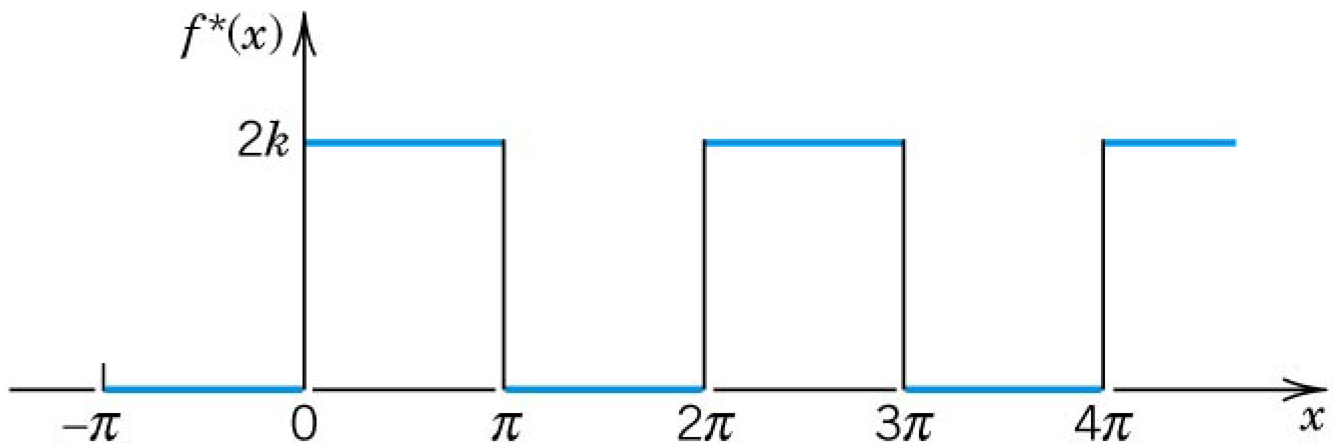
Fourier Series: Sum of Functions

- **Thm**. The Fourier coefficients of a sum f_1+f_2 are the sums of the corresponding Fourier coefficients of f_1 and f_2.



$$f(x) = \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

$$f^{*}(x) = k + \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right)$$

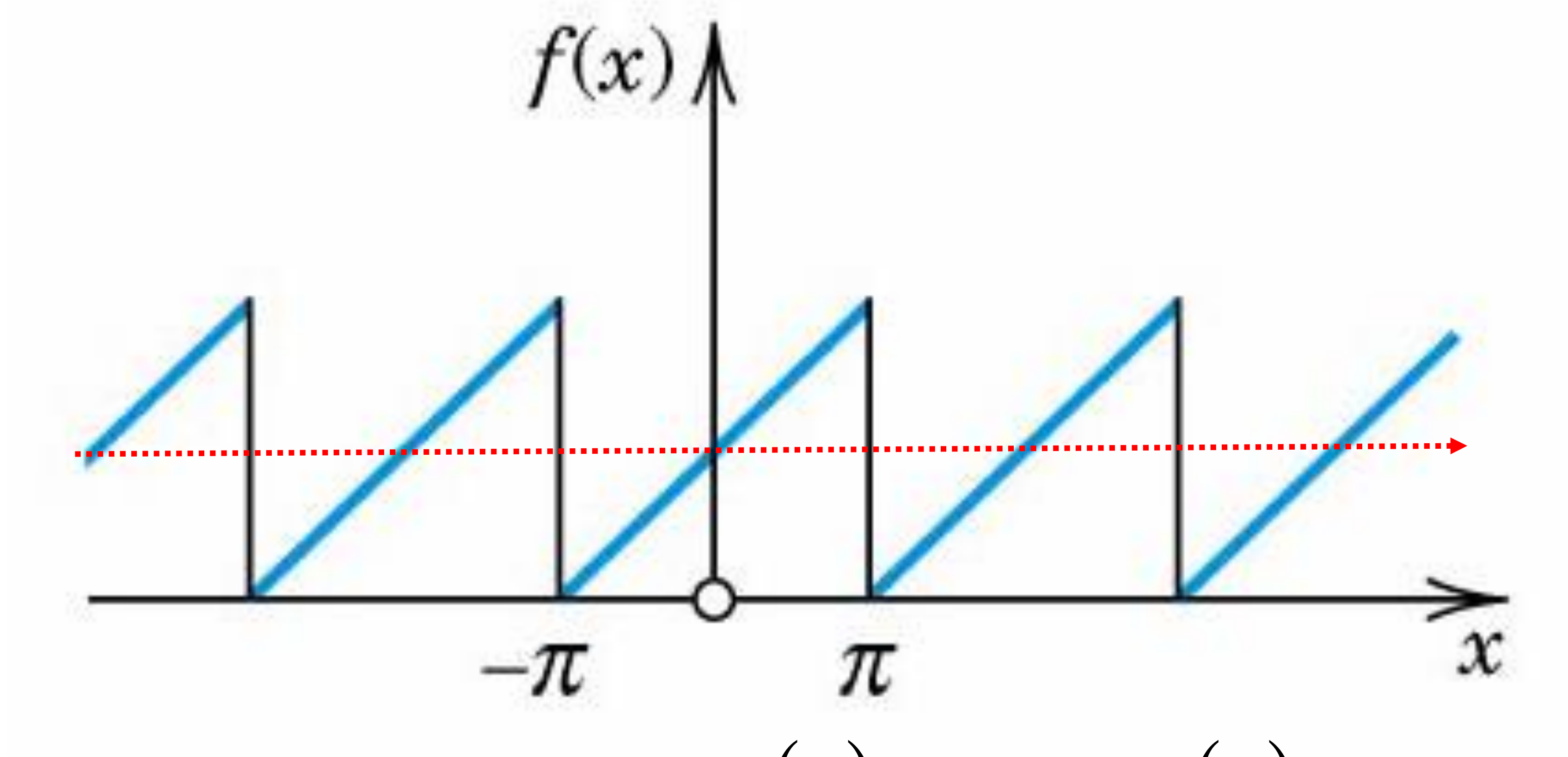


Ph.D.

Sawtooth Wave

- Ex. Find the Fourier series of the function

$$f(x) = x + \pi, \quad -\pi < x < \pi, \quad f(x + 2\pi) = f(x)$$



$$f = f_1 + f_2 \Rightarrow f_1(x) = x, \quad f_2(x) = \pi$$

Sawtooth Wave

- Since f_1 is odd, $a_n=0$ for $n=1,2,3,\ldots$ and

$$b_n = \frac{2}{L}\int_0^L f_1(x)\sin\frac{n\pi x}{L}dx$$

$$= \frac{2}{\pi}\int_0^\pi x\sin nx dx = \frac{2}{n\pi}\int_0^\pi x d(-\cos nx)$$

$$= \frac{2}{n\pi}x(-\cos nx)\Big|_0^\pi - \frac{2}{n\pi}\int_0^\pi(-\cos nx)dx$$

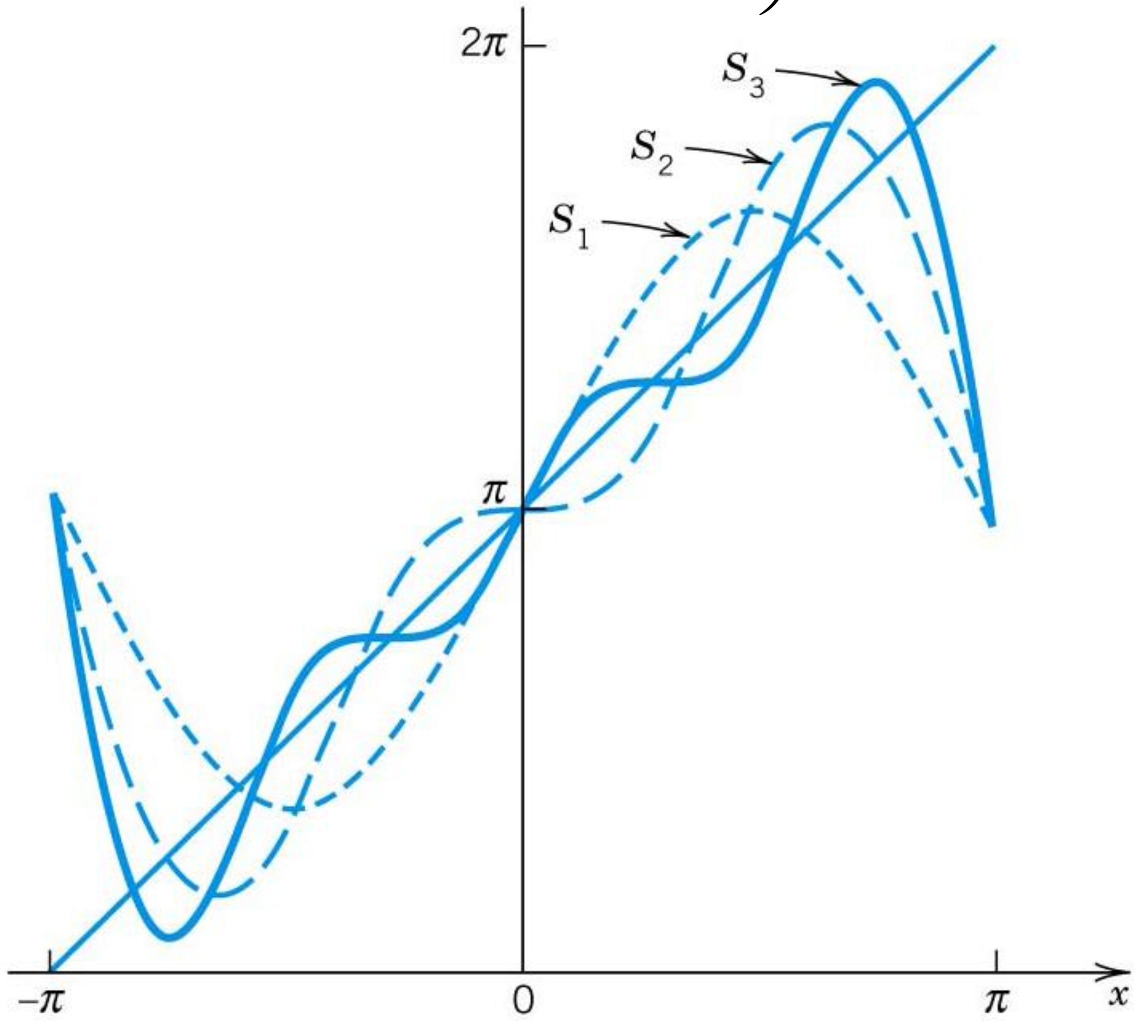
$$= \frac{-2}{n\pi}\pi\cos n\pi + \frac{2}{n^2\pi}\sin nx\Big|_0^\pi = -\frac{2}{n}\cos n\pi$$

$$f_1(x) = \sum_{n=1}^\infty b_n \sin nx = \sum_{n=1}^\infty -\frac{2}{n}\cos n\pi \sin nx$$

Sawtooth Wave

$$f(x) = \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - + \cdots \right)$$

$$= f_1(x) + f_2(x)$$



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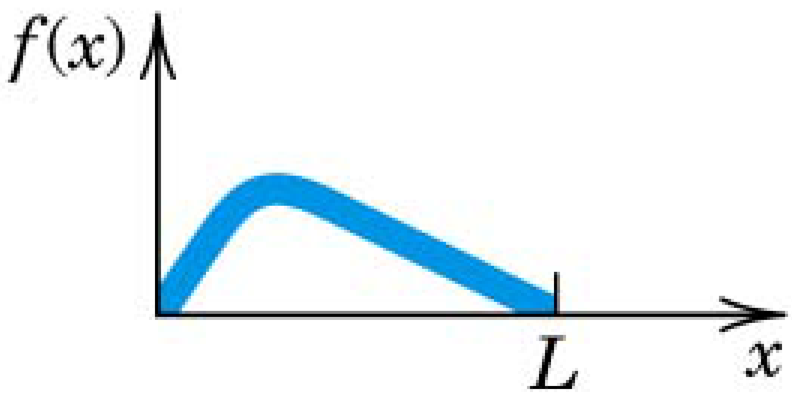
Half-Range Expansion

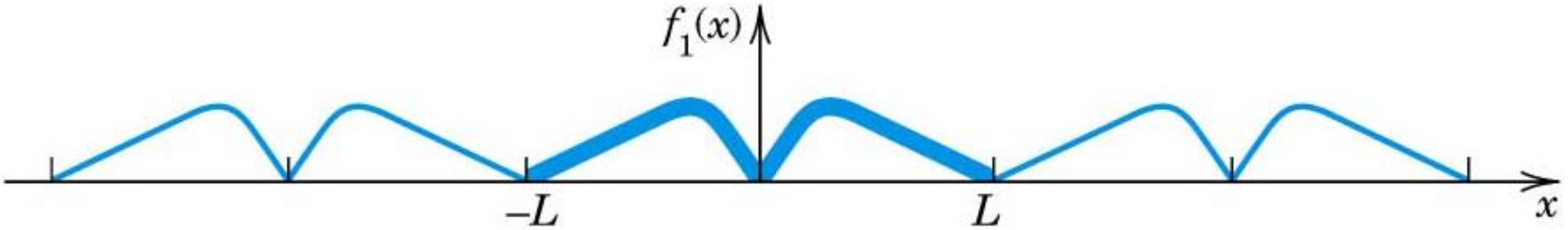
- In some applications, we often need to employ a Fourier series for a function $f(x)$ that is given only on some interval, say $0 \leqq x \leqq L$.



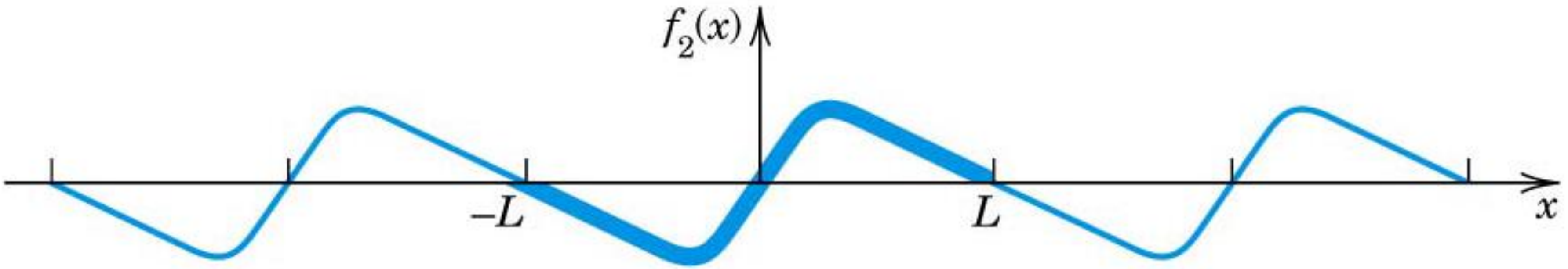
Suppose that we have a choice!! We can use even or odd periodic expansion.

Half-Range Expansion

- If we apply even periodic extension, we obtain a Fourier cosine series:



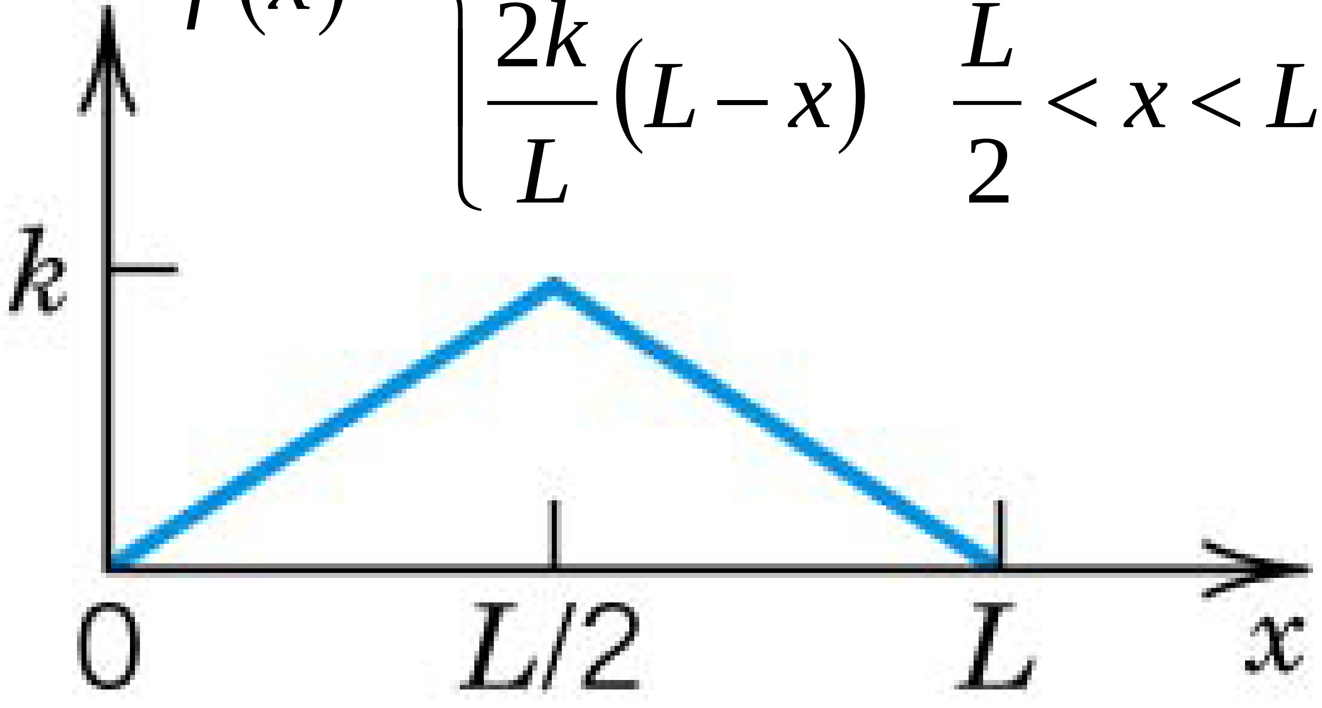
- If we apply odd periodic extension, we obtain a Fourier sine series:



Triangle Function: Half-Range Expansion

- **<u>Ex</u>**. Find the two half-range expansion of the triangle function

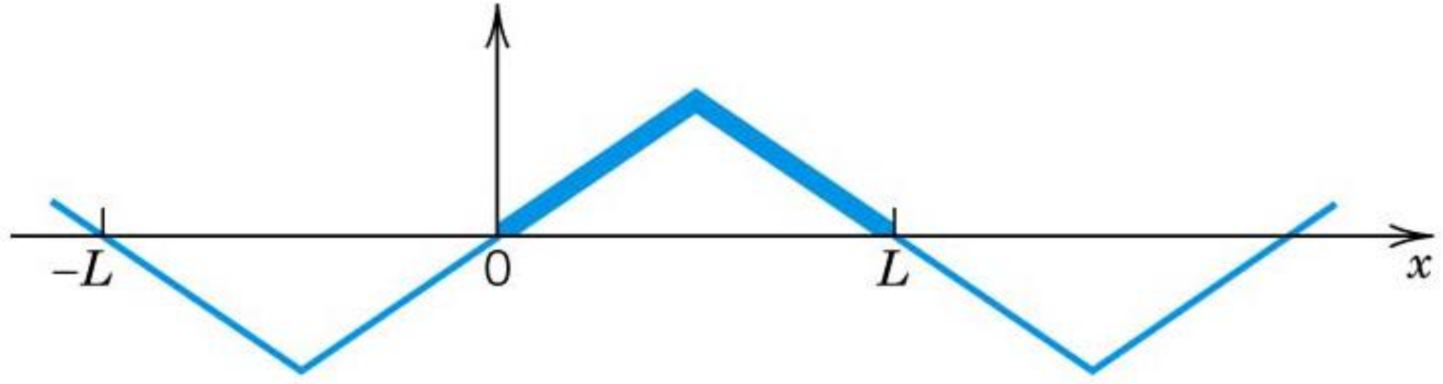
$$f(x)=\begin{cases}\dfrac{2k}{L}x & 0<x<\dfrac{L}{2}\\[2ex] \dfrac{2k}{L}(L-x) & \dfrac{L}{2}<x<L\end{cases}$$



Triangle Function: Half-Range Expansion

Even Expansion

Odd Expansion



Triangle Function: Even Periodic Expansion

$$f(x)=\begin{cases}\dfrac{2k}{L}x & 0<x<\dfrac{L}{2}\\[2mm] \dfrac{2k}{L}(L-x) & \dfrac{L}{2}<x<L\end{cases}$$

$$a_0=\frac{1}{2L}\int_{-L}^{L}e_f(x)dx=\frac{2}{2L}\int_0^L e_f(x)dx$$

$$=\frac{1}{L}\left[\frac{2k}{L}\int_0^{L/2}x\,dx+\frac{2k}{L}\int_{L/2}^{L}(L-x)dx\right]$$

$$=\frac{1}{L}\left[\frac{k}{L}\left(\frac{L}{2}\right)^2+\frac{k}{L}\left(\frac{L}{2}\right)^2\right]=\frac{k}{2}$$

$$a_n=\frac{1}{L}\int_{-L}^{L}e_f(x)\cos\frac{n\pi x}{L}dx=\frac{2}{L}\int_0^L e_f(x)\cos\frac{n\pi x}{L}dx$$

$$=\frac{2}{L}\left[\frac{2k}{L}\int_0^{L/2}x\cos\frac{n\pi x}{L}dx+\frac{2k}{L}\int_{L/2}^{L}(L-x)\cos\frac{n\pi x}{L}dx\right]$$

Triangle Function: Even Periodic Expansion

$$\int_0^{L/2} x\cos\frac{n\pi x}{L}dx = \frac{L}{n\pi}x\sin\frac{n\pi x}{L}\Bigg|_0^{L/2} - \frac{L}{n\pi}\int_0^{L/2}\sin\frac{n\pi x}{L}dx$$

$$= \frac{L^2}{2n\pi}\sin\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)$$

$$\int_{L/2}^{L}(L-x)\cos\frac{n\pi x}{L}dx$$

$$= \frac{L}{n\pi}(L-x)\sin\frac{n\pi x}{L}\Bigg|_{L/2}^{L} + \frac{L}{n\pi}\int_{L/2}^{L}\sin\frac{n\pi x}{L}dx$$

$$= -\left(\frac{L^2}{2n\pi}\sin\frac{n\pi}{2}\right) - \frac{L^2}{n^2\pi^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)$$

Triangle Function: Even Periodic Expansion

$$a_n = \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right)$$

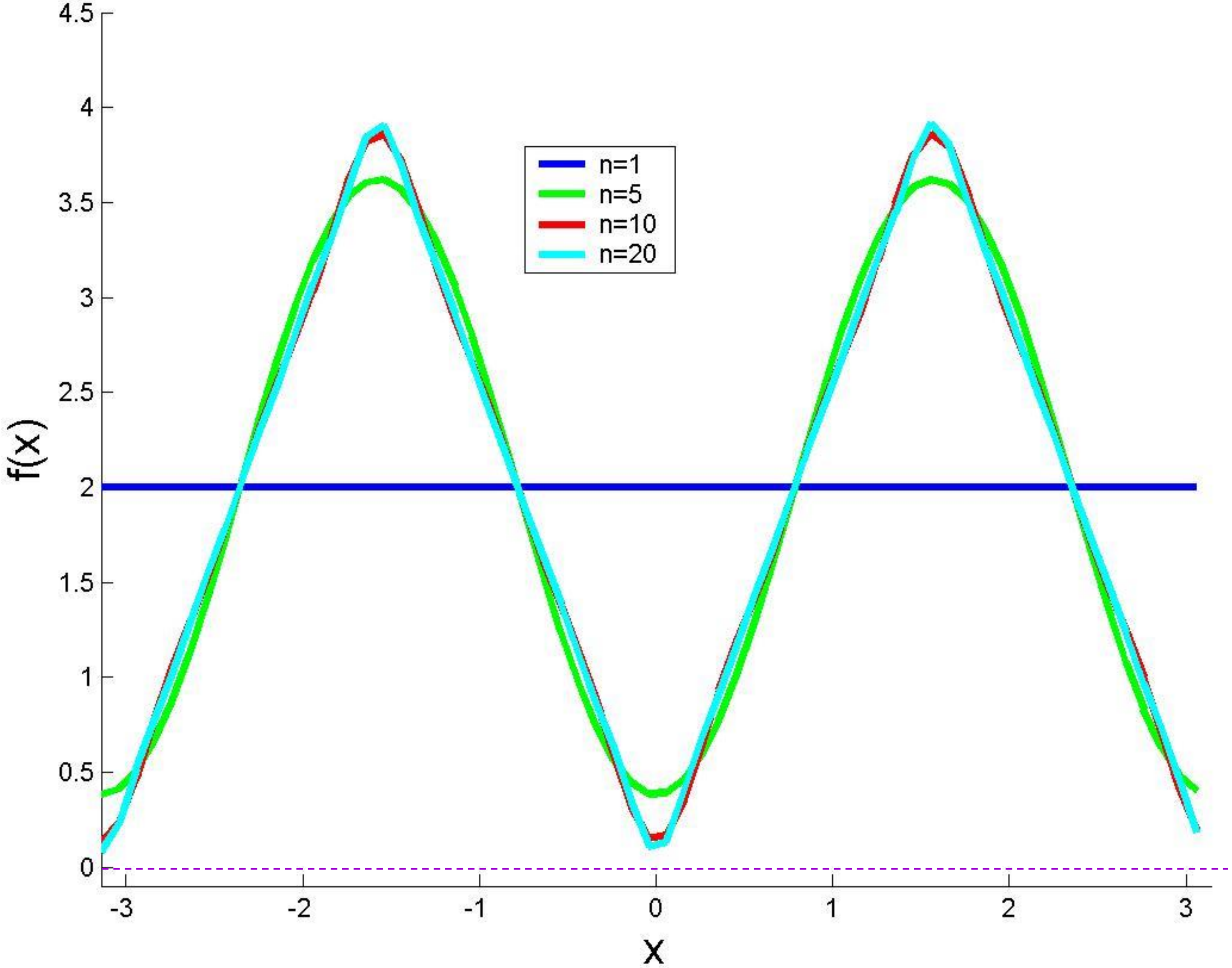
$$a_1 = 0, \quad a_2 = \frac{-16k}{2^2\pi^2}, \quad a_3 = 0, \quad a_4 = 0, \quad a_5 = 0$$

$$a_6 = \frac{-16k}{6^2\pi^2}, \quad a_{10} = \frac{-16k}{10^2\pi^2}, \cdots$$

$$a_n \neq 0, \quad n = 2,6,10,14,\cdots$$

$$\boxed{f(x) = \frac{k}{2} - \frac{-16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi}{L}x + \frac{1}{6^2}\cos\frac{6\pi}{L}x + \frac{1}{10^2}\cos\frac{10\pi}{L}x + \cdots\right)}$$

Triangle Function: Even Periodic Expansion



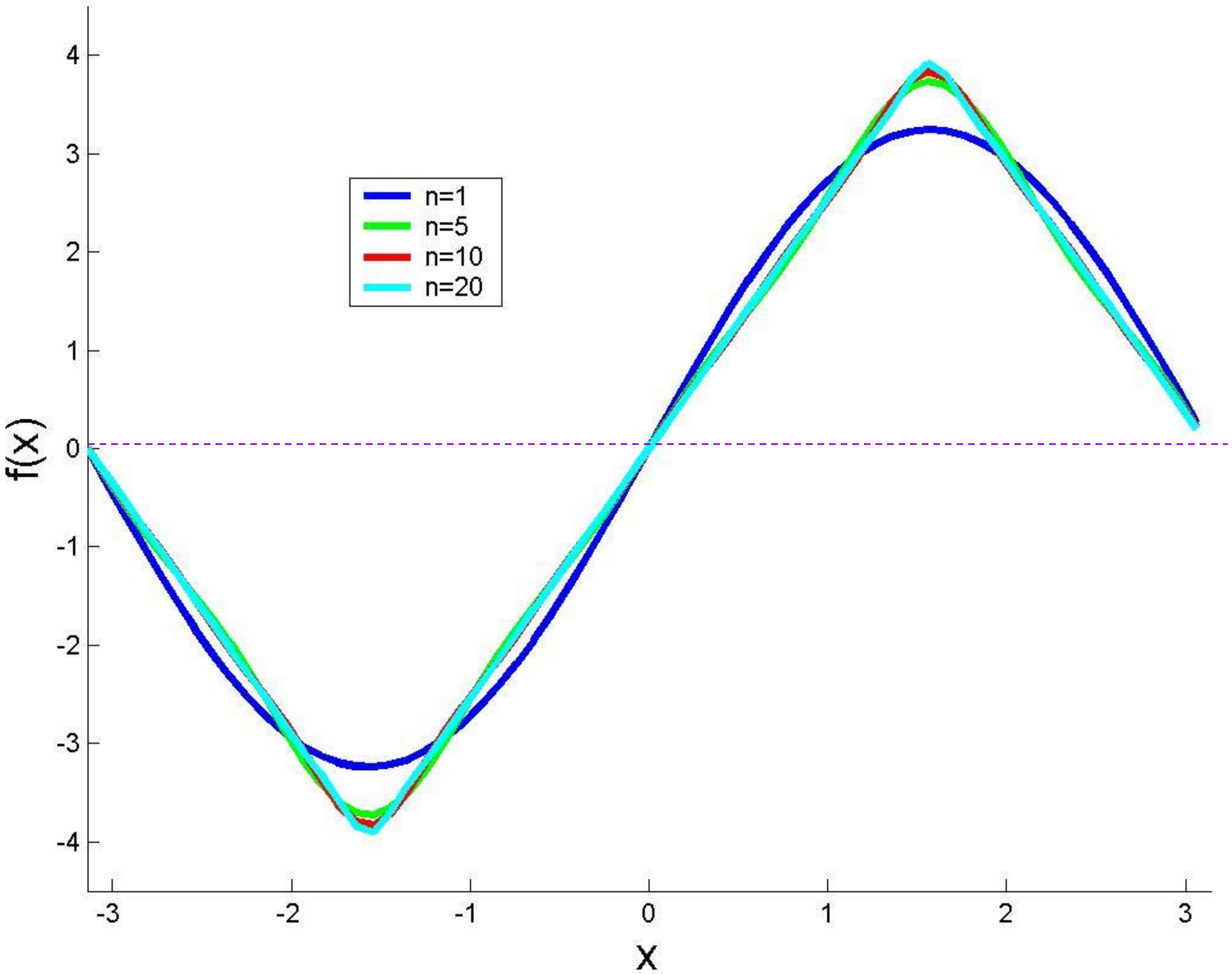
Triangle Function: Odd Periodic Expansion

$$b_n = \frac{1}{L}\int_{-L}^{L} o_f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{0}^{L} o_f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\boxed{f(x) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}x - \frac{1}{3^2}\sin\frac{3\pi}{L}x + \frac{1}{5^2}\sin\frac{5\pi}{L}x - + \cdots\right)}$$

Triangle Function: Odd Periodic Expansion



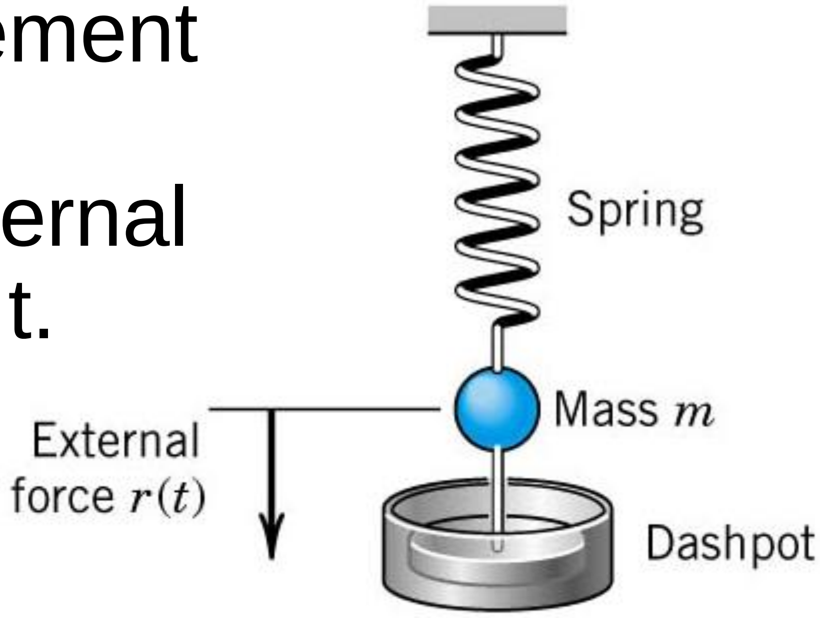
Ph.D.

Forced Oscillations

- Consider the forced oscillation of a body of mass m on spring of modulus are governed by the equation

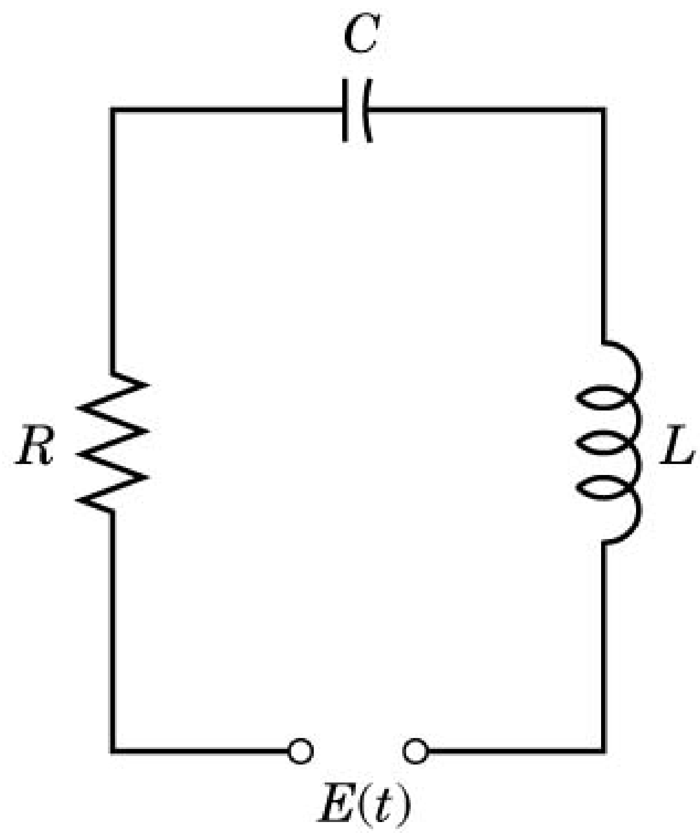
$$\boxed{my''(t)+cy'(t)+ky(t)=r(t)}$$

where $y(t)$ is the displacement from rest, c the damping constant, and $r(t)$ the external force depending on time t.



Forced Oscillations

- The electric analog to the spring is *RLC*-circuit: $$\boxed{LI''(t)+RI'(t)+\frac{1}{C}I(t)=E'(t)}$$



Forced Oscillations

- **Ex**. Let m=1(g), c=0.02(g/s), k=25(g/s^2), so that

$$y''(t)+0.02y'(t)+25y(t)=r(t)$$

where $r(t)$ is measured in $g\cdot\mathrm{cm}/s^2$. Let

$$r(t)=\begin{cases} t+\dfrac{\pi}{2} & -\pi<t<0 \\ -t+\dfrac{\pi}{2} & 0<t<\pi \end{cases}, \quad r(t+2\pi)=r(t)$$

Find the steady-state solution $y(t)$.

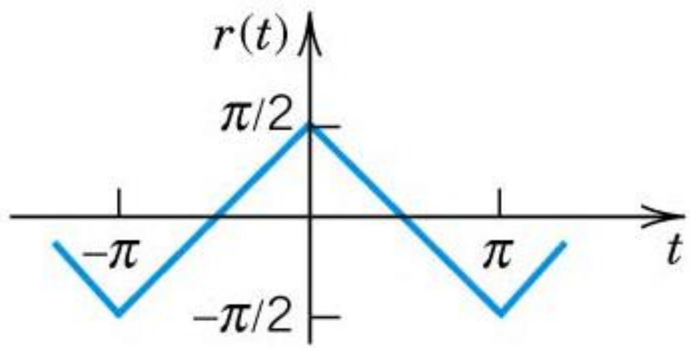
$$y_h(t)\approx Ae^{-0.01t}\cos 5t+Be^{-0.01t}\sin 5t$$

$$t\to\infty\Rightarrow y_h(t)\approx 0$$

Forced Oscillations

The Fourier series of $r(t)$ is

$$r(t) = \frac{4}{\pi}\left(\frac{1}{1^2}\cos t + \frac{1}{3^2}\cos 3t + \frac{1}{5^2}\cos 5t + \cdots\right)$$



Now we the following linear differential equation $y''(t) + 0.02y'(t) + 25y(t)$

$$= r(t) = \sum_{k=1}^{\infty} \frac{4}{(2k-1)^2\pi}\cos(2k-1)t$$

Forced Oscillations

The steady-state (particular) solution $y_n(t)$ corresponding to the sinusoidal input $\cos(2k\text{-}1)t=\cos nt$ can be computed using the method of undetermined coefficients

$$y_n(t) = A_n \cos nt + B_n \sin nt$$

$$A_n = \frac{4\left(25-n^2\right)}{n^2 \pi D}, \quad B_n = \frac{0.08}{n\pi D}$$

$$D = \left(25-n^2\right)^2 + \left(0.02n\right)^2$$

$$y_p(t) = y_1(t) + y_2(t) + y_3(t) + \cdots$$

Forced Oscillations

Let us find the amplitude $C_n = \sqrt{A_n^2 + B_n^2}$

We have

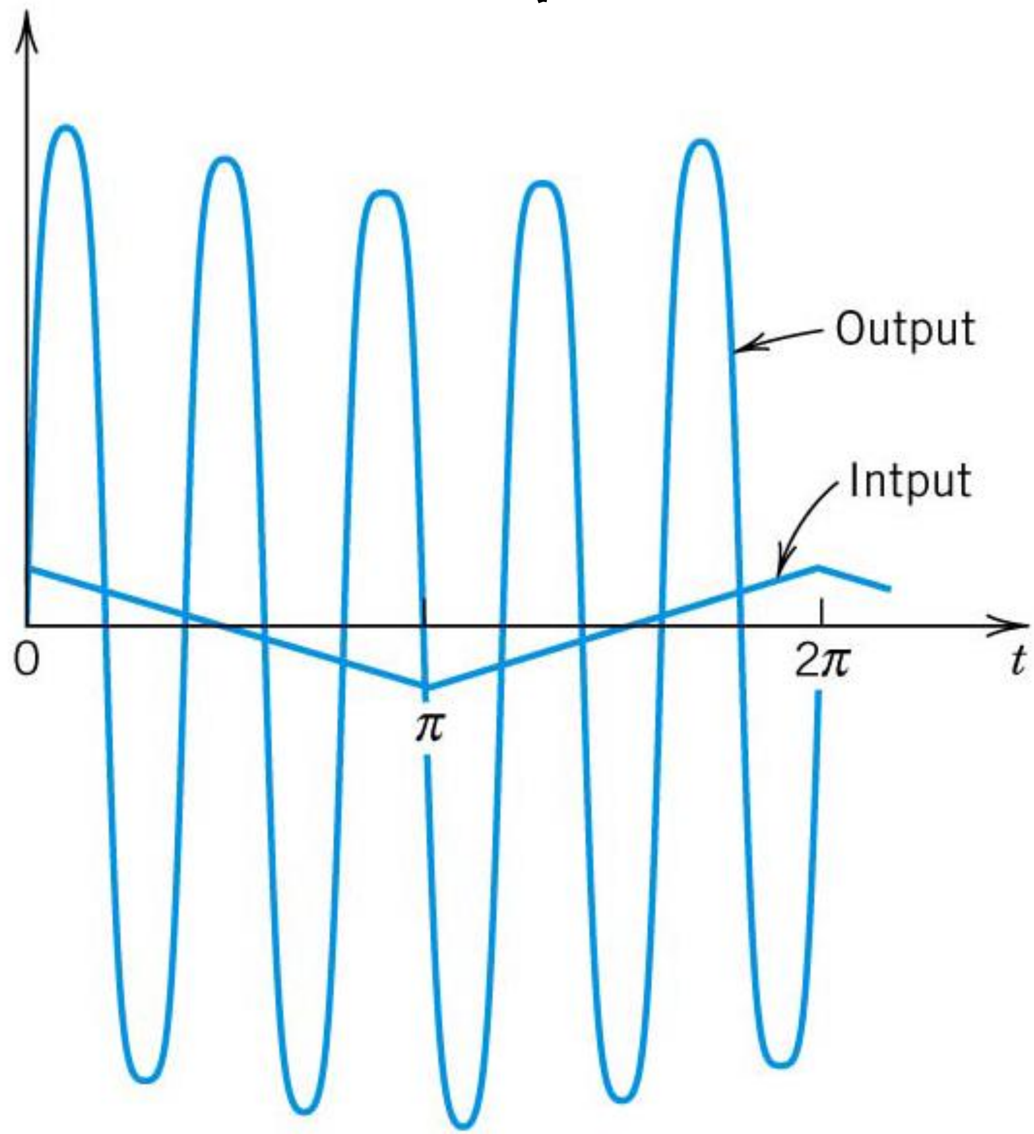
$C_1 = 0.0530$

$C_3 = 0.0088$

$\boxed{C_5 = 0.5100}$

$C_7 = 0.0011$

$C_9 = 0.0003$



Parseval's Theorem

Fourier Series Formulae

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx$$

Parseval's Theorem

- Show that:

$$\frac{1}{L}\int_{-L}^{L}\left[f(x)\right]^2 dx = 2a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

$$f^2(x) = \left[a_0 + \sum_{m=1}^{\infty}\left(a_m \cos\frac{m\pi}{L}x + b_m \sin\frac{m\pi}{L}x\right)\right]$$

$$\cdot\left[a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)\right]$$

$$\int_{-L}^{L} a_m b_n \cos\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$=\left(\frac{1}{2}a_m b_n\right)\int_{-L}^{L}\left[\sin\frac{(n+m)\pi}{L}x+\sin\frac{(n-m)\pi}{L}x\right]dx$$

$$=0,\quad \forall m,n$$

$$\int_{-L}^{L} a_0 a_0\, dx = 2La_0^2$$

$$\boxed{m \geq 1, \quad n \geq 1}$$

$$\int_{-L}^{L} a_m a_n \cos\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)dx = 0, \quad m \neq n$$

$$\int_{-L}^{L} a_n a_n \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)dx$$

$$= a_n^2 \int_{-L}^{L} \cos^2\left(\frac{n\pi}{L}x\right)dx = a_n^2 \int_{-L}^{L} \frac{1}{2}\left[1+\cos\left(\frac{2n\pi}{L}x\right)\right]dx$$

$$= La_n^2, \quad m = n$$

$$\boxed{m \geq 1, \quad n \geq 1}$$

$$\int_{-L}^{L} b_m b_n \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx = 0, \quad m \neq n$$

$$\int_{-L}^{L} b_n b_n \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$= b_n^2 \int_{-L}^{L} \sin^2\left(\frac{n\pi}{L}x\right)dx = b_n^2 \int_{-L}^{L} \frac{1}{2}\left[1-\cos\left(\frac{2n\pi}{L}x\right)\right]dx$$

$$= Lb_n^2, \quad m \neq n$$

$$f^2(x)=\left[a_0+\sum_{n=1}^{\infty}\left(a_m\cos\frac{m\pi}{L}x+b_m\sin\frac{m\pi}{L}x\right)\right]$$

$$\cdot\left[a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}x+b_n\sin\frac{n\pi}{L}x\right)\right]$$

$$=a_0^2+\sum_{n=1}^{\infty}\left(a_n^2\cos^2\frac{n\pi}{L}x+b_n^2\sin^2\frac{n\pi}{L}x\right)$$

$$+2\sum_{m\neq n}\left(a_m\cos\frac{m\pi}{L}x\right)\cdot\left(b_n\sin\frac{n\pi}{L}x\right)$$

$$+\sum_{m\neq n,m\geq 1,n\geq 1}\left(a_m\cos\frac{m\pi}{L}x\right)\cdot\left(a_n\cos\frac{n\pi}{L}x\right)$$

$$+\sum_{m\neq n,m\geq 1,n\geq 1}\left(b_m\sin\frac{m\pi}{L}x\right)\cdot\left(b_n\sin\frac{n\pi}{L}x\right)$$

$$\int_{-L}^{L} f^2(x)dx = \int_{-L}^{L} a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 \cos^2 \frac{n\pi}{L} x + b_n^2 \sin^2 \frac{n\pi}{L} x \right) dx$$

$$= L\left(2a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\right)$$

$$\frac{1}{L}\int_{-L}^{L} f^2(x)dx = 2a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

$$\boxed{\frac{1}{2L}\int_{-L}^{L} f^2(x)dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)}$$

Complex Fourier Series

Functions of Any Period $p=2L$

- A function $f(x)$ of period $p=2L$ has a Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$n = 1,2,3,\cdots$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

Euler Formulae

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right), \quad \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

$$a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

$$= \frac{1}{2} a_n \left(e^{i\frac{n\pi x}{L}} + e^{-i\frac{n\pi x}{L}}\right) + \frac{1}{2i} b_n \left(e^{i\frac{n\pi x}{L}} - e^{-i\frac{n\pi x}{L}}\right)$$

$$= \frac{1}{2}\left(a_n - ib_n\right)e^{i\frac{n\pi x}{L}} + \frac{1}{2}\left(a_n + ib_n\right)e^{-i\frac{n\pi x}{L}}$$

Complex Fourier Series

$$\frac{1}{2}\left(a_n - ib_n\right) = \frac{1}{2L}\int_{-L}^{L} f\left(x\right)\left(\cos\frac{n\pi x}{L} - i\sin\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{2L}\int_{-L}^{L} f\left(x\right)e^{-i\frac{n\pi x}{L}}dx = c_n$$

$$\frac{1}{2}\left(a_n + ib_n\right) = \frac{1}{2L}\int_{-L}^{L} f\left(x\right)\left(\cos\frac{n\pi x}{L} + i\sin\frac{n\pi x}{L}\right)dx$$

$$= \frac{1}{2L}\int_{-L}^{L} f\left(x\right)e^{-i\frac{(-n)\pi x}{L}}dx = c_{-n}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f\left(x\right)e^{-i\frac{(0)\pi x}{L}}dx = c_0$$

Complex Fourier Series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

$$= a_0 + \sum_{n=1}^{\infty} \frac{1}{2}(a_n - ib_n) e^{i\frac{n\pi x}{L}} + \sum_{n=1}^{\infty} \frac{1}{2}(a_n + ib_n) e^{-i\frac{n\pi x}{L}}$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{i\frac{n\pi x}{L}} + \sum_{n=1}^{\infty} c_{-n} e^{-i\frac{n\pi x}{L}} = c_0 + \sum_{n=1}^{\infty} c_n e^{i\frac{n\pi x}{L}} + \sum_{n=1}^{\infty} c_{-n} e^{i\frac{(-n)\pi x}{L}}$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{i\frac{n\pi x}{L}} + \sum_{n=-1}^{-\infty} c_n e^{i\frac{n\pi x}{L}}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$$

$$\boxed{c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{-i\frac{n\pi x}{L}} dx}$$

Let Us Play Fourier Series!!

Functions of Any Period $p=2L$

- A function $f(x)$ of period $p=2L$ has a Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx$$

$$n = 1, 2, 3, \cdots$$

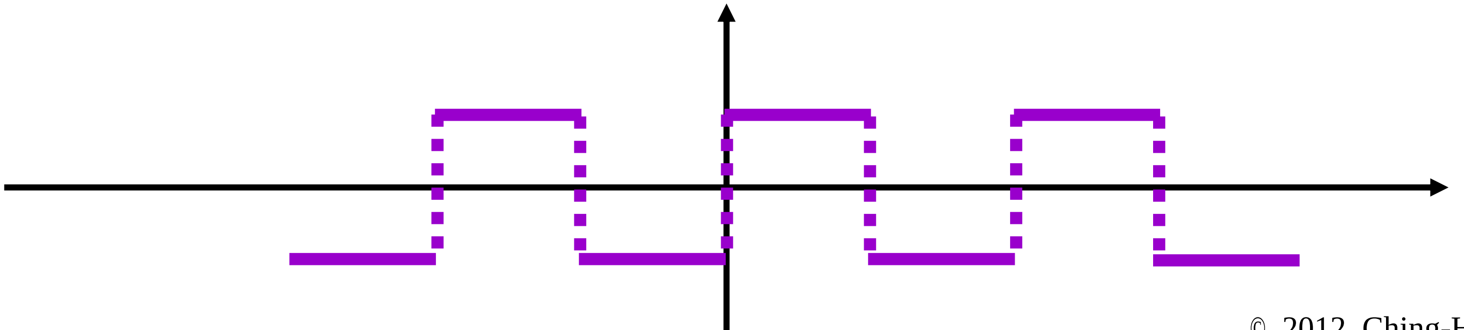
$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} dx \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} dx$$

Periodic Square Wave (I)

- **Ex**. Find the Fourier series of the function

$$f(x)=\begin{cases}1 & 0<x<L\\ -1 & -L<x<0\end{cases}$$

Since the function $f(x)$ is odd, there are no constant and cosine terms in the Fourier's series, i.e., $a_0=a_1=a_2=\ldots=0$!!



Periodic Square Wave (I)

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

$$= \frac{-1}{L}\int_{-L}^{0} 1\cdot\sin\frac{n\pi x}{L}dx + \frac{1}{L}\int_{0}^{L} 1\cdot\sin\frac{n\pi x}{L}dx$$

$$= \frac{2}{L}\int_{0}^{L} 1\cdot\sin\frac{n\pi x}{L}dx = \frac{2L}{Ln\pi}\left(-\cos\frac{n\pi x}{L}\right)\Bigg|_0^L$$

$$= \frac{2}{n\pi}(1-\cos n\pi)$$

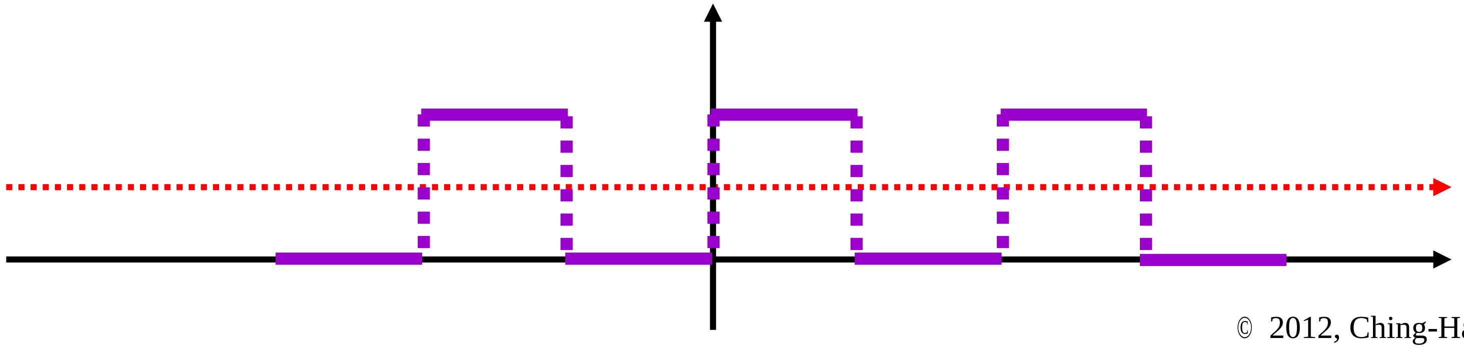
$$f(x) = \sum_{n=1}^{\infty}\left[\frac{2(1-\cos n\pi)}{n\pi}\cdot\sin\frac{n\pi}{L}x\right]$$

Periodic Square Wave (II)

- **Ex**. Find the Fourier series of the function

$$g(x) = \begin{cases} 1 & 0 < x < L \\ 0 & -L < x < 0 \end{cases}$$

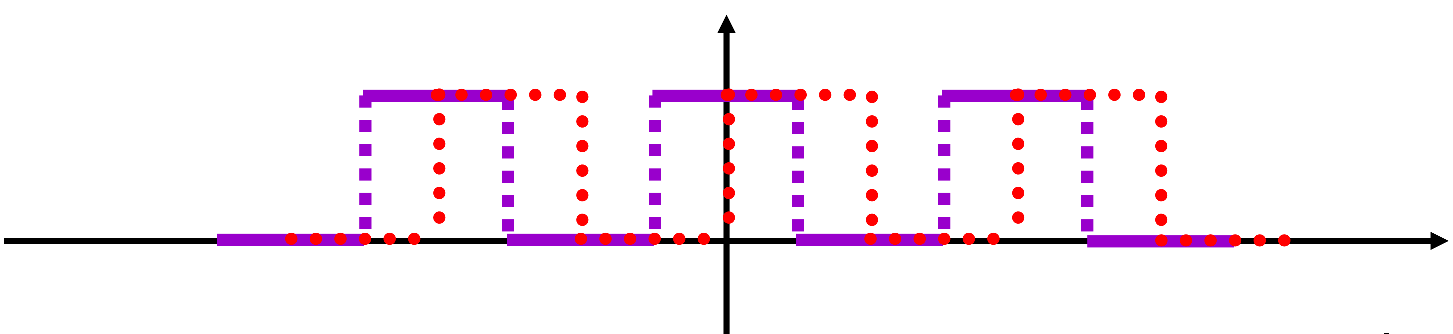
$$g(x) = \frac{1}{2}\left(1 + f(x)\right) = \frac{1}{2} + \sum_{n=1}^{\infty}\left[\frac{(1-\cos n\pi)}{n\pi} \cdot \sin\frac{n\pi}{L}x\right]$$



Periodic Square Wave (III)

- **<u>Ex</u>**. Find the Fourier series of the function

$$h(x)=\begin{cases}0 & L/2<x<L \\ 1 & -L/2<x<L/2 \\ 0 & -L<x<-L/2\end{cases}$$



Periodic Square Wave (III)

$$h(x) = g\left(x + \frac{L}{2}\right)$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty}\left[\frac{(1-\cos n\pi)}{n\pi}\cdot\sin\frac{n\pi}{L}\left(x + \frac{L}{2}\right)\right]$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty}\left[\frac{(1-\cos n\pi)}{n\pi}\cdot\left(\sin\frac{n\pi}{L}x\cos\frac{n\pi L}{2L} + \cos\frac{n\pi}{L}x\sin\frac{n\pi L}{2L}\right)\right]$$

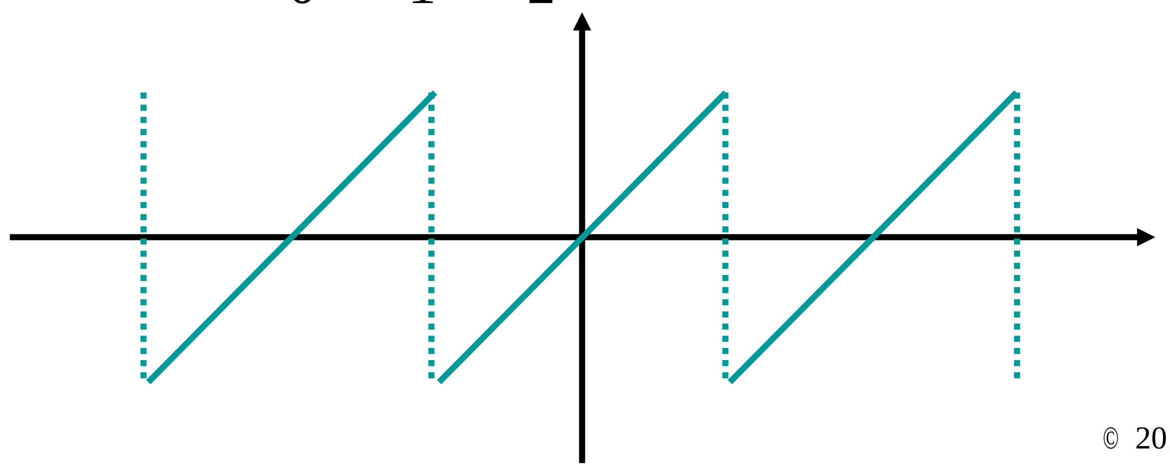
$$= \frac{1}{2} + \sum_{n=1}^{\infty}\left[\frac{(1-\cos n\pi)}{n\pi}\cdot\sin\frac{n\pi}{2}\cdot\cos\frac{n\pi}{L}x\right]$$

Triangle Wave (I)

- **<u>Ex</u>**. Find the Fourier series of the function

$$f(x) = x, \quad -L \leq x \leq L$$

Since the function $f(x)$ is odd, there are no constant and cosine terms in the Fourier's series, i.e., $a_0 = a_1 = a_2 = \ldots = 0$!!



$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = \frac{1}{L}\int_{-L}^{L} x\sin\frac{n\pi x}{L}dx$$

$$= \frac{2}{L}\int_{0}^{L} x\sin\frac{n\pi x}{L}dx = -\frac{2}{L}\cdot\frac{L}{n\pi}\cdot\int_{0}^{L} x d\cos\frac{n\pi x}{L}$$

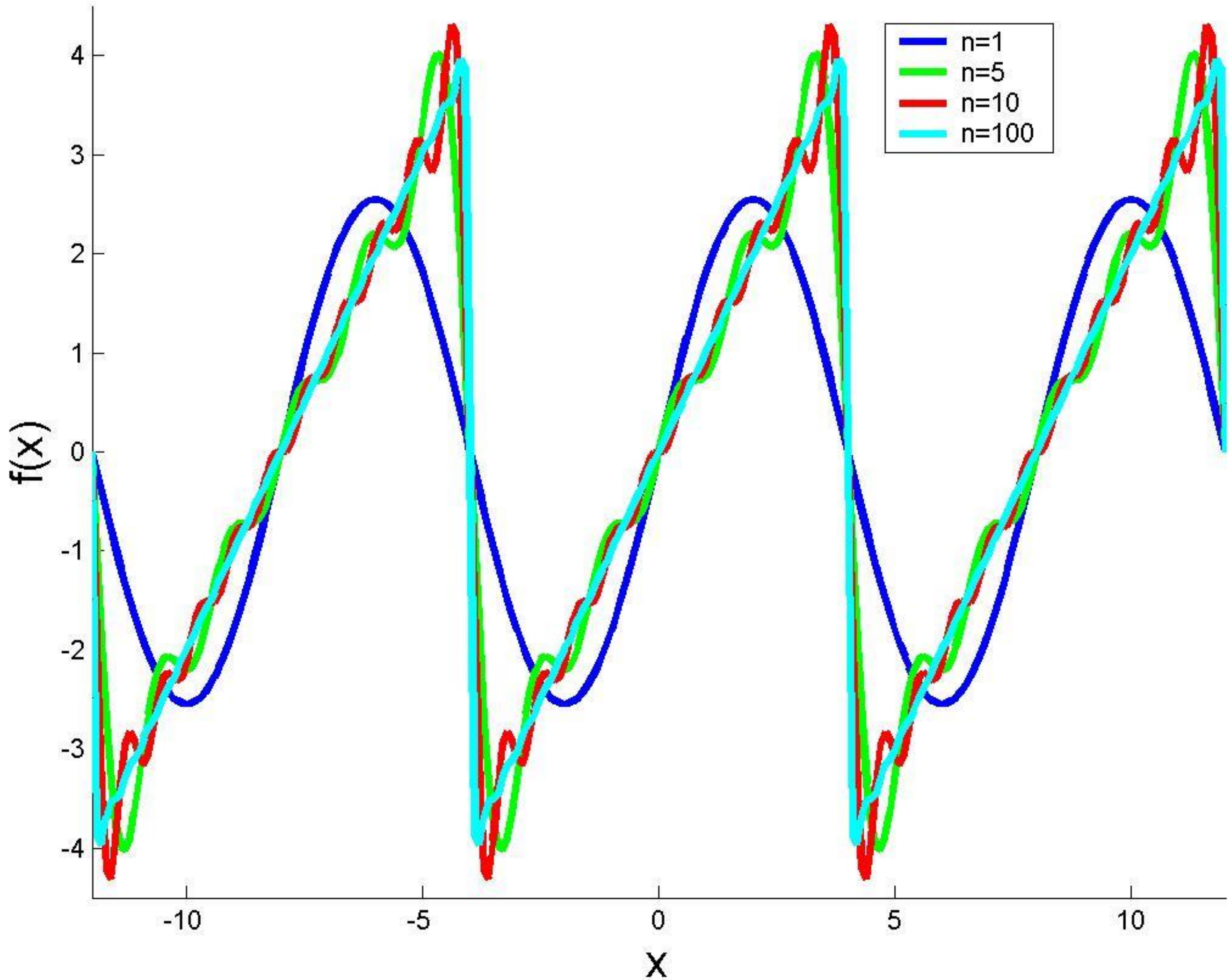
$$= -\frac{2}{n\pi}x\cos\frac{n\pi x}{L}\Big|_0^L + \frac{2}{n\pi}\int_0^L \cos\frac{n\pi x}{L}dx$$

$$= -\frac{2L}{n\pi}\cos n\pi + \frac{2}{n\pi}\cdot\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_0^L$$

$$= -\frac{2L}{n\pi}\cos n\pi$$

$$\boxed{f(x) = \sum_{n=1}^{\infty}\left[\frac{-2L}{n\pi}\cos n\pi \sin\frac{n\pi}{L}x\right]}$$

Triangle Wave (I)



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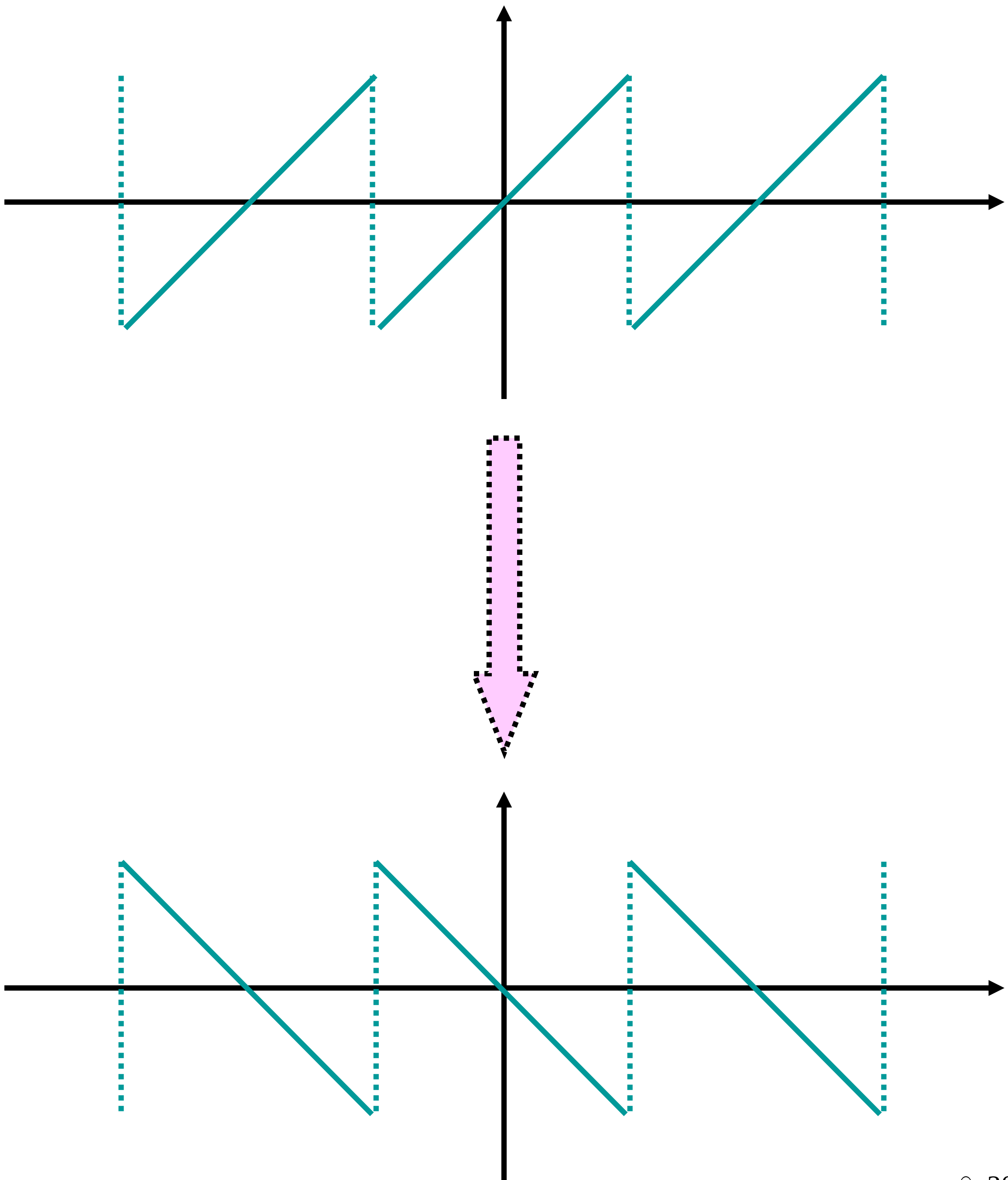
Triangle Wave (II)

- **<u>Ex</u>**. Find the Fourier series of the function

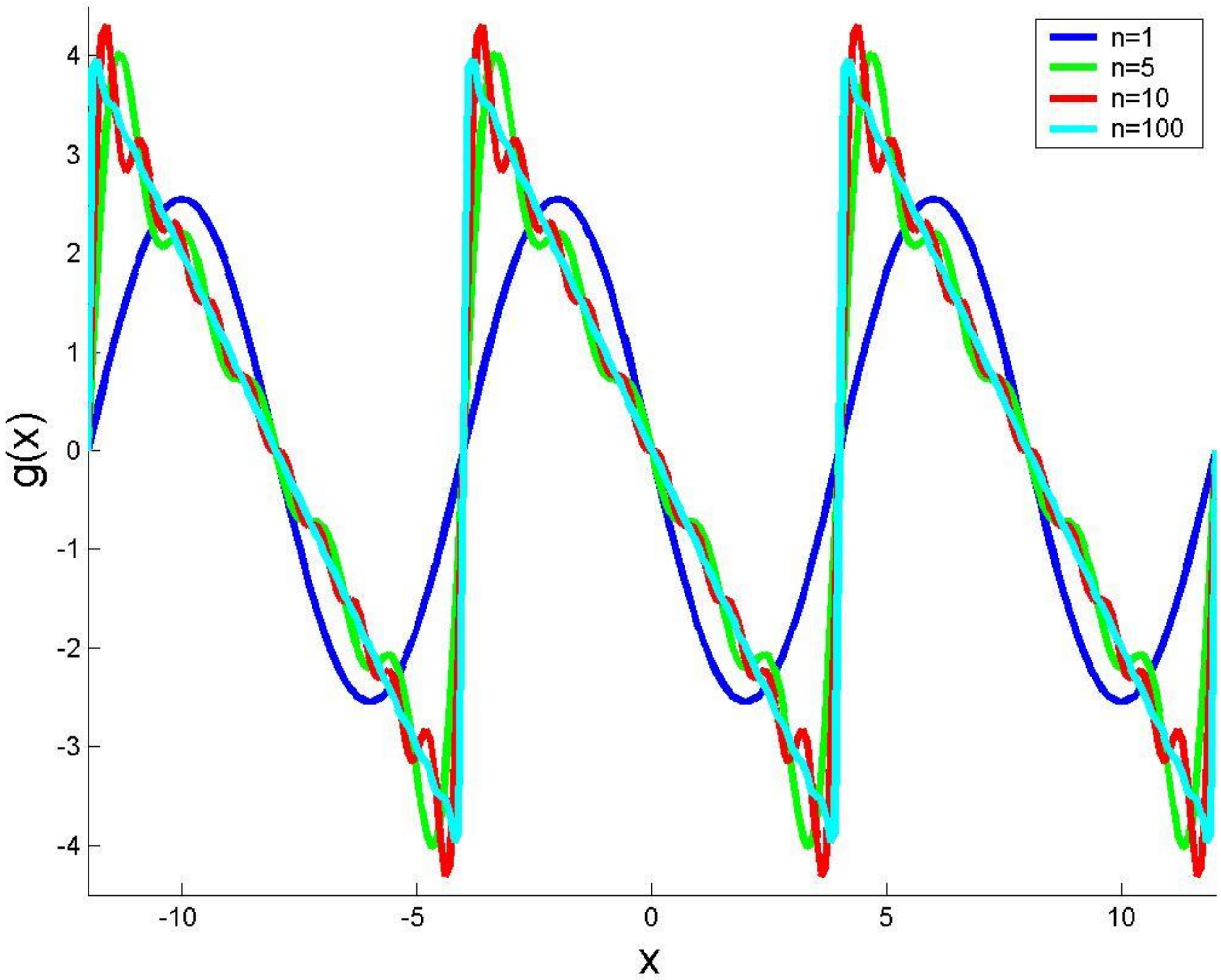
$$g(x) = f(-x) = -x, \quad -L \le x \le L$$

Since the function $f(x)$ is odd, there are no constant and cosine terms in the Fourier's series, i.e., $a_0 = a_1 = a_2 = \ldots = 0$!!

$$\boxed{g(x) = \sum_{n=1}^{\infty} \left[\frac{2L}{n\pi} \cos n\pi \sin \frac{n\pi}{L} x \right]}$$



Triangle Wave (II)

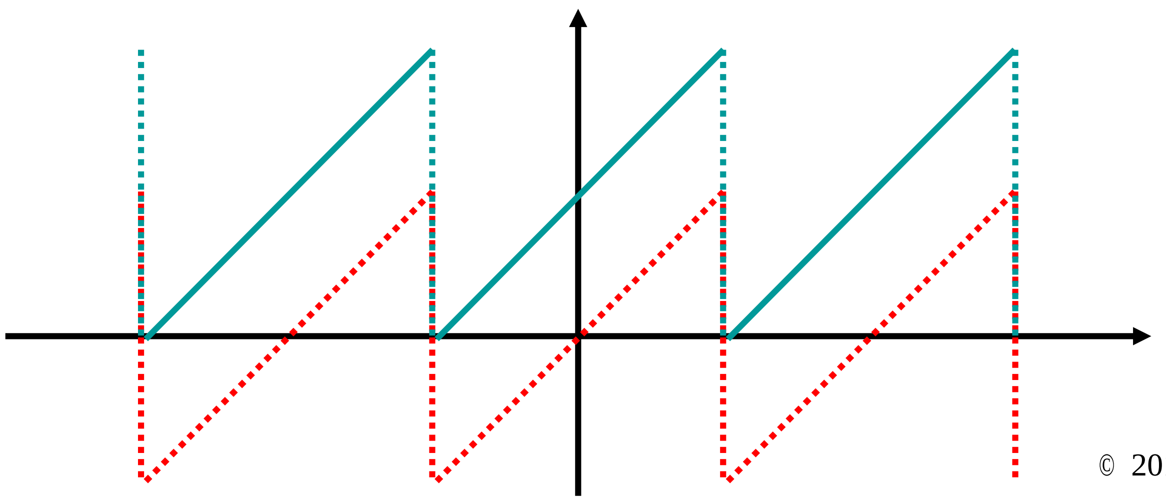


Triangle Wave (III)

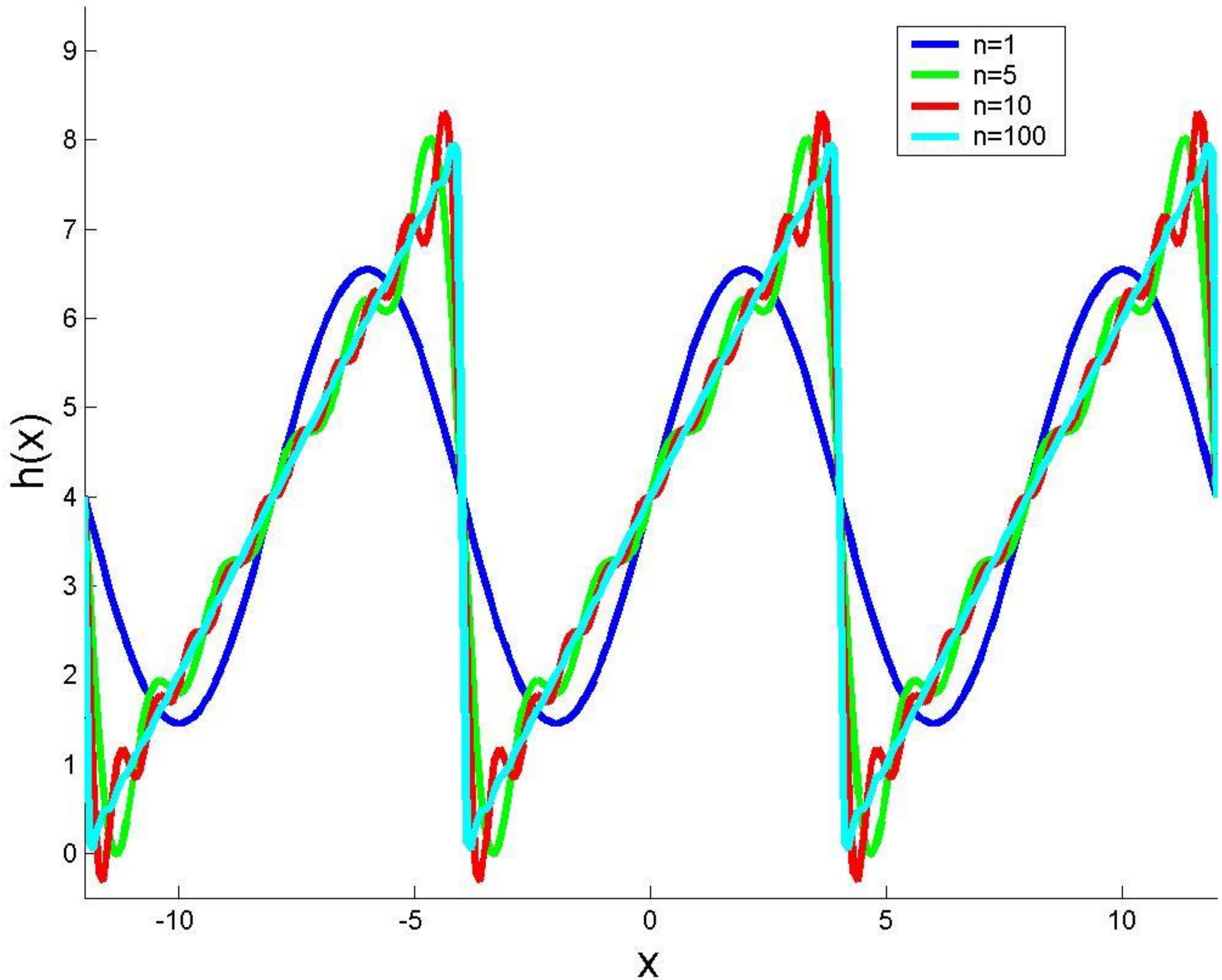
- **<u>Ex</u>**. Find the Fourier series of the function

$$h(x) = f(x) + L = x + L, \quad -L \le x \le L$$

$$\boxed{h(x) = L + \sum_{n=1}^{\infty}\left[\frac{-2L}{n\pi}\cos n\pi \sin\frac{n\pi}{L}x\right]}$$



Triangle Wave (III)

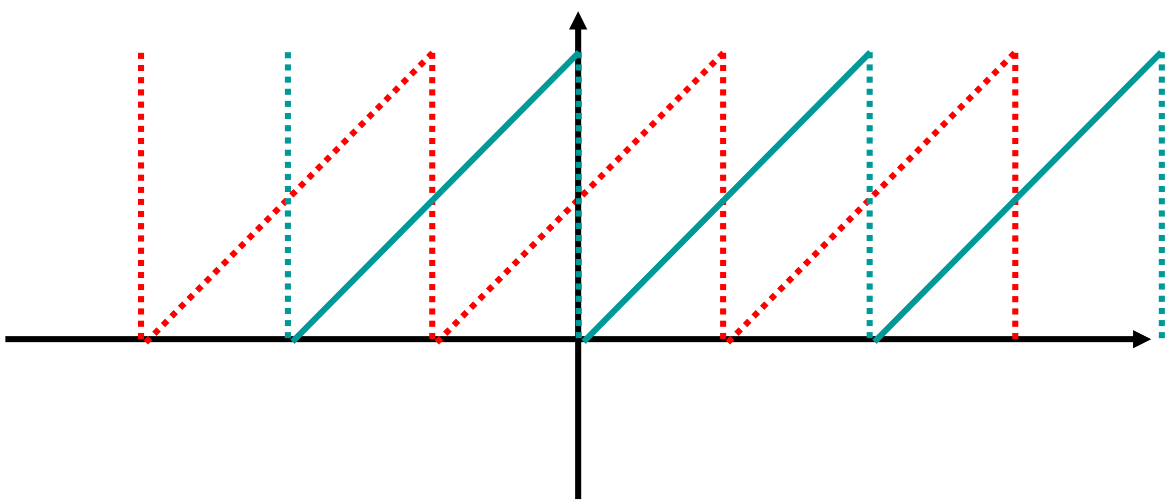


Triangle Wave (IV)

- **Ex**. Find the Fourier series of the function

$$p(x) = h(x - L) = x,$$

$$-L \le x - L \le L \Rightarrow 0 \le x \le 2L$$



$$p(x) = h(x - L)$$

$$= L + \sum_{n=1}^{\infty}\left[\frac{-2L}{n\pi}\cos n\pi \sin\frac{n\pi}{L}(x - L)\right]$$

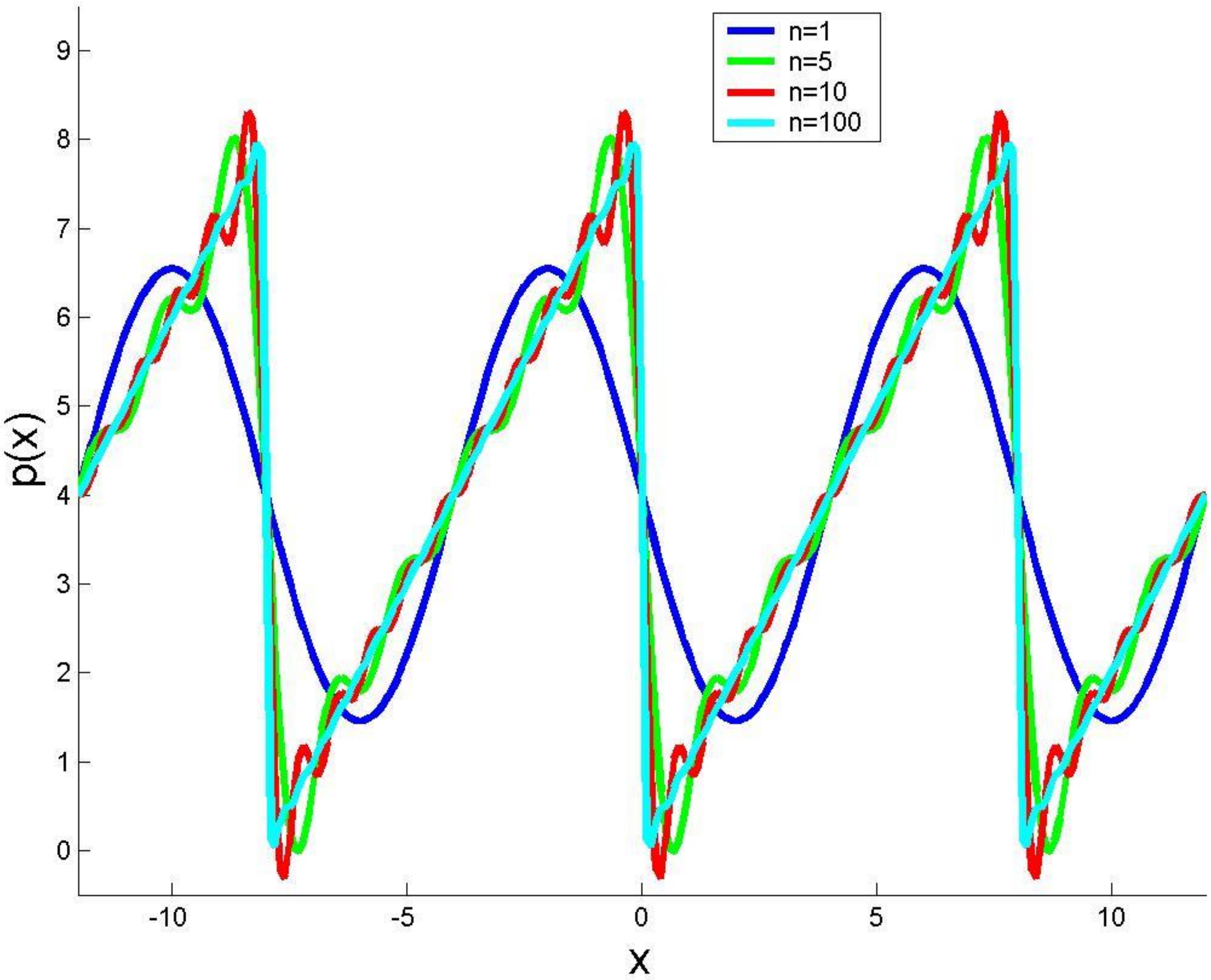
$$\sin\frac{n\pi}{L}(x - L)$$

$$= \sin\frac{n\pi x}{L}\cos\frac{n\pi L}{L} - \sin\frac{n\pi L}{L}\cos\frac{n\pi x}{L}$$

$$= \cos n\pi \sin\frac{n\pi x}{L}$$

$$\boxed{p(x) = L + \sum_{n=1}^{\infty}\left[\frac{-2L}{n\pi}\sin\frac{n\pi}{L}x\right]}$$

Triangle Wave (IV)



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